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Symmetry Groups in Quantum Mechanics and the Theorem of Wigner on the Symmetry Transformations

Gianni Cassinelli
Department of Physics and I.N.F.N., University of Genoa
Via Dodecaneso 33, 16146 Genoa, Italy
E-mail: cassinelli@ ge.infn.it

Ernesto DeVito
Department of Physics and I.N.F.N., University of Genoa
Via Dodecaneso 33, 16146 Genoa, Italy
E-mail: devito@ ge.infn.it

Pekka J. Lahti
Department of Physics, University of Turku
FIN-20014 Turku, Finland
E-mail: pekka.lahti@ utu.fi

Alberto Levrero
Department of Physics and I.N.F.N., University of Genoa
Via Dodecaneso 33, 16146 Genoa, Italy
E-mail: levrero@ ge.infn.it

Abstract

Various mathematical formulations of the symmetry group in quantum mechanics are investigated and shown to be mutually equivalent. A new proof of the theorem of Wigner on the symmetry transformations is worked out.

1. Introduction. The Hilbert space formulation of quantum mechanics points out several mathematical objects whose physical meaning is connected with the probabilistic structure of the theory. Among them there are:

1) the set of pure states $\mathbf{P}$ with the notion of *transition probability*,

2) the *convex* set $\mathbf{S}$ of states,

3) the *orthomodular lattice* $\mathbf{L}$ of the closed subspaces,

4) the *partial algebra* $\mathbf{E}$ of the positive operators bounded by the unit operator,

5) the *Jordan algebra* $\mathbf{B}_r$ of the self-adjoint bounded operators,

6) the C^**-algebra* $\mathbf{B}$ of the bounded operators.

The automorphisms of these sets, that is, the one to one maps from a given set onto itself preserving the corresponding relevant structure (transition probability, convexity, ...) are natural candidates to represent the symmetries of quantum mechanics. Moreover, the set of the automorphisms of any of the previous mathematical objects forms a group, denoted by $\mathrm{Aut}(\cdot)$, under the natural composition of mappings. So there are several groups which can be used to represent the symmetries of quantum mechanics. This poses the question on the equivalence of these groups, that is, if they are isomorphic in some natural way.

It is well known that the answer to the above question is positive (at least if the dimension of the Hilbert space is greater than two). Moreover, the Wigner theorem shows that one can associate to each element of $\mathrm{Aut}\,(\mathbf{P})$ a unitary or antiunitary operator which is unique up to a phase factor. In this way the previous groups $\mathrm{Aut}(\cdot)$ are shown to be isomorphic to the *symmetry group* $\Sigma(\mathcal{H})$ of the Hilbert space, that is, the group of unitary or antiunitary operators modulo the phase group. This allows using the theory of unitary representations of groups in order to implement symmetry in quantum mechanics.

Much of these facts are well known since long time and there is a rich literature on this topic. However, up to our knowledge there is no complete concise and simple treatment of the relations among these different symmetry groups.

The present paper is devoted to fill this gap with special care on the following aspects:

1) to point out the two dimensional case where some of the previous identifications fail to be true;

2) to discuss the topological properties of these isomorphisms that are indispensable in applying the theory of unitary representations;

3) to give a new simple proof of the Wigner theorem, whose need has been emphasised by Weinberg in his recent book [1].

We base our treatement on the following works which we believe are the essential contributions to the problems we are concerned with.

1) Wigner's book [2]: it contains the original idea on the isomorphism between $\mathrm{Aut}\,(\mathbf{P})$ and $\Sigma(\mathcal{H})$.

2) Uhlhorn's paper [3]: it proves Wigner theorem with a weaker assumption, but assuming that the dimension of the Hilbert space is greater than two. It also studies some relations between $\mathrm{Aut}\,(\mathbf{P})$ and $\mathrm{Aut}(\mathbf{B})$.

3) Bargmann's paper [4]: it gives the first complete proof of the Wigner theorem without any assumption on the dimension of $\mathcal{H}$.

4) Varadarajan's book [5]: it discusses the isomorphisms of some of the automorphism groups and the Wigner theorem using the fundamental theorem of projective geometry. This makes these results less accessible and they are also subject to the dimension limitation $\dim(\mathcal{H}) \geq 3$.

5) Simon's review [6]: it makes a survey of the relations among $\mathrm{Aut}\,(\mathbf{P})$, $\mathrm{Aut}\,(\mathbf{S})$, $\mathrm{Aut}\,(\mathbf{B}_r)$ and $\Sigma(\mathcal{H})$ and it points out the case of the time evolution as a one

parameter group of symmetries.

6) Ludwig's book [7]: it discusses in great detail the properties of the various automorphisms on $\mathbf{E}$, assuming again that $\dim(\mathcal{H}) \geq 3$.

The paper is organised in the following way. In section 2 we define the previous groups of automorphisms and we endow them with some *natural* topologies arising from the probabilistic structure of quantum mechanics. In the following section we give a new proof of the Wigner theorem based on the idea of positive cones in the Hilbert space. In section 4 we show that the groups Aut $(\mathbf{P})$, Aut $(\mathbf{S})$, Aut $(\mathbf{L})$, Aut $(\mathbf{E})$, Aut $(\mathbf{B}_r)$, Aut$(\mathbf{B})$ and $\Sigma(\mathcal{H})$ are isomorphic. In the last section we discuss the topological properties of these groups and we prove that they are second countable metrisable topological groups and that the previous isomorphisms are in fact homeomorphisms.

As a final introductory comment we notice that when the Hilbert space is one dimensional all the above automorphism groups have only one element whereas the group $\Sigma(\mathcal{H})$ has two elements. From now onwards we assume that $\dim \mathcal{H} \geq 2$.

2. The automorphism groups. We first introduce some notations. Let $\mathcal{H}$ be a complex separable Hilbert space associated with a quantum system. The inner product $\langle \mid \rangle$ is taken to be linear with respect to the second argument. We let $\mathrm{tr}\left[\cdot\right]$ denote the trace functional and $P[\varphi]$ the projection on the one-dimensional subspace $[\varphi]$ generated by the nonzero vector $\varphi \in \mathcal{H}$ (so that for any $\psi \in \mathcal{H}$, $P[\varphi]\psi = \frac{\langle \varphi \mid \psi \rangle}{\langle \varphi \mid \varphi \rangle}\varphi$). We denote by $\mathbf{B}_1$ the set of trace class operators on $\mathcal{H}$ and by $\mathbf{B}_{1,r}$ the set of self-adjoint trace class operators on $\mathcal{H}$.

We introduce next the various mathematical objects with their specific structures and the corresponding groups of automorphisms.

2.1. Let $\mathbf{P}$ be the set of one-dimensional projections on $\mathcal{H}$. From physical point of view $\mathbf{P}$ is the set of pure states of a quantum system. We endow $\mathbf{P}$ with the notion of transition probability:

$$\mathbf{P} \times \mathbf{P} \ni (P_1, P_2) \mapsto \mathrm{tr}\big[P_1 P_2\big] \in [0,1].$$

The corresponding automorphisms are the bijective maps $\alpha : \mathbf{P} \to \mathbf{P}$ that satisfy the condition

$$\mathrm{tr}\big[\alpha(P_1)\alpha(P_2)\big] = \mathrm{tr}\big[P_1 P_2\big]$$

for all $P_1, P_2 \in \mathbf{P}$. We call them $\mathbf{P}$-*automorphisms.* The set of such maps is denoted by $\mathrm{Aut}\,(\mathbf{P})$ and it forms a group.

If α satisfies only the weaker condition

$$\mathrm{tr}\big[\alpha(P_1)\alpha(P_2)\big] = 0 \Longleftrightarrow \mathrm{tr}\big[P_1 P_2\big] = 0 \qquad P_1, P_2 \in \mathbf{P},$$

we call it a *weak* $\mathbf{P}$-*automorphism.* In this way, a weak $\mathbf{P}$-automorphism is a bijective map preserving only the zero probabilities. The group of weak $\mathbf{P}$-automorphisms is denoted by $\mathrm{Aut}_w\,(\mathbf{P})$. Clearly $\mathrm{Aut}\,(\mathbf{P})$ is a subgroup of $\mathrm{Aut}_w\,(\mathbf{P})$. We endow both $\mathrm{Aut}\,(\mathbf{P})$ and $\mathrm{Aut}_w\,(\mathbf{P})$ with the initial topology defined by the following set of functions

$$f^{\mathbf{P}}_{P_1,P_2} : \alpha \mapsto \mathrm{tr}\big[P_1 \alpha(P_2)\big], \quad P_1, P_2 \in \mathbf{P}.$$

This is the natural topology with respect to transition probabilities.

2.2. Let $\mathbf{S}$ be the set of positive trace class operators of trace one,

$$\mathbf{S} = \{T \in \mathbf{B}_1 \,|\, T \geq O, \mathrm{tr}\big[T\big] = 1\}.$$

The elements of $\mathbf{S}$ represent the states of the quantum system. The set $\mathbf{S}$ is a convex subset of the vector space $\mathbf{B}_1$, that is, if $T_1, T_2 \in \mathbf{S}$ and $0 \leq w \leq 1$, then $wT_1 + (1-w)T_2 \in \mathbf{S}$. We observe that $\mathbf{P}$ is a subset of $\mathbf{S}$, in fact it is the set of extremal points of $\mathbf{S}$. The relevant automorphisms are the bijective maps $V : \mathbf{S} \to \mathbf{S}$ such that

$$V\big(wT_1 + (1-w)T_2\big) = wV(T_1) + (1-w)V(T_2)$$

for all $T_1, T_2 \in \mathbf{S}$, and for all $0 \leq w \leq 1$. These are the $\mathbf{S}$-*automorphisms* and they form a group denoted by $\mathrm{Aut}\,(\mathbf{S})$. We endow $\mathrm{Aut}\,(\mathbf{S})$ with the initial topology defined by the following set of functions

$$f^{\mathbf{S}}_{A,T} : V \mapsto \mathrm{tr}\big[AV(T)\big], \quad A \in \mathbf{B}_r,\ T \in \mathbf{S},$$

that are related to the probabilistic interpretation of the elements of $\mathbf{S}$.

We have the following properties of the $\mathbf{S}$-automorphisms:

LEMMA 2.1. *Let $V \in Aut\,(\mathbf{S})$.*

1) V is the restriction of a trace-norm preserving linear operator on $\mathbf{B}_{1,r}$;

2) if $P \in \mathbf{P}$, then $V(P) \in \mathbf{P}$;

3) if $V(P) = P$ for all $P \in \mathbf{P}$, then V is the identity.

PROOF: 1) We recall that V has a unique extension to a positive trace-preserving bijective linear map $\hat{V}$ on $\mathbf{B}_{1,r}$. For any $T \in \mathbf{B}_{1,r}$, write $T = T^+ - T^-$, where $T^{\pm} = \frac{1}{2}(|T| \pm T)$. Then

$$\begin{aligned}
\left\|\hat{V}(T)\right\|_1 &= \left\|\hat{V}(T^+ - T^-)\right\|_1 \\
&= \left\|\hat{V}(T^+) - \hat{V}(T^-)\right\|_1 \\
&\leq \left\|\hat{V}(T^+)\right\|_1 + \left\|\hat{V}(T^-)\right\|_1 \\
&= \|T^+\|_1 + \|T^-\|_1 \\
&= \|T\|_1 .
\end{aligned}$$

Since V^{-1} has the same properties than V we conclude that $\left\|\hat{V}(T)\right\|_1 = \|T\|_1$. 2) Let $P \in \mathbf{P}$ and assume that $V(P) = wT_1 + (1-w)T_2$ for some $0 < w < 1, T_1, T_2 \in \mathbf{S}$. Then $P = wV^{-1}(T_1) + (1-w)V^{-1}(T_2)$, so that $P = V^{-1}(T_1) = V^{-1}(T_2)$ and thus $V(P) = T_1 = T_2$ showing that $V(P) \in \mathbf{P}$. 3) Any $T \in \mathbf{S}$ can be expressed as $T = \sum_i w_i P_i$ for some sequence (w_i) of weights $[0 \leq w_i \leq 1, \sum w_i = 1]$ and for some sequence of elements $(P_i) \subset \mathbf{P}$ with the series converging in the trace norm.

The continuity of V thus gives $V(T) = T$ for all $T \in \mathbf{S}$ whenever $V(P) = P$ for all $P \in \mathbf{P}$.

2.3. Let $\mathbf{L}$ be the set of the closed subspaces of $\mathcal{H}$. This set can be described also as the set of projections on $\mathcal{H}$, identifying the closed subspace $M \in \mathbf{L}$ with the corresponding projection (denoted by the same symbol) onto M. $\mathbf{L}$ is a complete orthocomplemented (orthomodular) lattice with respect to the subspace inclusion as the order relation and the orthogonal complement $M \mapsto M^{\perp}$ as the orthocomplementation. By the previous identification, $\mathbf{P}$ is also a subset of $\mathbf{L}$. In fact, it is precisely the set of the minimal elements of $\mathbf{L}$. From a physical point of view the elements of $\mathbf{L}$ can be interpreted as the propositions on the system.

A function $\tau : \mathbf{L} \to \mathbf{L}$ is an $\mathbf{L}$-*automorphism* if it is bijective and preserves the orthogonality and the order on $\mathbf{L}$, that is, for all $M, M_1, M_2 \in \mathbf{L}$,

$$\tau(M^{\perp}) = \tau(M)^{\perp},$$
$$M_1 \subset M_2 \iff \tau(M_1) \subset \tau(M_2).$$

The group $\mathrm{Aut}\,(\mathbf{L})$ of the $\mathbf{L}$-automorphisms is a topological space with respect to the initial topology defined by the functions

$$f^{\mathbf{L}}_{T,M} : \tau \mapsto \mathrm{tr}\big[T\tau(M)\big],\ T \in \mathbf{S},\ M \in \mathbf{L}.$$

We recall that $\mathrm{tr}\big[MT\big]$ can be interpreted as the probability that the proposition M is true in the state T.

We list some properties of the $\mathbf{L}$-automorphisms in the following lemma.

LEMMA 2.2. *Let $\tau \in Aut\,(\mathbf{L})$.*

1) Let $(M_i)_{i\in I}$ be any family in $\mathbf{L}$, then

$$\tau\left(\sup_{i\in I} M_i\right) = \sup_{i\in I} \tau(M_i)$$
$$\tau\left(\inf_{i\in I} M_i\right) = \inf_{i\in I} \tau(M_i).$$

2) If $P \in \mathbf{P}$, then $\tau(P) \in \mathbf{P}$.

3) If $\tau(P) = P$ for all $P \in \mathbf{P}$, then τ is the identity map.

PROOF: 1) It suffices to show only the first relation. Since τ preserves the order we have

$$\begin{aligned} M_k &\subset \sup_{i\in I} M_i \\ \tau(M_k) &\subset \tau\left(\sup_{i\in I} M_i\right) \\ \sup_{i\in I} \tau(M_i) &\subset \tau\left(\sup_{i\in I} M_i\right). \end{aligned}$$

Since τ^{-1} shares the properties of τ we conclude that $\tau\left(\sup_{i\in I} M_i\right) = \sup_{i\in I} \tau(M_i)$. 2) As τ preserves the order it maps one dimensional projections into one dimensional projections. 3) This statement follows from the first one by observing that any element M of $\mathbf{L}$ is the supremum of the one dimensional subspaces contained in it.

2.4. Let $\mathbf{E}$ be the set of operators E on $\mathcal{H}$ such that $O \le E \le I$. $\mathbf{E}$ is a *partial algebra* with respect to the sum. We define an $\mathbf{E}$-*automorphism* as a bijective map f from $\mathbf{E}$ onto $\mathbf{E}$ preserving the partially defined sum, that is, satisfying

$$E + F \le I \Longleftrightarrow f(E) + f(F) \le I$$

and, in this case,

$$f(E+F) = f(E) + f(F).$$

We denote by Aut$(\mathbf{E})$ the group of the $\mathbf{E}$-automorphisms and we endow it with the initial topology defined by the following functions

$$f^{\mathbf{E}}_{E,T} : f \mapsto \mathrm{tr}\big[Tf(E)\big], \quad E \in \mathbf{E},\ T \in \mathbf{S}.$$

LEMMA 2.3. *Let $f \in Aut(\mathbf{E})$. Then*

1) f is order preserving, that is,

$$E \le F \Longleftrightarrow f(E) \le f(F);$$

2) if $(E_i)_{i\in I}$ *is any family of elements of* $\mathbf{E}$ *such that*

$$\sup_{i\in I} E_i \in \mathbf{E} \quad \text{and} \quad \sup_{i\in I} f(E_i) \in \mathbf{E}$$

then

$$\sup_{i\in I} f(E_i) = f\left(\sup_{i\in I} E_i\right);$$

3) $f(O) = O$ *and* $f(I) = I$*;*

4) if $(E_i)_{i\in I}$ *is an increasing net of elements of* $\mathbf{E}$*, then*

$$\sup_{i\in I} E_i \in \mathbf{E} \quad \text{and} \quad \sup_{i\in I} f(E_i) \in \mathbf{E}$$

and

$$\sup_{i\in I} f(E_i) = f\left(\sup_{i\in I} E_i\right).$$

PROOF: 1) If $E \leq F$ then $F = (F - E) + E$, with $F - E \in \mathbf{E}$, hence $f(F) = f(F - E) + f(E)$, so that $f(E) \leq f(F)$. Since f^{-1} shares the properties of f, the converse is also true. 2) See the proof of point 1) of Lemma 2.2. 3) This follows from the bijectivity of f and from the fact that $O = \inf \mathbf{E}$ and $I = \sup \mathbf{E}$. 4) Since $(E_i)_{i\in I}$ is an increasing net and it is norm bounded, it is a standard result that $\sup_{i\in I} E_i$ exists in $\mathbf{E}$. Due to 1) the same holds for $(f(E_i))_{i\in I}$ so that by 2) the proof is complete.

2.5. Let $\mathbf{B}_r$ be the set of self-adjoint operators on $\mathcal{H}$. $\mathbf{B}_r$ is a commutative algebra with respect to the *Jordan product*

$$\mathbf{B}_r \times \mathbf{B}_r \ni (A, B) \mapsto \frac{AB + BA}{2} \in \mathbf{B}_r.$$

A function $S : \mathbf{B}_r \to \mathbf{B}_r$ is a $\mathbf{B}_r$*-automorphism* if it is a linear bijection and preserves the Jordan product, that is, for any $A, B \in \mathbf{B}_r$,

$$S(\tfrac{1}{2}(AB + BA)) = \tfrac{1}{2}(S(A)S(B) + S(B)S(A)).$$

We denote the group of $\mathbf{B}_r$-automorphisms by Aut $(\mathbf{B}_r)$. We put on Aut $(\mathbf{B}_r)$ the initial topology defined by

$$f^{\mathbf{B}_r}_{T,A} : S \mapsto \mathrm{tr}\big[TS(A)\big], \quad T \in \mathbf{S},\ A \in \mathbf{B}_r.$$

One readily observes that a linear bijection $S : \mathbf{B}_r \to \mathbf{B}_r$ is a $\mathbf{B}_r$-automorphism if and only if satisfies

$$S(A^2) = S(A)^2, \quad A \in \mathbf{B}_r.$$

In fact, if $S \in$ Aut $(\mathbf{B}_r)$, then $S(A^2) = S(A)^2$ for all $A \in \mathbf{B}_r$. Conversely,

$$\begin{aligned} S((A+B)^2) &= S(A^2) + S(AB) + S(BA) + S(B^2) \\ &= (S(A+B))^2 \\ &= S(A)^2 + S(A)S(B) + S(B)S(A) + S(B)^2 \end{aligned}$$

gives $S(A)S(B) + S(B)S(A) = S(AB + BA)$.

REMARK 2.1. *In his paper [6] Simon defines also a weak* $\mathbf{B}_r$*-automorphism as a linear bijection preserving the Jordan product for pairs of commuting (in the algebra* $\mathbf{B}$*) bounded self-adjoint operators. The previous observation shows immediately that weak* $\mathbf{B}_r$*-automorphisms are in fact* $\mathbf{B}_r$*-automorphisms.*

The following lemma collects the basic properties of the $\mathbf{B}_r$-automorphisms.

LEMMA 2.4. *Let* $S \in Aut\,(\mathbf{B}_r)$*. Then*

1) for any $A, B \in \mathbf{B}_r$, $A \leq B$ *if and only if* $S(A) \leq S(B)$*,*

2) for any $E \in \mathbf{E}$, $S(E) \in \mathbf{E}$.

PROOF: 1) Since S is a linear bijection it suffices to show that S preserves the positivity. But if $A \geq O$, then $A = (A^{1/2})^2$ and $S(A) = S(A^{1/2})^2 \geq O$. 2) Taking into account point 1) it is sufficient to prove that $S(I) = I$. Since $S(M^2) = S(M)^2$

for all $M \in \mathbf{L}$ and S is a bijective order preserving map, it sends the greatest projection to the greatest projection, that is, $S(I) = I$.

2.6. The set $\mathbf{B}$ of bounded operators is a unital C^*-algebra. A function $\Phi : \mathbf{B} \to \mathbf{B}$ is a $\mathbf{B}$-*automorphism* if it is a linear or antilinear bijection and satisfies for all $A, B \in \mathbf{B}$ the conditions

$$\Phi(A^*) = \Phi(A)^*$$
$$\Phi(AB) = \Phi(A)\Phi(B).$$

In the linear case the notion of a $\mathbf{B}$-automorphism is the usual notion of a C^*-isomorphism.

Let $\mathrm{Aut}\,(\mathbf{B})$ denote the group of $\mathbf{B}$-automorphism with the initial topology defined by the functions

$$f^{\mathbf{B}}_{T,A} : \Phi \mapsto \mathrm{tr}\big[T\Phi(A)\big], \quad T \in \mathbf{B}_1,\ A \in \mathbf{B}.$$

This topology is the natural one since $\mathbf{B}$ is the dual of $\mathbf{B}_1$.

3. The Wigner theorem. We go on to prove the Wigner theorem. We emphasise that the proof does not depend on the dimension of the Hilbert space.

THEOREM 3.1. *Let $\alpha \in Aut\,(\mathbf{P})$. There is a unitary or an antiunitary operator U on $\mathcal{H}$ such that for any $P \in \mathbf{P}$,*

$$\alpha(P) = UPU^*.$$

U is unique up to a phase factor.

PROOF: Fix $\alpha \in \mathrm{Aut}\,(\mathbf{P})$. Let $\omega \in \mathcal{H}, \omega \neq 0$, be a fixed vector and define

$$\mathcal{O}_\omega := \big\{\varphi \in \mathcal{H} \mid \langle\, \omega \,|\, \varphi \,\rangle > 0\big\}.$$

We observe that $\mathcal{O}_\omega$ is a cone, that is, $\mathcal{O}_\omega + \mathcal{O}_\omega \subset \mathcal{O}_\omega$ and $\lambda\mathcal{O}_\omega \subset \mathcal{O}_\omega, \lambda > 0$. Let ω' be a vector in the range of the projection $\alpha(P[\omega])$ such that $\|\omega'\| = \|\omega\|$ and define the cone $\mathcal{O}_{\omega'}$. The proof of the theorem will now be split in five parts.

Part 1. We show that there is a function

$$T_\omega : \mathcal{O}_\omega \to \mathcal{O}_{\omega'}$$

such that for all $\varphi, \varphi_1, \varphi_2 \in \mathcal{O}_\omega, \lambda > 0$,

(a) $$\|T_\omega \varphi\| = \|\varphi\|,$$

(b) $$T_\omega(\lambda\varphi) = \lambda T_\omega \varphi,$$

(c) $$T_\omega(\varphi_1 + \varphi_2) = T_\omega\varphi_1 + T_\omega\varphi_2,$$

(d) $$P[T_\omega\varphi] = \alpha(P[\varphi]).$$

To define T_ω we observe first that for any vector $\varphi \in \mathcal{O}_\omega$, there is a unique vector $\psi \in \mathcal{O}_{\omega'}$, $\|\psi\| = \|\varphi\|$, such that $\alpha(P[\varphi]) = P[\psi]$. We denote $\psi = T_\omega\varphi$. This defines a function $T_\omega : \mathcal{O}_\omega \to \mathcal{O}_{\omega'}$. Observe that $T_\omega\omega = \omega'$. By definition, T_ω is norm preserving, positively homogeneous, and $\alpha(P[\varphi]) = P[T_\omega\varphi]$. Also for any $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$,

(+) $$|\langle T_\omega\varphi_1 \,|\, T_\omega\varphi_2 \rangle| = |\langle \varphi_1 \,|\, \varphi_2 \rangle|.$$

We prove next the additivity of T_ω. Let $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$. By the definition of $\mathcal{O}_\omega$, φ_1 and φ_2 are linearly dependent (over $\mathbf{C}$) if and only if $\varphi_1 = \lambda\varphi_2$ for some $\lambda > 0$. If $\varphi_1 = \lambda\varphi_2$ then $T_\omega(\varphi_1 + \varphi_2) = T_\omega\big((\lambda+1)\varphi_2\big) = (\lambda+1)T_\omega\varphi_2 = \lambda T_\omega\varphi_2 + T\varphi_2 = T_\omega\varphi_1 + T_\omega\varphi_2$. Assume now that φ_1, φ_2 are linearly independent. We observe first that for any $\psi \in \mathcal{H}$, if $\langle T_\omega\varphi_i \,|\, \psi \rangle = 0, i = 1,2$, then $\langle \varphi_i \,|\, \gamma \rangle = 0, i = 1,2$, for any $\gamma \in \alpha^{-1}(P[\psi])$, and thus $\langle T_\omega(\varphi_1 + \varphi_2) \,|\, \psi \rangle = 0$. Hence

$$T_\omega(\varphi_1 + \varphi_2) = z_1 T_\omega\varphi_1 + z_2 T_\omega\varphi_2$$

for some $z_1, z_2 \in \mathbf{C}$. Since φ_1, φ_2 are linearly independent there are two uniquely defined vectors θ_1, θ_2 in $[\varphi_1, \varphi_2]$, the subspace generated by the vectors φ_1, φ_2, such that $\langle \theta_i \,|\, \varphi_j \rangle = \delta_{ij}, i, j = 1, 2$. In fact, they are

$$\theta_i = \big(\langle \varphi_j \,|\, \varphi_j \rangle \varphi_i - \langle \varphi_j \,|\, \varphi_i \rangle \varphi_j\big) / \big(\langle \varphi_j \,|\, \varphi_j \rangle \langle \varphi_i \,|\, \varphi_i \rangle - \langle \varphi_i \,|\, \varphi_j \rangle \langle \varphi_j \,|\, \varphi_i \rangle\big),$$

$i = 1, 2, i \neq j$. Let $\theta_i' \in \mathcal{H}$ such that $\|\theta_i'\| = \|\theta_i\|$ and $P[\theta_i'] = \alpha(P[\theta_i])$. Writing $\varphi = \varphi_1 + \varphi_2$,

$$1 = \langle \varphi \,|\, \theta_i \rangle = |\langle \varphi \,|\, \theta_i \rangle|^2 = |\langle T_\omega \varphi \,|\, \theta_i' \rangle|^2 = |z_i|^2$$

so that that $|z_i| = 1$. Since $\varphi_1, \varphi_2, \varphi \in \mathcal{O}_\omega$ and $T_\omega\varphi_1, T_\omega\varphi_2, T_\omega\varphi \in \mathcal{O}_{\omega'}$ one has $\langle \omega \,|\, \varphi \rangle = |\langle \omega \,|\, \varphi \rangle| = |\langle \omega' \,|\, T_\omega \varphi \rangle| = \langle \omega' \,|\, T_\omega \varphi \rangle$, which gives

$$(*) \qquad \langle \omega \,|\, \varphi_1 \rangle + \langle \omega \,|\, \varphi_2 \rangle = z_1 \langle \omega \,|\, \varphi_1 \rangle + z_2 \langle \omega \,|\, \varphi_2 \rangle .$$

But then

$$\begin{aligned} \langle \omega \,|\, \varphi_1 \rangle + \langle \omega \,|\, \varphi_2 \rangle &= \big|\langle \omega \,|\, \varphi_1 \rangle + \langle \omega \,|\, \varphi_2 \rangle\big| \\ &= \big|z_1 \langle \omega \,|\, \varphi_1 \rangle + z_2 \langle \omega \,|\, \varphi_2 \rangle\big| \\ &\leq \big|z_1 \langle \omega \,|\, \varphi_1 \rangle\big| + \big|z_2 \langle \omega \,|\, \varphi_2 \rangle\big| \\ &= \langle \omega \,|\, \varphi_1 \rangle + \langle \omega \,|\, \varphi_2 \rangle , \end{aligned}$$

which shows that $z_1 \langle \omega \,|\, \varphi_1 \rangle = \lambda z_2 \langle \omega \,|\, \varphi_2 \rangle$ for some $\lambda \in \mathbf{R}$. Therefore, $0 < z_1 \langle \omega \,|\, \varphi_1 \rangle + z_2 \langle \omega \,|\, \varphi_2 \rangle = (1+\lambda) z_2 \langle \omega \,|\, \varphi_2 \rangle$, which shows that the imaginary part of z_2 equals 0 and one thus has $z_2 = \pm 1$. Similarly, one gets $z_1 = \pm 1$. From Eq. (*), where $\langle \omega \,|\, \varphi_1 \rangle, \langle \omega \,|\, \varphi_2 \rangle > 0$, one finally gets $z_1 = z_2 = 1$. This completes the proof of the additivity of T_ω.

Part 2. Let $\psi \in \mathcal{H}, \psi \neq 0$, and assume that T is any function $\mathcal{O}_\psi \to \mathcal{H}$, having the properties (a)-(d). Then for any $\varphi \in \mathcal{O}_\omega \cap \mathcal{O}_\psi$,

$$(\mathrm{e}) \qquad T(\varphi) = z T_\omega(\varphi),$$

for some $z \in \mathbf{T}$. Indeed, by the property (d), it holds that for any $\varphi \in \mathcal{O}_\omega \cap \mathcal{O}_\psi$, $T\varphi = f(\varphi) T_\omega \varphi$, with $f(\varphi) \in \mathbf{T}$, and it remains to be shown that $f(\varphi)$ is constant on $\mathcal{O}_\omega \cap \mathcal{O}_\psi$. For any $\lambda > 0$ and $\varphi \in \mathcal{O}_\omega \cap \mathcal{O}_\psi$, $T(\lambda\varphi) = f(\lambda\varphi) T_\omega(\lambda\varphi) = \lambda f(\lambda\varphi) T_\omega \varphi$ and $T(\lambda\varphi) = \lambda T\varphi = \lambda f(\varphi) T_\omega \varphi$. Hence $\lambda f(\lambda\varphi) T_\omega \varphi = \lambda f(\varphi) T_\omega \varphi$. Since $T_\omega \varphi \neq 0$ for $\varphi \neq 0$, this gives $f(\varphi) = f(\lambda\varphi)$. Consider next vectors $\varphi_1, \varphi_2 \in \mathcal{O}_\omega \cap \mathcal{O}_\psi$ such that $\varphi_1 \neq \lambda \varphi_2$ for any $\lambda > 0$ (so that φ_1, φ_2 are linearly independent over $\mathbf{C}$). Then

$T(\varphi_1+\varphi_2) = f(\varphi_1+\varphi_2)T_\omega(\varphi_1+\varphi_2) = f(\varphi_1)T_\omega\varphi_1 + f(\varphi_2)T_\omega\varphi_2$. Using again the above vectors θ_1, θ_2, associated with φ_1, φ_2 one easily gets, e.g., $f(\varphi_1+\varphi_2) = f(\varphi_1)$ for any $\varphi_2 \in \mathcal{O}_\omega \cap \mathcal{O}_\psi$. Hence $f(\varphi)$ is constant on $\mathcal{O}_\omega \cap \mathcal{O}_\psi$ and thus T_ω is unique modulo a phase on the cone $\mathcal{O}_\omega$.

Part 3. Let $\omega \in \mathcal{H}, \omega \neq 0$, and let $T_\omega : \mathcal{O}_\omega \to \mathcal{O}_{\omega'}$ be defined as in part 1. We show next that T_ω has one of the following two properties, either

$$\text{(f)} \qquad \langle\, T_\omega\varphi_1 \,|\, T_\omega\varphi_2 \,\rangle = \langle\, \varphi_1 \,|\, \varphi_2 \,\rangle$$

for all $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$, or

$$\text{(g)} \qquad \langle\, T_\omega\varphi_1 \,|\, T_\omega\varphi_2 \,\rangle = \langle\, \varphi_2 \,|\, \varphi_1 \,\rangle$$

for all $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$. First of all, let $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$. Then $\langle\, T_\omega(\varphi_1+\varphi_2) \,|\, T_\omega(\varphi_1+\varphi_2) \,\rangle = \langle\, \varphi_1+\varphi_2 \,|\, \varphi_1+\varphi_2 \,\rangle$. Using the additivity of T_ω and the inner product this shows, in view of (+), that either $\langle\, T_\omega\varphi_1 \,|\, T_\omega\varphi_2 \,\rangle = \langle\, \varphi_1 \,|\, \varphi_2 \,\rangle$ or $\langle\, T_\omega\varphi_1 \,|\, T_\omega\varphi_2 \,\rangle = \langle\, \varphi_2 \,|\, \varphi_1 \,\rangle$. We show next that for a fixed $\varphi \in \mathcal{O}_\omega$, either $\langle\, T_\omega\varphi \,|\, T_\omega\psi \,\rangle = \langle\, \varphi \,|\, \psi \,\rangle$ or $\langle\, T_\omega\varphi \,|\, T_\omega\psi \,\rangle = \langle\, \psi \,|\, \varphi \,\rangle$ for all $\psi \in \mathcal{O}_\omega$. To prove this assume on the contrary that there are vectors $\varphi_1, \varphi_2 \in \mathcal{O}_\omega$ such that $\langle\, T_\omega\varphi \,|\, T_\omega\varphi_1 \,\rangle = \langle\, \varphi \,|\, \varphi_1 \,\rangle (\neq \langle\, \varphi_1 \,|\, \varphi \,\rangle)$ and $\langle\, T_\omega\varphi \,|\, T_\omega\varphi_2 \,\rangle = \langle\, \varphi_2 \,|\, \varphi \,\rangle (\neq \langle\, \varphi \,|\, \varphi_2 \,\rangle)$. By a direct computation of $\langle\, T_\omega\varphi \,|\, T_\omega(\varphi_1+\varphi_2) \,\rangle$ one observes that this leads to a contradiction. By a similar counter argument one shows finally that either $\langle\, T_\omega\varphi \,|\, T_\omega\psi \,\rangle = \langle\, \varphi \,|\, \psi \,\rangle$ for all $\varphi, \psi \in \mathcal{O}_\omega$ or $\langle\, T_\omega\varphi \,|\, T_\omega\psi \,\rangle = \langle\, \psi \,|\, \varphi \,\rangle$ for all $\psi \in \mathcal{O}_\omega$.

Part 4. We construct next a unitary or antiunitary operator U of $\mathcal{H}$ for which $\alpha(P) = UPU^*$ for all $P \in \mathbf{P}$.

Let $\omega \in \mathcal{H}$ and $T_\omega : \mathcal{O}_\omega \to \mathcal{O}_{\omega'}$ be given as in part one. Let $M = [\omega]^\perp$ and $M' = [\omega']^\perp$ and define a function $S : M \to M'$ by

$$S\varphi := T_{\omega+\varphi}\varphi, \quad \varphi \neq 0$$
$$S\varphi := 0, \qquad \varphi = 0$$

where $T_{\omega+\varphi}$ is the operator on the cone $\mathcal{O}_{\omega+\varphi}$ with the choice of the phase given by $T_{\omega+\varphi}\omega = \omega'$. S is well-defined since for any $\varphi \in M$, $\varphi \neq 0$, we have $\varphi \in \mathcal{O}_{\omega+\varphi}$.

Moreover, for any two $\varphi, \psi \in M$, $T_{\omega+\varphi} = T_{\omega+\psi}$ on the cone $\mathcal{O}_{\omega+\varphi} \cap \mathcal{O}_{\omega+\psi}$, which contains at least the vector ω for which $T_{\omega+\varphi}\omega = T_{\omega+\psi}\omega$. According to part 3 any $T_{\omega+\varphi}$, $\varphi \in M$, has either the property (f) or the property (g). Due to the fact that for all $\varphi, \psi \in M$, $T_{\omega+\varphi} = T_{\omega+\psi}$ on the intersection of their defining cones, all the operators $T_{\omega+\varphi}, \varphi \in M$, are of the type (f) or they all are of the type (g). We proceed to show that S is in the first case a unitary operator and in the second case an antiunitary operator. In fact the proofs of the two different cases are similar and we treat only the case that all $T_{\omega+\varphi}, \varphi \in M$, are of the type (f).

We show first that for any $\varphi \in M, \lambda \in \mathbf{C}$, $S(\lambda\varphi) = \lambda S\varphi$. In fact, if $\lambda\varphi = 0$, the result is obvious, otherwise we have

$$\begin{aligned}
\langle\, T_\omega(\omega + \lambda\varphi) \,|\, T_\omega(\omega+\varphi)\,\rangle &= \langle\, \omega + \lambda\varphi \,|\, \omega + \varphi\,\rangle \\
&= \|\omega\|^2 + \bar{\lambda}\,\langle\, \varphi \,|\, \varphi\,\rangle \\
\langle\, T_\omega(\omega + \lambda\varphi) \,|\, T_\omega(\omega+\varphi)\,\rangle &= \langle\, T_{\omega+\lambda\varphi}(\omega + \lambda\varphi) \,|\, T_{\omega+\varphi}(\omega+\varphi)\,\rangle \\
&= \langle\, T_{\omega+\lambda\varphi}\omega + T_{\omega+\lambda\varphi}(\lambda\varphi) \,|\, T_{\omega+\varphi}\omega + T_{\omega+\varphi}\varphi\,\rangle \\
&= \langle\, \omega' + S(\lambda\varphi) \,|\, \omega' + S\varphi\,\rangle \\
&= \|\omega'\|^2 + \langle\, S(\lambda\varphi) \,|\, S\varphi\,\rangle .
\end{aligned}$$

Since $\|\omega\| = \|\omega'\|$ this gives $\langle\, S(\lambda\varphi) \,|\, S\varphi\,\rangle = \bar{\lambda}\,\langle\, \varphi \,|\, \varphi\,\rangle$. But $S(\lambda\varphi) = T_{\omega+\lambda\varphi}(\lambda\varphi) \in \alpha(P[\lambda\varphi])$ and $S\varphi \in \alpha(P[\varphi])$, which shows that $S(\lambda\varphi) = zS\varphi$ for some $z \in \mathbf{C}$. Therefore, $\bar{\lambda}\,\langle\, \varphi \,|\, \varphi\,\rangle = \langle\, S(\lambda\varphi) \,|\, S\varphi\,\rangle = \bar{z}\,\langle\, S\varphi \,|\, S\varphi\,\rangle = \bar{z}\,\langle\, \varphi \,|\, \varphi\,\rangle$, which gives $\bar{z} = \bar{\lambda}$, and thus $S(\lambda\varphi) = \lambda S\varphi$.

To show the additivity of S on M, let $\varphi_1, \varphi_2 \in M$. If $\varphi_1 = \lambda\varphi_2, \lambda \in \mathbf{C}$, then the homogeneity of S gives the additivity. Therefore, assume that φ_1, φ_2 are linearly independent. Let θ_1, θ_2 be the unique vectors in $[\varphi_1, \varphi_2]$ such that $\langle\, \theta_i \,|\, \varphi_j\,\rangle = \delta_{ij}$. Then

$$\begin{aligned}
S(\varphi_1 + \varphi_2) &= T_{\omega+\varphi_1+\varphi_2}(\varphi_1 + \varphi_2) \\
&= T_{\omega+\theta_1+\theta_2}(\varphi_1 + \varphi_2) \\
&= T_{\omega+\theta_1+\theta_2}\varphi_1 + T_{\omega+\theta_1+\theta_2}\varphi_2 \\
&= T_{\omega+\varphi_1}\varphi_1 + T_{\omega+\varphi_2}\varphi_2 = S\varphi_1 + S\varphi_2.
\end{aligned}$$

Hence $S : M \to M'$ is a linear map. It is also isometric since for any $\varphi \in M$, $\varphi \neq 0$, $\langle S\varphi \,|\, S\varphi \rangle = \langle T_{\omega+\varphi}\varphi \,|\, T_{\omega+\varphi}\varphi \rangle = \langle T_\varphi \varphi \,|\, T_\varphi \varphi \rangle = \langle \varphi \,|\, \varphi \rangle$. Moreover, for any unit vector $\varphi \in M$ one has $P[S\varphi] = \alpha(P[\varphi])$. To show the surjectivity of S, let $\psi \in M', \psi \neq 0$. Since α is surjective there is a unit vector $\varphi \in M$ such that $\alpha(P[\varphi]) = P[\psi]$. Hence $S\varphi = \lambda\psi$ for some $\lambda \in \mathbf{C}$. Since $\|\varphi\| = 1$, also $\|S\varphi\| = 1$ so that $\lambda \neq 0$ and thus $S(\frac{\varphi}{\lambda}) = \psi$. This concludes the proof of the unitarity of S.

We now have $\mathcal{H} = [\omega] \oplus M = [\omega'] \oplus M'$ and we define $U : \mathcal{H} \to \mathcal{H}$ such that $U(\lambda\omega + \varphi) = \lambda\omega' + S\varphi$ for all $\lambda \in \mathbf{C}, \varphi \in M$. If S is antiunitary we define U instead by $U(\lambda\omega + \varphi) = \bar{\lambda}\omega' + S\varphi$. Clearly, the operator U is unitary (antiunitary) and it is related to the function α according to $\alpha(P) = UPU^*$ for any $P \in \mathbf{P}$.

Part 5. Let $V : \mathcal{H} \to \mathcal{H}$ be related to α according to $\alpha(P) = VPV^*$, $P \in \mathbf{P}$. By change of phase we may assume that $V\omega = \omega'$. Let $\varphi \in M$. The operator V has, in particular, the properties (a)-(d) on $\mathcal{O}_{\omega+\varphi}$ so that V, when restricted on $\mathcal{O}_{\omega+\varphi}$, equals with $zT_{\omega+\varphi}$ for some $z \in \mathbf{T}$. But since $V\omega = \omega' = zT_{\omega+\varphi}\omega = z\omega'$, one has that for any $\varphi \in M$, $V|_{\mathcal{O}_{\omega+\varphi}} = T_{\omega+\varphi}$, that is, $V\varphi = S\varphi$ on M. Therefore, V equals with U on M, showing that $V = U$ whenever $M \neq \{0\}$. In other words, U is unique modulo a phase factor and the unitary or the antiunitary nature of U is completely determined by $\alpha \in \mathrm{Aut}\,(\mathbf{P})$ (apart from the trivial case of $\mathcal{H}$ being one-dimensional). Moreover, the operator U does not depend on the choice of the vector ω. This ends the proof of the theorem.

The content of the Wigner theorem suggests to introduce another group of quantum symmetries.

Let $\mathbf{U} \cup \bar{\mathbf{U}}$ denote the group of unitary and antiunitary operators on $\mathcal{H}$. It is a metrisable second countable topological group with respect to the induced strong operator topology. Let $\mathbf{T} = \{zI \,|\, z \in \mathbf{T}\}$ be the phase group which is the closed centre of $\mathbf{U} \cup \bar{\mathbf{U}}$.

Let $\Sigma(\mathcal{H})$ denote the quotient group $\mathbf{U} \cup \bar{\mathbf{U}}/\mathbf{T}$, endowed with the quotient topology. We call it the *symmetry group* on $\mathcal{H}$ and denote its elements by $[U]$, with

$U \in \mathbf{U} \cup \bar{\mathbf{U}}$. $\Sigma(\mathcal{H})$ is a metrisable topological group satisfying the second axiom of countability.

4. The group isomorphisms. We now proceed to show that the groups of automorphisms introduced in section 2 are isomorphic. To work out this plan, we are going to prove that the arrows in the following diagram

$$
\begin{array}{ccccccc}
\mathrm{Aut}\,(\mathbf{L}) & \xrightarrow{9} & \mathrm{Aut}\,(\mathbf{S}) & \xleftarrow{5} & \mathrm{Aut}\,(\mathbf{E}) & \xleftarrow{4} & \mathrm{Aut}\,(\mathbf{B}_r) \\
\uparrow{\scriptstyle 8} & & \downarrow{\scriptstyle 6} & & & & \uparrow{\scriptstyle 3} \\
\mathrm{Aut}_w\,(\mathbf{P}) & \xleftarrow{7} & \mathrm{Aut}\,(\mathbf{P}) & \xrightarrow{1} & \Sigma(\mathcal{H}) & \xrightarrow{2} & \mathrm{Aut}\,(\mathbf{B})
\end{array}
$$

are injective group homomorphisms. The diagram contains two loops, one on the right hand side and one on the left hand side. We prove that the maps obtained by composing the arrows along both loops are the identity. From this it follows that all the maps are isomorphisms.

The arrows between the automorphism groups are *natural* in the sense that they are defined in terms of some *natural* relations between the sets which the automorphisms act on. In particular, the arrows 3, 4, 6 and 8 are induced by the inclusions

$$\mathbf{E} \subset \mathbf{B}_r \subset \mathbf{B} \qquad \mathbf{P} \subset \mathbf{S} \qquad \mathbf{P} \subset \mathbf{L}.$$

Since a $\mathbf{P}$-automorphism is a weak $\mathbf{P}$-automorphism the arrow 7 is the natural inclusion. The arrows 5 and 9 are based on the duality between $\mathbf{B}$ and $\mathbf{B}_1$. The arrows 1 and 2 reflect the natural action of the unitary (or antiunitary) group on $\mathbf{P}$ and $\mathbf{B}$, respectively.

We notice that there are other natural relations giving rise to (a priori) different homomorphisms, for example the inclusion of $\mathbf{L}$ in $\mathbf{E}$; however, our choice is motivated by the aim of presenting as simple a proof as possible for the various isomorphisms. We come back to this issue at the end of the section.

A particular care is needed to define the arrow 9 since we have to assume that the dimension of $\mathcal{H}$ is at least three. This will be clarified by the discussion after Corollary 4.2.

We consider first the right hand side of the diagram. The proofs that the arrows 1, 2 and 3 are injective homomorphisms are immediate. We summarise these results in the following propositions.

PROPOSITION 4.1. *Any $\alpha \in Aut\,(\mathbf{P})$ defines (via the Wigner theorem) an equivalence class of unitary or antiunitary operators $[U_\alpha]$ such that*

$$\alpha(P) = U_\alpha P U_\alpha^*, \qquad P \in \mathbf{P}.$$

The map $Aut\,(\mathbf{P}) \ni \alpha \mapsto [U_\alpha] \in \Sigma(\mathcal{H})$ is an injective group homomorphism.

PROPOSITION 4.2. *Any $[U] \in \Sigma(\mathcal{H})$ defines a $\Phi_U \in Aut\,(\mathbf{B})$ by $\Phi_U(A) = UAU^*$, $A \in \mathbf{B}$. The map $\Sigma(\mathcal{H}) \ni [U] \mapsto \Phi_U \in Aut\,(\mathbf{B})$ is an injective group homomorphism.*

PROPOSITION 4.3. *Any $\Phi \in Aut\,(\mathbf{B})$, when restricted on $\mathbf{B}_r$, is a $\mathbf{B}_r$-automorphism S_Φ. The map $Aut\,(\mathbf{B}) \ni \Phi \mapsto S_\Phi \in Aut\,(\mathbf{B}_r)$ is an injective group homomorphism.*

Now we turn to the fourth arrow.

PROPOSITION 4.4. *Any $S \in Aut\,(\mathbf{B}_r)$, when restricted on $\mathbf{E}$, is an $\mathbf{E}$-automorphism f_S. The map $Aut\,(\mathbf{B}_r) \ni S \mapsto f_S \in Aut\,(\mathbf{E})$ is an injective group homomorphism.*

PROOF: By Lemma 2.4 f_S is a well defined bijective map from $\mathbf{E}$ onto $\mathbf{E}$. Since S is linear it follows that f_S is an $\mathbf{E}$-automorphism. The map $S \mapsto f_S$ is obviously a group homomorphism. To show that it is injective suppose that $f_S(E) = E$ for all

$E \in \mathbf{E}$. Let $A \in \mathbf{B}_r$, then

$$\begin{aligned} S(A) &= S(A_+ - A_-) \\ &= S(A_+) - S(A_-) \\ &= \|A_+\| f_S\left(\frac{A_+}{\|A_+\|}\right) - \|A_-\| f_S\left(\frac{A_-}{\|A_-\|}\right) \\ &= A_+ - A_-, \end{aligned}$$

so that S is the identity.

To define mapping 5 we need a lemma.

LEMMA 4.1. *Let $p : \mathbf{E} \to [0,1]$ be a function with the following properties:*

1) if $E + F \leq I$, then $p(E+F) = p(E) + p(F)$,

2) if $(E_i)_{i\in I}$ is an increasing net in $\mathbf{E}$, then

$$p\left(\sup_{i\in I} E_i\right) = \sup_{i\in I} p(E_i).$$

Then there is a unique positive trace class operator T such that for all $E \in \mathbf{E}$

$$p(E) = tr\big[TE\big].$$

PROOF: We notice first that $p(E) = p(E+O) = p(E) + p(O)$, so that $p(O) = 0$. We prove next that for all $E \in \mathbf{E}$ and $0 < \lambda < 1$,

$$(**) \qquad p(\lambda E) = \lambda p(E).$$

If λ is rational this follows from the additivity of p. Let $0 < \lambda < 1$ and let (r_n) be an increasing sequence of positive rationals converging to λ. Then

$$\sup_n (r_n E) = \lambda E$$

and this implies that

$$\begin{aligned} p(\lambda E) &= p\left(\sup_n \{r_n E\}\right) \\ &= \sup_n p(r_n E) \\ &= \lambda p(E). \end{aligned}$$

We now extend p first to the set of positive operators $\mathbf{B}_+$, defining

$$p^+(A) = \|A\|\, p\left(\frac{A}{\|A\|}\right), \qquad A \in \mathbf{B}_+,$$

and then to the set of self-adjoint operators $\mathbf{B}_r$, letting

$$\hat{p}(A) = \frac{1}{2}\left(p^+\left(A + |A|\right) - p^+\left(|A| - A\right)\right), \qquad A \in \mathbf{B}_r.$$

From the additivity of p and from property (**) it follows that $\hat{p}$ is linear. Moreover, by construction $\hat{p}$ is positive and it is the unique linear extension of p to $\mathbf{B}_r$.

The linear map $\hat{p}$ is, in fact, normal. If $(A_i)_{i\in I}$ is an increasing norm bounded positive net in $\mathbf{B}_r$, then, letting $c = \sup_i \|A_i\|$, $(\frac{A_i}{c})_{i\in I}$ is an increasing net in $\mathbf{E}$ and we have

$$\begin{aligned} \hat{p}\left(\sup_i A_i\right) &= cp\left(\sup_i \frac{A_i}{c}\right) \\ &= c \sup_i p\left(\frac{A_i}{c}\right) \\ &= \sup_i \hat{p}(A_i). \end{aligned}$$

Hence $\hat{p}$ is a linear positive normal function on $\mathbf{B}_r$. It is well known that $\hat{p}$ defines a unique positive trace class operator T such that

$$\hat{p}(A) = \mathrm{tr}\big[TA\big], \qquad A \in \mathbf{B}_r.$$

Since $\hat{p}$ is uniquely defined by its restriction p on $\mathbf{E}$ the proof is complete.

PROPOSITION 4.5. *Let $f \in Aut\,(\mathbf{E})$. There is a unique $V_f \in Aut\,(\mathbf{S})$ such that $f(P) = V_f(P)$ for all $P \in \mathbf{P}$. Moreover, the correspondence $Aut\,(\mathbf{E}) \ni f \mapsto V_f \in Aut\,(\mathbf{S})$ is an injective group homomorphism.*

PROOF: Let $f \in \mathrm{Aut}\,(\mathbf{E})$. For all $T \in \mathbf{S}$ define the map from $\mathbf{E}$ to $[0,1]$ by

$$E \mapsto \mathrm{tr}\big[Tf^{-1}(E)\big].$$

Using now the statement 4) of Lemma 2.3 and Lemma 4.1 there is a positive trace class operator T' such that

$$\mathrm{tr}\big[Tf^{-1}(E)\big] = \mathrm{tr}\big[T'E\big], \qquad E \in \mathbf{E}.$$

Taking $E = I$ we have $\mathrm{tr}\big[T'\big] = 1$, hence $T' \in \mathbf{S}$. We define V_f from $\mathbf{S}$ to $\mathbf{S}$ as $V_f(T) = T'$ so that

$$\mathrm{tr}\big[V_f(T)E\big] = \mathrm{tr}\big[Tf^{-1}(E)\big], \qquad E \in \mathbf{E}.$$

Using this formula it is straightforward to prove that $V_f \in \mathrm{Aut}\,(\mathbf{S})$ and that $f \mapsto V_f$ is a group homomorphism. Moreover, suppose that $V_f(T) = T$ for all $T \in \mathbf{S}$, then

$$\mathrm{tr}\big[T(E - f^{-1}(E))\big] = 0, \qquad E \in \mathbf{E},\ T \in \mathbf{S}.$$

Hence $E = f^{-1}(E)$ for all $E \in \mathbf{E}$, that is, f is the identity. This shows the injectivity of the map $f \mapsto V_f$ and ends the proof.

Finally we have

PROPOSITION 4.6. *Any $V \in Aut\,(\mathbf{S})$ restricted to $\mathbf{P}$ defines an element $\alpha_V \in Aut\,(\mathbf{P})$. The function $Aut\,(\mathbf{S}) \ni V \mapsto \alpha_V \in Aut\,(\mathbf{P})$ is an injective group homomorphism.*

PROOF: Let $V \in \mathrm{Aut}\,(\mathbf{S})$. By Lemma 2.1 its restriction α_V on $\mathbf{P}$ is well-defined and bijective. Let $\hat{V}$ be the trace-norm preserving linear extension of V given by Lemma 2.1. Let $P_1, P_2 \in \mathbf{P}$. A simple calculation shows that

$$\begin{aligned} 2\sqrt{1 - \mathrm{tr}\big[P_1P_2\big]} &= \|P_1 - P_2\|_1 \\ &= \left\|\hat{V}\,(P_1 - P_2)\right\|_1 \\ &= \|V(P_1) - V(P_2)\|_1 \\ &= 2\sqrt{1 - \mathrm{tr}\big[V(P_1)V(P_2)\big]}, \end{aligned}$$

so that α_V preserves the transition probabilities. The map $V \mapsto \alpha_V$ is clearly a group homomorphism and its injectivity follows from Lemma 2.1.

Let G denote any of the six groups on the right hand side of the diagram. Starting from G and composing the injective group homomorphisms one obtains an injective group homomorphism ϕ_G of G into G.

COROLLARY 4.1. *The map ϕ_G is the identity on G.*

PROOF: It is sufficient to prove the statement for a particular choice of G. Choosing $G = \Sigma(\mathcal{H})$ the proof is immediate. In fact, let $[U] \in \Sigma(\mathcal{H})$; a simple computation shows that the image of $[U]$ with respect to the composition of the first three homomorphisms is the $\mathbf{E}$-automorphism

$$\mathbf{E} \ni E \mapsto UEU^* \in \mathbf{E}.$$

Using the properties of the trace this is mapped to the $\mathbf{S}$-automorphism

$$\mathbf{S} \ni T \mapsto UTU^* \in \mathbf{S}$$

and then to the $\mathbf{P}$-automorphism

$$\mathbf{P} \ni P \mapsto UPU^* \in \mathbf{P}.$$

The statement follows now from the Wigner theorem.

The previous proposition implies that the six injections on the right hand side of the diagram are all isomorphisms. We stress that this holds without any assumption on the dimension of the Hilbert space.

Now we consider the left hand side. The homomorphism 6 is defined in Proposition 4.6 while the homomorphism 7 is trivial. In fact we have the following statement.

PROPOSITION 4.7. *The natural immersion $Aut(\mathbf{P}) \hookrightarrow Aut_w(\mathbf{P})$ is an injective group homomorphism.*

The following proposition describes the homomorphism 8.

PROPOSITION 4.8. *Let $\alpha \in Aut_w(\mathbf{P})$. There is a unique $\tau_\alpha \in Aut(\mathbf{L})$ such that $\tau_\alpha(P) = \alpha(P)$ for all $P \in \mathbf{P}$. Moreover, the map $Aut_w(\mathbf{P}) \ni \alpha \mapsto \tau_\alpha \in Aut(\mathbf{L})$ is an injective group homomorphism.*

PROOF: Let $\alpha \in \mathrm{Aut}_w(\mathbf{P})$. For all $M \subset \mathcal{H}$, $M \neq \{0\}$, let

$$\tau_\alpha(M) = \{\psi \in \alpha([\phi]) \ : \phi \in M, \ \phi \neq 0\},$$

and put $\tau_\alpha(\{0\}) = \{0\}$. We observe that

$$\begin{aligned}\tau_{\alpha^{-1}}(\tau_\alpha(M)) &= \{\Phi \in \alpha^{-1}([\psi]) \ : \psi \in \tau_\alpha([\phi]), \ \phi \in M, \phi \neq 0\} \\ &= \{\Phi \in \alpha^{-1}(\alpha[\phi]) \ : \phi \in M, \ \phi \neq 0\} \\ &= \mathbf{C}M.\end{aligned}$$

In the same way we have that $\tau_\alpha(\tau_{\alpha^{-1}}(M)) = \mathbf{C}M$.

Let now $M \in \mathbf{L}$. We then have $\tau_\alpha(M^\perp) = \tau_\alpha(M)^\perp$. In fact, if $\phi \in M$ and $\psi \in M^\perp$ are nonzero vectors, then $\alpha(P[\phi]) \perp \alpha([\psi])$. Hence

$$\begin{gathered}\tau_\alpha(M) \perp \tau_\alpha(M^\perp) \\ \tau_\alpha(M^\perp) \subset \tau_\alpha(M)^\perp\end{gathered}$$

and, since $M = \tau_{\alpha^{-1}}(\tau_\alpha(M))$, one concludes that $\tau_\alpha(M^\perp) = \tau_\alpha(M)^\perp$. Moreover, since M is a closed subspace,

$$\begin{aligned}\tau_\alpha(M) &= \tau_\alpha((M^\perp)^\perp) \\ &= (\tau_\alpha(M^\perp))^\perp,\end{aligned}$$

proving that $\tau_\alpha(M)$ is a closed subspace.

We denote by τ_α the map from $\mathbf{L}$ to $\mathbf{L}$ sending M to $\tau_\alpha(M)$. Obviously τ_α is bijective and preserves the order and the orthogonality, that is, $\tau_\alpha \in \mathrm{Aut}(\mathbf{L})$. Finally, by construction, $\tau_\alpha(P) = \alpha(P)$ for all $P \in \mathbf{P}$. A standard calculation shows that the map $\alpha \mapsto \tau_\alpha$ is a group homomorphism. The statement 3 of Lemma 2.2 shows that it is also injective. This concludes the proof.

We end with the following proposition where the assumption on the dimension of the Hilbert space is essential.

Proposition 4.9. *Let* $\dim(\mathcal{H}) \geq 3$. *Given* $\tau \in Aut(\mathbf{L})$ *there is a unique* $V_\tau \in Aut(\mathbf{S})$ *such that*

$$V_\tau(P) = \tau(P)$$

for all $P \in \mathbf{P}$. *Moreover, the map* $Aut(\mathbf{L}) \ni \tau \mapsto V_\tau \in Aut(\mathbf{S})$ *is an injective group homomorphism.*

Proof: Let $\tau \in \mathrm{Aut}(\mathbf{L})$. Since τ is a lattice orthoisomorphism on $\mathbf{L}$ the mapping

$$\mathbf{L} \ni M \mapsto \mathrm{tr}\big[T\tau^{-1}(M)\big] \in [0,1]$$

is a generalised probability measure on $\mathbf{L}$ for all $T \in \mathbf{S}$. According to a theorem of Gleason [8] (which holds if the dimension of $\mathcal{H}$ is greater than 2) there is a unique $T' \in \mathbf{S}$ such that $\mathrm{tr}\big[T'M\big] = \mathrm{tr}\big[T\tau^{-1}(M)\big]$ for all $M \in \mathbf{L}$. The induced function $T \mapsto T' =: V_\tau(T)$ is one-to-one onto and it preserves the convex structure of $\mathbf{S}$, that is, $V_\tau \in \mathrm{Aut}(\mathbf{S})$. Clearly the map $\tau \mapsto V_\tau$ is a group homomorphism.

We show now that $V_\tau(P) = \tau(P)$ for all $P \in \mathbf{P}$. It is sufficient to prove that

$$\mathrm{tr}\big[V_\tau(P_1)P_2\big] = \mathrm{tr}\big[\tau(P_1)P_2\big], \qquad P_1, P_2 \in \mathbf{P}.$$

Since V_τ, restricted to $\mathbf{P}$, is a $\mathbf{P}$-automorphism we have

$$\begin{aligned}
\mathrm{tr}\big[V_\tau(P_1)P_2\big] &= \mathrm{tr}\big[P_1 V_\tau^{-1}(P_2)\big] \\
&= \mathrm{tr}\big[P_1 V_{\tau^{-1}}(P_2)\big] \\
&= \mathrm{tr}\big[\tau(P_1)P_2\big].
\end{aligned}$$

Suppose now that $V_\tau(T) = T$ for all $T \in \mathbf{S}$. Then, in particular, $\tau(P) = P$ for all $P \in \mathbf{P}$ so that by Lemma 2.2 τ is the identity. This shows the injectivity of the map $\tau \mapsto V_\tau$.

Similarly to Corollary 4.1 we have the following statement.

Corollary 4.2. *Let* $\dim \mathcal{H} \geq 3$. *The composition map of the arrows 6 to 9 is the identity on each group of automorphisms.*

PROOF: We compose the maps starting from $\mathrm{Aut}\,(\mathbf{S})$. Let $V \in \mathrm{Aut}\,(\mathbf{S})$. Its restriction α_V to $\mathbf{P}$ is a (weak) $\mathbf{P}$-automorphism. Hence, by Proposition 4.8, α_V defines an $\mathbf{L}$-automorphism τ_{α_V} such that $\tau_{\alpha_V}(P) = V(P)$ for all $P \in \mathbf{P}$. Hence the corresponding $\mathbf{S}$-automorphism given by Proposition 4.9 is V on $\mathbf{P}$.

From Corollaries 4.1 and 4.2 we conclude that if the dimension of the Hilbert space is greater that two all the injections of the diagram are isomorphisms and all the groups are isomorphic.

On the other hand, if the dimension of $\mathcal{H}$ is 2, the groups on the right hand side of the diagram are still isomorphic, while for the left hand side we will prove that in the diagram

$$\mathrm{Aut}\,(\mathbf{S}) \xrightarrow{6} \mathrm{Aut}\,(\mathbf{P}) \xrightarrow{7} \mathrm{Aut}_w\,(\mathbf{P}) \xrightarrow{8} \mathrm{Aut}\,(\mathbf{L}),$$

the maps 6 and 8 are still surjective while the range of the injection 7 is a proper subset of $\mathrm{Aut}_w\,(\mathbf{P})$. As a consequence one obtains that the assumption on the dimension of $\mathcal{H}$ in Proposition 4.9 cannot be avoided.

The fact that the injection 6 is surjective follows directly from Corollary 4.1. The surjectivity of the arrow 8 is the content of the following proposition.

COROLLARY 4.3. *The homomorphism $\alpha \mapsto \tau_\alpha$ defined in Proposition 4.8 is surjective (without any assumption on the dimension of $\mathcal{H}$.)*

PROOF: Let $\tau \in \mathrm{Aut}\,(\mathbf{L})$. Its restriction to $\mathbf{P}$ is a weak $\mathbf{P}$-automorphism since τ preserves orthogonality, hence Proposition 4.8 gives the result.

The fact that $\mathrm{Aut}\,(\mathbf{P})$ is a proper subset of $\mathrm{Aut}_w\,(\mathbf{P})$ was first shown by Uhlhorn [3]. The following is a simplified version of his example.

EXAMPLE 4.1. *Consider the two dimensional Hilbert space $\mathcal{H} = \mathbf{C}^2$. The set $\mathbf{P}$ of one-dimensional projections on $\mathbf{C}^2$ consists exactly of the operators of the form $\frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$, where $\vec{a} \in \mathbf{R}^3, \|\vec{a}\| = 1$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli* *matrices. Therefore, any* $\alpha : \mathbf{P} \to \mathbf{P}$ *is of the form* $\frac{1}{2}(I + \vec{a}\cdot\vec{\sigma}) \mapsto \frac{1}{2}(I + \vec{a}'\cdot\vec{\sigma})$ *so that* α *is bijective if and only if* $\vec{a} \mapsto \vec{a}' =: f(\vec{a})$ *is a bijection on the unit sphere of* $\mathbf{R}^3$*. Writing* $\vec{a} = (1, \theta, \phi)$, $\theta \in [0, \pi], \phi \in [0, 2\pi]$ *we define a function* f *such that* $f(1, \theta, \phi) = (1, \theta, \phi)$ *whenever* $\theta \neq \frac{\pi}{2}$ *and we write* $f(1, \frac{\pi}{2}, \phi) = (1, \frac{\pi}{2}, g(\phi))$, *with* $g(\phi) = \phi^2/\pi$ *for* $0 \leq \phi \leq \pi$ *and* $g(\phi) = (\phi - \pi)^2/\pi + \pi$ *for* $\pi \leq \phi \leq 2\pi$*. The function* $\alpha : \mathbf{P} \to \mathbf{P}$ *defined by* f *is clearly bijective. Using the fact that* $tr\big[\frac{1}{2}(I + \vec{a}\cdot\vec{\sigma})\frac{1}{2}(I + \vec{b}\cdot\vec{\sigma})\big] = \frac{1}{2}(1 + \vec{a}\cdot\vec{b})$ *one immediately observes that* α *preserves transition probability zero but not, in general, other transition probabilities. Hence* $\alpha \in Aut_w(\mathbf{P})$, *but* $\alpha \notin Aut(\mathbf{P})$.

We noticed at the beginning of the section that there exist some other *natural* ways to define isomorphisms between the various groups. However, they lead to the same isomorphism we obtained composing the arrows of the diagram. Consider for instance the following examples.

1) Composing the homomorphisms 5, 6, 7, 8 we obtain an injective group homomorphism from $\mathrm{Aut}(\mathbf{E})$ to $\mathrm{Aut}(\mathbf{L})$. This is exactly the map induced by the inclusion $\mathbf{L} \subset \mathbf{E}$. This homomorphism is surjective if and only if the dimension of $\mathcal{H}$ is greater than two.

2) Starting from any group $\mathrm{Aut}(\mathbf{X})$ one can obtain an isomorphism onto $\mathrm{Aut}(\mathbf{P})$ (if $\mathbf{X} = \mathbf{L}$ we assume $\dim\mathcal{H} \geq 3$). This is the map induced by the inclusion $\mathbf{P} \subset \mathbf{X}$.

6. Conclusion. Using the results of the previous two sections we shall describe the various groups of automorphisms in terms of unitary or antiunitary operators, taking into account also the topological properties.

Let $\mathbf{X}$ denote one of the sets $\mathbf{S}$, $\mathbf{P}$, $\mathbf{E}$, $\mathbf{B}_r$, $\mathbf{B}$ or $\mathbf{L}$. In the case $\mathbf{X} = \mathbf{L}$ we suppose that the dimension of $\mathcal{H}$ is greater than 2. We denote by $\mathrm{Aut}(\mathbf{X})$ the group of automorphisms of $\mathbf{X}$, endowed with the topology defined in section 2. By the results of section 4 $\mathrm{Aut}(\mathbf{X})$ is isomorphic to $\Sigma(\mathcal{H})$ and for any element

$\chi \in \mathrm{Aut}\,(\mathbf{X})$ there is a unitary or anti-unitary operator, uniquely defined by χ up to a phase factor, such that

$$\chi(A) = UAU^* := \chi_U(A), \qquad A \in \mathbf{X}.$$

PROPOSITION 6.1. *The map $j_{\mathbf{X}} : \Sigma(\mathcal{H}) \mapsto Aut\,(\mathbf{X})$ defined as*

$$j_{\mathbf{X}}([U]) = \chi_U, \qquad U \in \mathbf{U} \cup \bar{\mathbf{U}},$$

is a group homeomorphism and $Aut\,(\mathbf{X})$ is a second countable, metrisable, topological group.

PROOF: Taking into account that the topological group $\Sigma(\mathcal{H})$ is second countable and metrisable, the only fact to be proven is that the map $j_{\mathbf{X}}$ is a homeomorphism.

We demonstrate first that the function $J_{\mathbf{X}} : \mathbf{U} \cup \bar{\mathbf{U}} \to \mathrm{Aut}\,(\mathbf{X}), U \mapsto J_{\mathbf{X}}(U) := \chi_U$ is continuous. Since $\mathbf{U} \cup \bar{\mathbf{U}}$ is second countable, it suffice to show that if $(U_n)_{n\geq 1}$ is a (strongly) convergent sequence in $\mathbf{U} \cup \bar{\mathbf{U}}$, then $(J_{\mathbf{X}}(U_n))_{n\geq 1}$ is convergent in $\mathrm{Aut}\,(\mathbf{X})$. As $U \mapsto U^{-1}$ is continuous in $\mathbf{U} \cup \bar{\mathbf{U}}$, we have, for instance for $\mathbf{X} = \mathbf{S}$,

$$\begin{aligned} \lim_{n\to\infty} f^{\mathbf{S}}_{A,T}(J_{\mathbf{S}}(U_n)) &= \lim_{n\to\infty} \mathrm{tr}\big[AJ_{\mathbf{S}}(U_n)(T)\big] \\ &= \lim_{n\to\infty} \mathrm{tr}\big[AU_nTU_n^{-1}\big] \\ &= \mathrm{tr}\big[AUTU^{-1}\big] = f^{\mathbf{S}}_{A,T}(U), \end{aligned}$$

for all $A \in \mathbf{B}_r, T \in \mathbf{S}$, which shows the continuity of $J_{\mathbf{S}}$. The other cases are shown as well. By definition of quotient topology, this proves also that $j_{\mathbf{X}}$ is continuous.

It remains to be shown that the inverse mapping $j_{\mathbf{X}}^{-1}$ is continuous. Consider the group $\mathrm{Aut}\,(\mathbf{X})$ and let $(\varphi_i)_{i\geq 1}$ be a dense sequence of unit vectors in $\mathcal{H}$. Since $\mathbf{P}$ is contained in $\mathbf{X}$, then the sequence of functions $\left(f^{\mathbf{X}}_{P[\varphi_i],P[\varphi_j]}\right)_{i,j\geq 1}$ gives $\mathrm{Aut}\,(\mathbf{X})$ a metrisable topology, which a priori is weaker than the one defined above for $\mathrm{Aut}\,(\mathbf{X})$. We shall show that $j_{\mathbf{X}}^{-1}$ is continuous in this weaker topology. It suffices again to consider only sequences. Let (γ_n) be a convergent sequence in $\mathrm{Aut}\,(\mathbf{X})$, with $\gamma_n \to \gamma$. We will show that $j_{\mathbf{X}}^{-1}(\gamma_n) \to j_{\mathbf{X}}^{-1}(\gamma)$ in $\Sigma(\mathcal{H})$. To proceed assume on the contrary that $j_{\mathbf{X}}^{-1}$ is not continuous so that there is an open set $O \subset \Sigma(\mathcal{H})$ such that $j_{\mathbf{X}}^{-1}(\gamma) \in O$ but $j_{\mathbf{X}}^{-1}(\gamma_{n_k}) \notin O$ for a subsequence (γ_{n_k}) of (γ_n). Let $U_k, U \in \mathbf{U} \cup \bar{\mathbf{U}}$ such that $j_{\mathbf{X}}([U_k]) = \gamma_{n_k}$ and $j_{\mathbf{X}}([U]) = \gamma$. The sequence (U_k) is bounded, so that it has a weakly convergent subsequence (U_{k_h}) in $\mathbf{U} \cup \bar{\mathbf{U}}$, with $U_{k_h} \to V$. But then $\mathrm{tr}\big[P[\varphi_i]\gamma_{n_{k_h}}(P[\varphi_j])\big] = |\left\langle \varphi_i \,|\, U_{n_{k_h}} \varphi_j \right\rangle|^2 \to |\langle \varphi_i \,|\, V\varphi_j \rangle|^2$ and $\mathrm{tr}\big[P[\varphi_i]\gamma_{n_{k_h}}(P[\varphi_j])\big] \to \mathrm{tr}\big[P[\varphi_i]\gamma(P[\varphi_j])\big] = |\langle \varphi_i \,|\, U\varphi_j \rangle|^2$, which shows that $[V] = [U]$. Since $U_{n_{k_h}} \to V$ also strongly we thus have $[U_{n_{k_h}}] \to [V] = [U]$ which is a contradiction. This shows that $j_{\mathbf{X}}^{-1} : \mathrm{Aut}\,(\mathbf{X}) \to \Sigma(\mathcal{H})$ is continuous. This ends the proof.

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