GALILEI INVARIANT WAVE EQUATIONS

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ABSTRACT. We describe the quantum states of an elementary particle invariant with respect to a symmetry group as solutions of an invariant wave equation. From the mathematical point of view the wave equation is a set of differential operators on the space of vector valued distributions over the space-time. As an application, we consider the Galilei invariant particles both in $3+1$ and in $2+1$ dimensions.

1. Introduction

In Quantum Mechanics the states of an elementary particle invariant with respect to a symmetry group G are described by rays of a Hilbert space (called state space) carrying a projective unitary representation of G. When G is either the Galilei group or the Poincaré group, this approach gives a description of the state space as the set of square integrable functions over momentum space. Nevertheless, there is the need of characterising the states as solutions of partial differential equations, usually called wave equations, in the space-time variables. The physical reason for this is that one can introduce the interaction with external fields as in the classical framework. In almost every cases, the wave equations are deduced from a Lagrangian function invariant with respect to a suitable representation of G and the states are regarded as classical (differentiable) fields. In doing so, one hides the Hilbert structure of the theory and the mathematical problems connected with the fact that states are not smooth functions, but only square integrable ones.

Following Schwartz, [9], we propose in Sec. 3 a definition of *wave equation* as a family of differential operators $(D_i)_1^N$ acting on a space of vector valued distributions over the space-time. The space of the distributions carries a *geometric* representation of the symmetry group G and the operators D_i are invariant with respect to this action. The state space of the quantum system is singled out by the family of equations $D_iT = 0$ and by the requirement that the action of G on it be unitary and irreducible (with respect to a suitable scalar product).

The definition of wave equation we propose is unsatisfactory for the Poincaré group. In that case the physical interesting wave equations (like the Dirac equation) are associated with *reducible* representation since they have to describe both the particle and its antiparticle. Nevertheless, for the

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Galilei case the theory we develop in Sec. 3 gives rise to some interesting and non-trivial results, which justify the definition of wave equation we give.

From a mathematical point of view, we assume that the vector space where the distributions take value is finite dimensional and that G is a semidirect product with abelian normal factor. With these hypotheses, using the theory of intertwining numbers for induced representations developed by Bruhat, [1], we reduce the problem of studying wave equations to the one of characterising commuting rings of finite dimensional representations of suitable subgroups of G.

In order to describe quantum symmetries, one has to take care of the projective character of the representations of the symmetry group G. To bypass this technical problem, in this paper we replace G by its *universal central extension*. This group has the property that there is a one to one correspondence between its unitary irreducible representations and the irreducible projective representations of G. This fact allows using freely the theory of (ordinary) representations.

In Sec. 4 we apply the theory developed in Sec. 3 to the Galilei groups in $3 + 1$ and in $2 + 1$ dimensions. In the first case, for each elementary particle, we find a corresponding wave equation (D_1, D_2). The operator D_1 takes into account the fact that the the unitary representation associated with the particle is truly projective (the usual Schrödinger equation is not invariant with respect to the Galilei group). The operator D_2 coincides, at least formally, with the wave equation found by Hurley with Lagrangian methods, [6]. The main feature of D_2, comparing it with the Schrödinger equation, is that D_2 is a differential operator of the first order in the space-time variables as the Dirac equation for the relativistic electron. Moreover, if the interaction with an electromagnetic field is introduced by means of the minimal coupling principle, the particle acquires an internal magnetic momentum with gyromagnetic ratio $g = \frac{1}{j}$, where j is the spin of the particle.

In the $2 + 1$ dimensional case we show that there are two classes of elementary particles: the first one does not admit wave equations at all (at least in the sense of our definition), whereas for the second one there is essentially only one wave equation such that the operators D_i are of first order and the vector space where the distributions take value has minimal dimension. The particles described by this wave equation interact with an electromagnetic field as three-dimensional 1/2-spin particles independently of their two-dimensional *spin*.

2. Preliminaries and notations

We briefly recall some basic facts we use freely in the following. As a standard reference we adopt [10].

By *locally convex vector space* we mean a locally convex, complete, Hausdorff, topological vector space over the complex field. By *Lie group* we mean a unimodular Lie group countable at infinity with Haar measure dg.

Let A be an abelian Lie group (A is of the form $\mathbb{R}^n \times \mathbf{T}^m \times \mathbb{Z}^p \times \Phi$ where Φ is a finite abelian group) and V a finite dimensional vector space. We denote by $\mathcal{D}(A;V)$ the space of C^∞-functions from A to V with compact support and by $\mathcal{S}(A;V)$ the space of C^∞-functions from A to V with each derivative rapidly decreasing at infinity. These spaces are endowed with the corresponding Schwartz topologies with respect to which they are locally convex Montel vector spaces and, in particular, reflexive. By $\mathcal{D}(A;V)'$ and $\mathcal{S}(A;V)'$ we mean the topological duals, *i.e.* the space of V-distributions and the space of tempered V-distributions. We denote by $\mathcal{F}$ the Fourier transform both on $\mathcal{S}(A;V)$ and $\mathcal{S}(A;V)'$. We say that a distribution $T \in \mathcal{D}(A;V)'$ is *a function* if there is a map ϕ from A to V', the dual of V, such that ϕ is locally integrable with respect to the Haar measure da and, for all $f \in \mathcal{D}(A;V)$,

$$\langle\, T, f \,\rangle = \int_A \langle\, \phi(a), f(a) \,\rangle \; da.$$

Let G be a Lie group and E a locally convex vector space. By *representation* of G on E we mean a continuous representation. When this does not cause any confusion, we denote the action of the representation on E by $g.v$ with $g \in G$ and $v \in E$.

Let E be a Banach space, $\|\cdot\|_E$ denote the norm and $\mathcal{L}(E)$ the Banach space of continuous operators on E.

If E and F are two locally convex vector spaces carrying two representations of G, we denote by $\mathrm{Hom}_G(E;F)$ the vector space of continuous intertwining operators from E to F.

We recall the definition of induced representation both in the differentiable and unitary cases.

Let G be a Lie group, H a unimodular closed subgroup of G and q the canonical projection of G onto G/H. Given a differentiable representation L of H on a locally convex vector space V, define ${}^LC^\infty(G;V)$ as the set of C^∞-functions $f : G \to V$ such that

1. $f(gh) = h^{-1}.f(g)$ for all $g \in G$, $h \in H$;
2. $q(\mathrm{supp}\, f)$ is compact in G/H.

The set ${}^LC^\infty(G;V)$ is a locally convex vector space with respect to its natural Schwartz topology, see, for example, p. 368 of [10]. For all $g \in G$ and $f \in {}^LC^\infty(G;V)$, let

$$(g.f)(g') = f(g^{-1}g') \qquad g' \in G,$$

then $g.f \in {}^LC^\infty(G;V)$ and this defines a differentiable representation, namely the C^∞-induced representation ${}^\infty\mathrm{Ind}_H^G(L)$. Moreover, let π from $\mathcal{D}(G;V)$ to ${}^LC^\infty(G;V)$

$$(\pi f)(g) = \int_H h.f(gh)\; dh \qquad g \in G, \tag{1}$$

then π is a strict morphism and intertwines the left regular representation with ${}^\infty\mathrm{Ind}_H^G(L)$.

If $\mathcal{K}$ is a Hilbert space and m is a unitary representation of H on $\mathcal{K}$, let $\mathcal{K}_\infty$ be the set of C^∞-vectors in $\mathcal{K}$ for m and m_∞ the restriction of m to $\mathcal{K}_\infty$, which is a C^∞-representation. We can define on ${}^{m_\infty}C^\infty(G;\mathcal{K}_\infty)$ a sesquilinear form

$$(f_1, f_2) = \int_{G/H} (f_1(g), f_2(g))_{\mathcal{K}}\, d\nu(q(g)),$$

where ν is the invariant measure on G/H. With respect to this form, ${}^{m_\infty}C^\infty(G;\mathcal{K}_\infty)$ is a pre-Hilbert space and ${}^\infty\mathrm{Ind}_H^G(m_\infty)$ is unitary. Let $\mathcal{K}^m$ be the completion of ${}^{m_\infty}C^\infty(G;\mathcal{K}_\infty)$, then ${}^\infty\mathrm{Ind}_H^G(m_\infty)$ extends to a unitary representation ${}^u\mathrm{Ind}_H^G(m)$, namely the representation unitarily induced by m.

We review the notion of physical symmetry in quantum mechanics. Let $\mathfrak{S}$ be a quantum system and $\mathcal{H}$ the corresponding Hilbert space. Given a group G, we say that G is a symmetry group of $\mathfrak{S}$ if there is a projective representation of G on $\mathcal{H}$. Moreover $\mathfrak{S}$ is *elementary* with respect to G if the projective representation is irreducible. The classification of the projective representations is usually traced back to the study of the (ordinary) unitary representations of the central extensions of the phase group by G. Alternatively, one can use a more economic procedure based on the following result, see [2] for the proof.

Theorem 1. *Let G be a connected Lie group. There is a connected Lie group $\overline{G}$ such that $\overline{G}$ is a central extension of an abelian group by G and there is a one to one correspondence between the irreducible unitary projective representations of G and the irreducible unitary representations of $\overline{G}$. The group $\overline{G}$ is called the universal central extension of G.*

In the hypotheses of the preceding theorem, we cannot conclude that the group $\overline{G}$ is unique (on this topic one can consult [8]). However, this fact does not constitute an obstacle for the solution of the problem addressed in this paper.

If G is the Poincaré group, $\overline{G}$ is its universal covering group. In Sec. 4 we give the explicit construction of $\overline{G}$ when G is the Galilei group in $3+1$ and $2+1$ dimensions. In both cases $\overline{G}$ is the semidirect product of an abelian normal closed subgroup A and a closed subgroup H and A contains the space-time translations.

Hence, due to the above theorem, in order to study elementary quantum systems invariant with respect to a given symmetry group, one can replace this group by its universal central extension, denoted simply by G in the following sections.

3. Invariant wave equations

In this section we propose a definition of wave equation in the case that the symmetry group is a semidirect product with abelian normal factor and we give a method to find wave equations by solving a finite dimensional problem.

Let G be the universal central extension of a given symmetry group. We assume that G is a connected Lie group such that G is semidirect product of an abelian normal closed subgroup A and a unimodular closed subgroup H, *i.e.* $G = A \times' H$. We write A additively and we denote by e the identity of H.

Notice that, since A is a connected abelian Lie group, A and its dual group $\hat{A}$ are of the form $\mathbb{R}^n \times \mathbf{T}^m$ and $\mathbb{R}^n \times \mathbb{Z}^m$, respectively. On A and, by duality, on $\hat{A}$ there are two natural actions of G. Indeed, since A is normal in G, the inner action

$$A \ni a \mapsto gag^{-1} \in A \tag{2}$$

is well defined. Moreover, since A is canonically identified with the quotient space G/H, one has also the action

$$A \ni a \mapsto hah^{-1} + b \in A \tag{3}$$

where $g = (b, h) \in G$. In the following we regard A with the action of G given by formula (3) and $\hat{A}$ with the action dual to the one defined by (2). In both cases the action of an element $g \in G$ on a point $a \in A$ (resp. $p \in \hat{A}$) is denoted simply by $g.a$ (resp. $g.p$). With this choice the product in G is

$$(a_1, h_1)(a_2, h_2) = (a_1 + h_1.a_2, h_1h_2) \qquad a_1, a_2 \in A, h_1, h_2 \in H.$$

Moreover we write the pairing between A and $\hat{A}$ as $\prec\cdot,\cdot\succ$. Finally, notice that, if $h \in H$,

$$\prec h.p, a\succ = \prec p, h^{-1}.a\succ \qquad a \in A,\ p \in \hat{A}.$$

Since H and G are unimodular, the Haar measure da on A is G-invariant and $dg = da \otimes dh$. If $p \in \hat{A}$, let G_p be the stability subgroup of G at p, $G.p$ the corresponding orbit and $H_p = G_p \cap H$. We assume that each orbit in $\hat{A}$ is locally closed (*i.e.* the semidirect product is regular) and G_p is unimodular (hence, the orbits admit G-invariant measures). Let $\hat{G}$ be the (unitary) dual group of G and for all $s \in \hat{G}$ denote by U_s a representative in the equivalence class s and by $\mathcal{H}_s$ the Hilbert space carrying U_s.

Let L be a (not necessarily unitary) representation of H on a finite dimensional Hilbert space V and extend L to a representation of G in a trivial way

$$L(a, h) := L(h) \qquad a \in A,\ h \in H.$$

Let V' be the dual space with the representation L' contragredient to L, (here we do not identify V' with V by means of the scalar product). We denote by $\langle\,\cdot,\cdot\,\rangle$ the pairing between V and V' and by $(\cdot,\cdot)$ the scalar product on V.

Since $\dim V' < \infty$, L' is differentiable and we consider the induced representation ${}^{\infty}\mathrm{Ind}_H^G(V')$ of G. Identifying G/H with A in the canonical way, one has the following natural isomorphism.

Lemma 1. *The map*

$$^{L}C^{\infty}(G;V') \ni f \mapsto f(\cdot,e) \in \mathcal{D}(A;V')$$

is a topological linear isomorphism and $^{\infty}\mathrm{Ind}_H^G(V')$ *acts on* $\mathcal{D}(A;V')$ *as*

$$(g.\phi)(a) = g.\phi(g^{-1}.a) \qquad a \in A,\ \phi \in \mathcal{D}(A;V'),\ g \in G.$$

Since $\mathcal{D}(A;V')$ is reflexive, the contragredient representation acting on $\mathcal{D}(A;V')'$ is differentiable too. On V we can consider also the action of G given by the representation adjoint to L. The corresponding representation on $\mathcal{D}(A;V')'$ will be denoted by $T \mapsto g^*.T$.

Remark 1. Identifying V'' with V, if $T \in \mathcal{D}(A;V')'$ is the function $\phi : A \to V$ and $g \in G$, then $g.T$ is the function

$$A \ni a \mapsto L(g)\phi(g^{-1}.a) \in V$$

and $g^*.T$ is the function

$$A \ni a \mapsto L^*(g)\phi(g^{-1}.a) \in V.$$

The following definition is the corner-stone of the paper.

Definition 1. *Let L be a representation of H on a finite dimensional Hilbert space V. An L-*invariant wave equation *is a family $(D_i)_{i=1}^N$ of continuous operators on $\mathcal{D}(A;V')'$ such that:*

1. *for all $T \in \mathcal{D}(A;V')'$ and $i = 1,\dots,N$,*

$$\mathrm{supp}\ D_iT \subset \mathrm{supp}\ T. \tag{4}$$

2. *For each $i = 1,\dots,N$, one of the following two conditions holds:*

$$g.D_i = D_i.g \qquad g \in G, \tag{5}$$
$$g^*.D_i = D_i.g \qquad g \in G. \tag{6}$$

3. *There is a unique subspace $\mathcal{H}$ of $\mathcal{D}(A;V')'$ and a unique, up to a constant, sequilinear form $\langle\,\cdot,\cdot\,\rangle_{\mathcal{H}}$ defined on $\mathcal{H}$ such that:*
   (a) *$(\mathcal{H}, \langle\,\cdot,\cdot\,\rangle_{\mathcal{H}})$ is a Hilbert space, the corresponding topology is finer than the original one, $\mathcal{H}$ is invariant with respect to the action of G and the restriction of this action, denoted by U, is unitary and irreducible.*
   (b) *for each $i = 1,\dots,N$, $D_i\mathcal{H} = 0$.*

Remark 2. A continuous operator on $\mathcal{D}(A;V')'$ that decreases supports, *i.e.* such that condition (4) holds, and satisfies condition (5) (resp. condition (6)) is called an *invariant wave operator* (resp. **-invariant wave operator*).

If $s \in \hat{G}$ is the equivalence class of $(\mathcal{H}, U)$ we say that the wave equation $(D_i)_{i=1}^N$ is *associated* with s.

The first condition in Def. 1 is the requirement that the physical system be described by *local* dynamical laws. The second one is the simplest way to assure that the kernels of the operators D_i be G-invariant. On the other hand, condition (6) implies that the *formal* Lagrangian function

$$\mathcal{L}_i(T) = \int_A (T(a), (D_i T)(a))_V \ da$$

is G-invariant. The last condition means that the wave equation

$$D_i T = 0 \qquad \forall i = 1, \dots, N$$

singles out exactly one elementary system, namely the one associated with $(\mathcal{H}, U)$.

Remark 3. The requirement that there be only one irreducible Hilbert G-module contained in the intersection of the kernels of the operators D_i is very restrictive. In fact, in the case of the Poincaré group, wave equations in the sense of Def. 1 do not exist. For example, the Dirac equation singles out two Hilbert spaces, one associated with the orbit of *positive energy* and the other associated with the orbit of *negative energy*. This phenomenon is due the the fact that orbits in the momentum space are not described by polynomial conditions. On the other hand, in the case of Galilei group, we show that there are wave equations in the sense of Def. 1, which are of physical interest. The different behaviour is a further evidence of the deep difference between the mathematical structures of the Galilei and Poincaré groups.

In the following, given a representation L of H and an equivalence class $s \in \hat{G}$, we characterise the L-invariant wave equations associated with s. Let V be the finite dimensional Hilbert space carrying L.

First of all, we study the invariant wave operators.

Proposition 1. *Let $M : \hat{A} \to \mathcal{L}(V)$ be such that*

(7) $\quad M$ *is a polynomial function over* $\hat{A}$

$$h.M(h^{-1}.p).h^{-1} = M(p) \qquad \text{for all } h \in H \text{ and } p \in \hat{A} \tag{8}$$

and define $D : \mathcal{D}(A; V')' \to \mathcal{D}(A; V')'$ as

$$\langle\, DT, f \,\rangle = \langle\, T, \mathcal{F}^{-1} M^t \mathcal{F} f \,\rangle \qquad T \in \mathcal{D}(A; V')', \ f \in \mathcal{D}(A; V'), \tag{9}$$

where $(M^t f)(p) := M(p)^t f(p)$. Then D is an invariant wave operator and all invariant wave operators are of this form.

The same result holds for $*$-invariant operators, replacing condition (8) *with*

$$h^*.M(h^{-1}.p).h^{-1} = M(p) \qquad \text{for all } h \in H \text{ and } p \in \hat{A}. \tag{10}$$

Remark 4. Recall that $\hat{A} = \mathbb{R}^n \times \mathbb{Z}^m$ so that a *polynomial function* over $\hat{A}$ is the restriction to $\hat{A}$ of a polynomial function on $\mathbb{R}^{n+m}$. Moreover, due to condition (7), for all $f \in \mathcal{D}(A; V')$, one has $\mathcal{F}^{-1} M^t \mathcal{F} f \in \mathcal{D}(A; V')$.

Proof of the proposition. Let D be an invariant operator. Since $\mathcal{D}(A;V')$ is reflexive, the transpose D^t of D is a continuous G-invariant operator on $\mathcal{D}(A;V')$. Moreover D^t decreases supports; indeed, let $f \in \mathcal{D}(A;V')$ and Ω be an open set in A such that f is zero on Ω, then for all $T \in \mathcal{D}(A;V')'$ such that $\operatorname{supp} T \subset \Omega$,

$$\langle\, T, D^t f \,\rangle = \langle\, DT, f \,\rangle = 0$$

since $\operatorname{supp} DT \subset \operatorname{supp} T \subset \Omega$. This implies that $D^t f$ is zero on Ω and, hence, $\operatorname{supp} D^t f \subset \operatorname{supp} f$.

The operator D^t can be extended to a linear G-invariant operator D^t_0 from $C^\infty(A,V')$ into itself. For all $a \in A$, let $\phi_a \in \mathcal{D}(A;\mathbb{C})$ be such that ϕ_a is 1 on a neighbourhood of a and define

$$(D^t_0 f)(a) = (D^t \phi_a f)(a) \qquad a \in A,\ f \in C^\infty(A,V').$$

Since D^t decreases supports, D^t_0 does not depend on ϕ_a and it is linear. Moreover D^t_0 commutes with the action of G on $C^\infty(A,V')$ given by

$$(g.f)(a) = g.f(g^{-1}.a) \qquad a \in A,\ f \in C^\infty(A,V'),\ g \in G.$$

Indeed, if $g \in G,\ f \in C^\infty(A,V')$, then

$$\begin{aligned}
(g.D^t_0 f)(a) &= g.(D^t_0 f)(g^{-1}.a) \\
&= g.(D^t \phi_{g^{-1}.a} f)(g^{-1}.a) \\
&= \left(D^t g.(\phi_{g^{-1}.a} f)\right)(a) \\
&= \Big(D^t(\phi^g_{g^{-1}.a} g.f)\Big)(a) \\
&= \left(D^t_0 g.f\right)(a),
\end{aligned}$$

since $\phi^g_{g^{-1}.a}$ is in $\mathcal{D}(A;\mathbb{C})$ and it is 1 on a neighbourhood of a. Moreover D^t_0 decreases supports. Indeed, let $f \in C^\infty(A,V')$ and Ω be an open set in A with compact closure such that f is zero on Ω. Choose $\phi \in \mathcal{D}(A;\mathbb{C})$ that is 1 on Ω, then for all $a \in \Omega$ $(D^t_0 f)(a) = D^t(\phi f)(a) = 0$, so that $\operatorname{supp} D^t_0 f \subset \operatorname{supp} f$.

Using a theorem of Peetre on the characterisation of differential operators, see, for example, Appendix A.2.1 of [10], one has that D^t_0 is a (vector valued) differential operator on A, precisely for all Ω open set in A homeomorphic to an open set of $\mathbb{R}^{n+m}$ (recall that $A = \mathbb{R}^n \times \mathbf{T}^m$), there exist $\alpha_\nu \in C^\infty(\Omega, \mathcal{L}(V'))$ such that on Ω

$$D^t_0 = \sum_\nu \alpha_\nu D^\nu$$

where ν runs on the A-invariant differential operators D^ν on A (*i.e.* the universal enveloping algebra of the Lie algebra of A) and, for every compact K of A, $\operatorname{supp} \alpha_\nu \cap K$ is empty for all but a finite number of ν. Moreover, since A does not act on V', D^t_0 is an A-invariant differential operator, *i.e.* the maps α_ν are constant and D^t_0 is of finite order. This implies that D^t_0, restricted to $\mathcal{S}(A;V')$, is a continuous operator into $\mathcal{S}(A;V')$. Hence, the operator $\mathcal{F} D^t_0 \mathcal{F}^{-1}$ acts on $\mathcal{S}(\hat{A};V')$ as a multiplication by a polynomial function $\hat{M}$ from $\hat{A}$ to $\mathcal{L}(V')$. Since D_0^t and $\mathcal{F}$ commute with the action of H, the map $\hat{M}$ satisfies

$$L'(h)\hat{M}(h^{-1}.p)L'(h^{-1}) = \hat{M}(p) \qquad h \in H,\ p \in \hat{A}.$$

Define $M(p) = \hat{M}(p)^t$, then M clearly satisfies Eq. (7), Eq. (8) and Eq. (9).

Conversely, if $M : \hat{A} \to \mathcal{L}(V)$ satisfies conditions (7) and (8), then one easily checks that D, defined as in Eq. (9), is an invariant wave operator.

The same proof holds for the *-invariant case. □

Remark 5. Let $p \in \hat{A}$. The condition (8) (resp. (10)) implies that $M(p) \in \mathrm{Hom}_{H_p}(V;V)$ (resp. $M(p) \in \mathrm{Hom}_{H_p}(V;V^*)$, where V^* denotes the vector space V endowed with the representation adjoint to L.) Moreover $M(p)$ defines uniquely M on the orbit $G.p$.

Remark 6. Let D be an invariant (*-invariant) wave operator and M the corresponding function given by the above proposition. Since M is polynomial, it is of the form

$$M = \sum_{\nu} C_\nu p_1^{\nu_1} \dots p_{n+m}^{\nu_{n+m}}$$

where $\nu = (\nu_1, \dots, \nu_{n+m})$ is a multiindex and $C_\nu \in \mathcal{L}(V)$. Then, we write

$$D = \sum_{\nu} (i)^{|\nu|} C_\nu \frac{\partial}{\partial^{\nu_1} a_1 \dots \partial^{\nu_{n+m}} a_{n+m}},$$

where we are assuming the natural identification between the dual group of $A = \mathbb{R}^n \times \mathbf{T}^m$ and $\mathbb{R}^n \times \mathbb{Z}^m$. Moreover, let $T \in \mathcal{D}(A;V')'$ such that T is a smooth function ϕ, then $DT = 0$ in the sense of distributions if and only if $D\phi = 0$ regarding D as a classical vector valued differential operator.

We now characterise the subspaces $\mathcal{H}$ of $\mathcal{D}(A;V')'$ satisfying condition (a) of item 3 of Def. 1. It is clear that this problem is equivalent to studying the space of intertwining operators $\mathrm{Hom}_G(\mathcal{H}_t;\mathcal{D}(A;V')')$ when t runs over $\hat{G}$.

To do this, observe that, since G is a regular semidirect product, every unitary irreducible representation of G is equivalent to an induced one. Hence, without loss of generality, for each $t \in \hat{G}$ we can assume that $\mathcal{H}_t$ carries a representation U_t of the form ${}^u\mathrm{Ind}_{G_{p_t}}^G(p_t \otimes m_t)$, where $p_t \in \hat{A}$ and m_t is an irreducible unitary representation of H_{p_t} acting on a Hilbert space $\mathcal{K}_t$

The following Lemma 2, based on the Bruhat theory of intertwining numbers, see [10] and the original paper of Bruhat, [1], characterises the space $\mathrm{Hom}_G(\mathcal{H};\mathcal{D}(A;V')')$ when $\mathcal{H}$ carries an induced unitary representation of G. An alternative approach to the problem of studying Hilbert subspaces of $\mathcal{D}(A;V')'$ is described in [9].

Let $p \in \hat{A}$, m be a unitary representation of H_p acting on a Hilbert space $\mathcal{K}$. We denote by $\mathcal{K}_\infty$ the vector space of C^∞-vectors for m and by m_∞ the restriction of m to $\mathcal{K}_\infty$. Denote $M = p \otimes m_\infty$. Let ν be the G-invariant measure on G/G_p and q the canonical projection from G onto G/G_p.

Remark 7. Since A is abelian, A acts trivially on $\hat{A}$, so that the map

$$H \ni h \mapsto q(0,h) \in G/G_p$$

is surjective and a function f over H can be regarded as a function over G/G_p provided that $f(hs) = f(h)$ for all $h \in H$ and $s \in H_p$.

Lemma 2. *Given $B \in \mathrm{Hom}_{H_p}(\mathcal{K}_\infty; V)$, the map T^B from ${}^M C^\infty(G;\mathcal{K}_\infty)$ to $\mathcal{S}(A;V')'$*

$$\langle\, T^B f, \phi \,\rangle \;:=\; \int_{G/G_p} \langle\, Bf(0,h), h^{-1}.(\mathcal{F}\phi)(h.p) \,\rangle \; d\nu(q(0,h)) \qquad \text{where } \phi \in \mathcal{S}(A;V') \tag{11}$$

is well defined and $T^B \in \mathrm{Hom}_G({}^M C^\infty(G;\mathcal{K}_\infty); \mathcal{S}(A;V')')$.

Moreover, taking into account that $\mathcal{S}(A;V')' \subset \mathcal{D}(A;V')'$, the map

$$\mathrm{Hom}_{H_p}(\mathcal{K}_\infty; V) \ni B \mapsto T^B \in \mathrm{Hom}_G({}^M C^\infty(G;\mathcal{K}_\infty); \mathcal{D}(A;V')')$$

is a linear bijection.

Finally if $P \in \mathrm{Hom}_G(\mathcal{K}^{p\otimes m}; \mathcal{D}(A;V')')$, then there exists a unique $B \in \mathrm{Hom}_{H_p}(\mathcal{K}_\infty; V)$ such that

$$Pf = T^B f \qquad f \in {}^M C^\infty(G;\mathcal{K}_\infty).$$

Proof. Let $B \in \mathrm{Hom}_{H_p}(\mathcal{K}_\infty; V)$. For all $f \in {}^M C^\infty(G;\mathcal{K}_\infty)$ and $\phi \in \mathcal{S}(A;V')$, the function

$$G/G_p \ni q(0,h) \mapsto \langle\, Bf(0,h), h^{-1}.(\mathcal{F}\phi)(h.p) \,\rangle \in \mathbb{C}$$

is well defined (see Remark 7), continuous and has compact support, hence it is integrable with respect to ν. Moreover, let ω be a compact subset of G and define

$$\hat{\omega} = \{h \in H \;:\; \exists\, a \in A \text{ such that } (a,h) \in \omega\},$$

which is compact. Let $f \in {}^M C^\infty(G;\mathcal{K}_\infty)$ such that $q(\mathrm{supp}\, f) \subset q(\omega)$, then

$$\begin{aligned}
|\langle\, T^B f, \phi \,\rangle| &\le \nu(q(\omega)) \sup_{q(0,h)\in q(\omega)} |\langle\, Bf(0,h), h^{-1}.(\mathcal{F}\phi)(h.p) \,\rangle| \\
&= \nu(q(\omega)) \sup_{(0,h)\in \omega G_p} |\langle\, Bf(0,h), h^{-1}.(\mathcal{F}\phi)(h.p) \,\rangle| \\
&= \nu(q(\omega)) \sup_{h\in\hat{\omega}} |\langle\, Bf(0,h), h^{-1}.(\mathcal{F}\phi)(h.p) \,\rangle| \\
&\le C \sup_{h\in\hat{\omega}} \|f(0,h)\|_{\mathcal{K}_\infty} \sup_{h\in\hat{\omega}} \left\|L'(h^{-1})\right\|_{\mathcal{L}(V')} \sup_{p'\in\hat{A}} \|(\mathcal{F}\phi)(p')\|_{V'} \\
&\le C' \sup_{h\in\hat{\omega}} \|f(0,h)\|_{\mathcal{K}_\infty} \sup_{p'\in\hat{A}} \|(\mathcal{F}\phi)(p')\|_{V'},
\end{aligned}$$

where $\|\cdot\|_{\mathcal{K}_\infty}$ is a suitable seminorm on $\mathcal{K}_\infty$, depending on B.

Fixed f, the above inequality shows that $T^B f \in \mathcal{S}(A;V')'$. Moreover the map T^B is continuous if and only if, for all ω compact subset of G, T^B is continuous when it is restricted to the subspace

$$\{f \in {}^M C^\infty(G;\mathcal{K}_\infty) \ : \ q(\text{supp } f) \subset q(\omega)\}$$

endowed with the Schwartz topology of $C^\infty(G,\mathcal{K}_\infty)$. Fixed ω, this latter topology is metrisable and $\mathcal{S}(A;V')'$ is a Montel space, so that it is sufficient to prove that, for all $\phi \in \mathcal{S}(A;V')$, the map $f \mapsto \langle\, T^B f, \phi\,\rangle$ is continuous and this last fact is now consequence of the above inequalities.

To prove the intertwining property of T^B, let $(a,h) = g \in G$, $f \in {}^M C^\infty(G;\mathcal{K}_\infty)$ and $\phi \in \mathcal{S}(A;V')$, then (to simplify the notation we do not write the measure ν in the integral)

$$\begin{aligned}
\langle\, T^B(g.f), \phi\,\rangle &= \int \langle\, Bf^g(0,h'), {h'}^{-1}.(\mathcal{F}\phi)(h'.p)\,\rangle \\
&= \int \langle\, Bf(-h^{-1}.a, h^{-1}h'), {h'}^{-1}.(\mathcal{F}\phi)(h'.p)\,\rangle \\
&= \int \langle\, Bf((0,h^{-1}h')(-{h'}^{-1}.a, e)), {h'}^{-1}.(\mathcal{F}\phi)(h'.p)\,\rangle \\
&= \int \prec h'.p, a\succ \langle\, Bf(0,h^{-1}h'), {h'}^{-1}.(\mathcal{F}\phi)(h'.p)\,\rangle \\
&= \int \prec hh'.p, a\succ \langle\, Bf(0,h'), {h'}^{-1}h^{-1}.(\mathcal{F}\phi)(hh'.p)\,\rangle \\
&= \int \langle\, Bf(0,h'), {h'}^{-1}.\left(\prec h'.p, h^{-1}.a\succ h^{-1}.(\mathcal{F}\phi)(h.h'.p)\right)\,\rangle \\
&= \int \langle\, Bf(0,h'), {h'}^{-1}.\left(\prec h'.p, h^{-1}.a\succ (\mathcal{F}h^{-1}.\phi)(h'.p)\right)\,\rangle \\
&= \int \langle\, Bf(0,h'), {h'}^{-1}.(\mathcal{F}g^{-1}.\phi)(h'.p)\,\rangle \\
&= \langle\, (g.T^B)f, \phi\,\rangle.
\end{aligned}$$

Obviously the map $B \mapsto T^B$ is linear, the fact that it is an isomorphism from $\mathrm{Hom}_{H_p}(\mathcal{K}_\infty;V)$ onto $\mathrm{Hom}_G({}^M C^\infty(G;\mathcal{K}_\infty);\mathcal{D}(A;V')')$ is consequence of the following result.

If $P \in \mathrm{Hom}_G({}^M C^\infty(G;\mathcal{K}_\infty);\mathcal{D}(A;V')')$, then there exists a unique $B \in \mathrm{Hom}_{H_p}(\mathcal{K}_\infty;V)$ such that $P = T^B$. Indeed, P defines uniquely an intertwining form (in the sense of Bruhat, see Section 4.1.3 of [10]) on ${}^M C^\infty(G;\mathcal{K}_\infty)\times \mathcal{D}(A;V')$

$$(f_1,f_2) \mapsto \langle\, Pf_1, f_2\,\rangle.$$

By a theorem of Bruhat on the intertwining forms for C^∞-induced representations, see, for example, Th. 5.3.2.1 of [10], there is a unique $\mathcal{K}_\infty \otimes V'$-distribution[1] Ω on G such that for all $f_1 \in \mathcal{D}(G; \mathcal{K}_\infty)$ and $f_2 \in \mathcal{D}(G; V')$

$$\int_G dg_2 \int_G f_1(g_2 g_1) \otimes f_2(g_2) d\Omega(g_1) = \langle\, P(\pi_1(f_1)), \pi_2(f_2) \,\rangle$$

where π_i are defined by Eq. (1) and we are using Lemma 1 to identify ${}^L C^\infty(G; V')$ with $\mathcal{D}(A; V')$. Moreover Ω is a *quasi-invariant* distribution with respect to the action of the group $H \times (A \times' H_p)$ on G given by $(h; a, s).g = hg(a, s)^{-1}$, namely for all $f \in \mathcal{D}(G; \mathcal{K}_\infty \otimes V')$

$$\langle\, \Omega, f^{(h;a,s)^{-1}} \,\rangle = \langle\, \Omega, \big(M(a, s) \otimes L'(h)\big)\, f \,\rangle.$$

The action is transitive on G, the stability subgroup at the identity is H_p, regarded as the diagonal subgroup of $H_p \times H_p$, and a natural section from G to $H \times (A \times' H_p)$ is

$$(a, h) \mapsto (h; -h^{-1}.a, e).$$

Hence, due to a theorem of Bruhat on quasi-invariant distributions, see Th. 5.2.1.4 of [10], there is a unique $B \in (\mathcal{K}_\infty \otimes V')'$ such that

1. $\langle\, B, m_\infty(s) \otimes L(s)w \,\rangle = \langle\, B, w \,\rangle$ for all $s \in H_p$, $w \in \mathcal{K}_\infty \otimes V'$;
2. for all $f \in \mathcal{D}(G; \mathcal{K}_\infty \otimes V')$

$$\langle\, \Omega, f \,\rangle = \int \langle\, B, M(h^{-1}.a, e) \otimes L'(h^{-1}) f(a, h) \,\rangle da \otimes dh$$

(of course the Haar measure of G is $H \times (A \times' H_p)$ invariant).

Observing that $(\mathcal{K}_\infty \otimes V')' \simeq \mathrm{Hom}_{H_p}(\mathcal{K}_\infty; V)$ and denoting, as usual, by $h.v = L'(h)v$ the action of H on V' defined by L', one has that, if $f_1 \in$

[1] Since $\dim V' < \infty$ the tensor product is simply the algebraic one.

$\mathcal{D}(G;\mathcal{K}_\infty)$ and $f_2 \in \mathcal{D}(G;V')$,

$$
\begin{aligned}
\langle\, P(\pi_1(f_1)), \pi_2(f_2)\,\rangle \;&=\; \int_{G\times G} dg_2 dg_1 \prec h_1.p, a_1 \succ \langle\, Bf_1(g_2 g_1), h_1^{-1}.f_2(g_2)\,\rangle \\
&\quad (g_1 \mapsto g_2^{-1} g_1) \\
&= \int_{G\times G} da_2 dh_2 da_1 dh_1 \;\prec h_1.p, a_1 - a_2 \succ \\
&\qquad \langle\, Bf_1(a_1,h_1), h_1^{-1}h_2.f_2(a_2,h_2)\,\rangle \\
&\quad (\text{definition of } \pi_2) \\
&= \int_{A\times G} da_2 da_1 dh_1 \;\prec h_1.p, a_1 \succ \overline{\prec h_1.p, a_2 \succ} \\
&\qquad \langle\, Bf_1(a_1,h_1), h_1^{-1}.(\pi_2 f_2)(a_2)\,\rangle \\
&\quad (\text{Fourier transform on } a_2, a = a_1, h = h_1) \\
&= \int_{G} da\, dh \prec h.p, a \succ \langle\, Bf_1(a,h), h^{-1}.(\mathcal{F}\pi_2 f_2)(h.p)\,\rangle \\
&\quad (\text{Mackey} - \text{Bruhat formula}) \\
&= \int_{G/G_p} d\nu(q(0,h)) \int_{A\times' H_p} dbds \\
&\qquad \langle\, B(\prec p, b \succ f_1(hbs), s^{-1}h^{-1}.(\mathcal{F}\pi_2 f_2)(h.p)\,\rangle \\
&= \int_{G/G_p} d\nu(q(0,h)) \int_{A\times' H_p} dbds \\
&\qquad \langle\, B\,(\prec p, b \succ m_\infty(s))\, f_1(hbs), h^{-1}.(\mathcal{F}\pi_2 f_2)(h.p)\,\rangle \\
&\quad (\text{definition of } \pi_1) \\
&= \int_{G/G_p} \langle\, B(\pi_1 f_1)(0,h), h^{-1}.(\mathcal{F}\pi_2 f_2)(h.p)\,\rangle d\nu(q(0,h)).
\end{aligned}
$$

By surjectivity of the maps π_i one has that $P = T^B$.

Finally, let $P \in \mathrm{Hom}_G(\mathcal{K}^{p\otimes m};\mathcal{D}(A;V')')$, since ${}^M C^\infty(G;\mathcal{K}_\infty)$ is continuously and densely embedded in $\mathcal{K}^{p\otimes m}$, then P is uniquely defined by its restriction P^∞ to ${}^M C^\infty(G;\mathcal{K}_\infty)$ and $P^\infty \in \mathrm{Hom}_G({}^M C^\infty(G;\mathcal{K}_\infty);\mathcal{D}(A;V')')$. Then, by the previous results, there is a unique B such that $P^\infty = T^B$. □

The above lemma does not characterise completely $\mathrm{Hom}_G(\mathcal{K}^{p\otimes m};\mathcal{D}(A;V')')$, however, in the cases we are interested in, the following lemma can be applied, compare with [9]. We say that the measure ν, defined on $G.p$ is *tempered*, if its extension to $\hat{A}$ is a tempered Radon measure and that L is *at most of polynomial growth on* $G.p$ if there exists a section β from $G.p$ to H such that

$$\|L(\beta(p'))\|_{\mathcal{L}(V)} \le C(1+|p'|^k) \qquad p' \in G.p,$$

where $C > 0$ and $k \in \mathbb{N}$ are suitable constants and $|p'|$ is the euclidian norm of p' regarding p' as an element of $\mathbb{R}^{n+m}$.

Lemma 3. *Assume that $\mathcal{K}$ is finite dimensional, the measure ν is tempered and L is at most of polynomial growth on G_p. Let $B \in \mathrm{Hom}_{H_p}(\mathcal{K}; V)$, then the map T^B defined in Eq.* (11) *extends by continuity to $\mathcal{K}^{p\otimes m}$ and, if T^B denotes again the extension, $T^B \in \mathrm{Hom}_G(\mathcal{K}^{p\otimes m}; \mathcal{S}(A; V')')$.*

Proof. Since $\mathcal{K}$ is finite dimensional, $\mathcal{K}_\infty = \mathcal{K}$ and $M = p \otimes m$.

The hypotheses on ν and L assure that for all $\phi \in \mathcal{S}(A; V')$, the map

$$q(0, h) \mapsto \left\| B^t L(h)^t \mathcal{F}(\phi)(h.p) \right\|^2_{\mathcal{K}'}$$

is well defined (see Remark 7) and integrable with respect to ν. Then, if $f \in {}^{p\otimes m}C^\infty(G; \mathcal{K})$ and $\phi \in \mathcal{S}(A; V')$, using Schwarz inequality and taking into account that $L'(h^{-1}) = L(h)^t$,

$$|\langle\, T^B f, \phi \,\rangle| \leq \|f\|_{\mathcal{K}^{p\otimes m}} \left(\int_{G/G_p} \left\| B^t L(h)^t \mathcal{F}(\phi)(h.p) \right\|^2_{\mathcal{K}'} d\nu(q(0,h)) \right)^{\frac{1}{2}}. \tag{12}$$

Let $\phi \in \mathcal{S}(A; V')$ and $f \in \mathcal{K}^{p\otimes m}$. Choose a sequence $(f_n)_{n\in\mathbb{N}}$ in ${}^{p\otimes m}C^\infty(G; \mathcal{K})$ with limit f (in the norm of $\mathcal{K}^{p\otimes m}$) and define

$$\langle\, T^B f, \phi \,\rangle = \lim_{n\to\infty} \langle\, T^B f_n, \phi \,\rangle,$$

which is well defined by relation (12). Moreover, due to the fact that $\mathcal{S}(A; V')$ is a Montel space, $T^B f \in \mathcal{S}(A; V')'$, see, for example, corollary of Prop. 3.6.5 of [5], and $f \mapsto T^B f$ is continuous, see, for example, corollary of Prop. 3.9.2 of [5]. The intertwining property is evident, so $T^B \in \mathrm{Hom}_G(\mathcal{K}^{p\otimes m}; \mathcal{S}(A; V')')$. □

The following theorem characterises completely the L-invariant wave equations associated with s.

Theorem 2. *Let L be a finite dimensional representation of H acting on a Hilbert space V and $s \in \hat{G}$. Assume that for all $t \in \hat{G}$ the invariant measure on $G.p_t$ is tempered and L is at most of polynomial growth on $G.p_t$.*

With these hypotheses, let $(M_i)_1^N$ be a set of polynomial functions from $\hat{A}$ to $\mathcal{L}(V)$ satisfying either

$$h.M(h^{-1}.p).h^{-1} = M(p) \qquad \text{for all } h \in H \text{ and } p \in \hat{A} \tag{13}$$

or

$$h^*.M(h^{-1}.p).h^{-1} = M(p) \qquad \text{for all } h \in H \text{ and } p \in \hat{A}, \tag{14}$$

as well as the following two conditions

$$\begin{aligned} &\exists B \in \mathrm{Hom}_{H_{p_s}}(\mathcal{K}_s; V) \quad B \neq 0 \text{ unique up to a constant} \\ &\qquad\qquad \text{such that} \quad M_i(p_s)B = 0 \ \ \forall i = 1, \ldots, N, \end{aligned} \tag{15}$$

$$\begin{aligned} &\forall t \in \hat{G},\ t \neq s \quad \forall B \in \mathrm{Hom}_{H_{p_t}}(\mathcal{K}_t; V) \quad B \neq 0 \\ &\exists i = 1, \ldots, N \qquad \text{such that} \qquad M_i(p_t)B \neq 0. \end{aligned} \tag{16}$$

For each $i = 1 \dots N$ define $D_i : \mathcal{D}(A; V')' \to \mathcal{D}(A; V')'$ as

$$\langle\, D_i T, f \,\rangle = \langle\, T, \mathcal{F}^{-1} M_i^t \mathcal{F} f \,\rangle \qquad T \in \mathcal{D}(A; V')',\ f \in \mathcal{D}(A; V'), \tag{17}$$

then $(D_i)_1^N$ is an L-invariant wave equation associated with s and every L-invariant wave equation associated with s is of this form.

We need a technical lemma.

Lemma 4. *Let $t \in \hat{G}$ and assume that the invariant measure on $G.p_t$ is tempered and L is at most of polynomial growth on $G.p_t$, then the space $\mathrm{Hom}_G(\mathcal{H}_t; \mathcal{D}(A; V')')$ is non-zero if and only if L restricted to H_{p_t} contains m_t as a subrepresentation.*

Proof. Assume that $\dim \mathrm{Hom}_G(\mathcal{H}_t; \mathcal{D}(A; V')') \neq 0$. Then, by Lemma 2 $\dim \mathrm{Hom}_{H_{p_t}}(\mathcal{K}_{t_\infty}; V) \neq 0$. Hence, since m_{t_∞} is irreducible, L restricted to H_{p_t} contains m_{t_∞} as a subrepresentation. In particular, since V is finite dimensional, $\mathcal{K}_t$ is finite dimensional too and $m_{t_\infty} = m_t$. Conversely, if L restricted to H_{p_t} contains m_t as a subrepresentation, then $m_t = m_{t_\infty}$ and $\dim \mathrm{Hom}_{H_{p_t}}(\mathcal{K}_{t_\infty}; V) \neq 0$. We conclude applying Lemma 3. □

Proof of the theorem. Applying Prop. 1, one has a one to one correspondence between the set of wave operators $(D_i)_1^N$ and the set of functions $(M_i)_1^N$ satisfying conditions (7) and either (13) or (14). The conditions (18) and (19) are clearly equivalent to the two following facts:

$$\begin{aligned} &\exists P \in \mathrm{Hom}_G(\mathcal{H}_s; \mathcal{D}(A; V')') \quad P \neq 0 \text{ unique up to a constant} \\ &\qquad\qquad \text{such that} \qquad D_i P = 0 \ \ \forall i = 1, \dots, N \end{aligned} \tag{18}$$

$$\begin{aligned} &\forall t \in \hat{G},\ t \neq s \quad \forall P \in \mathrm{Hom}_G(\mathcal{H}_t; \mathcal{D}(A; V')') \quad P \neq 0 \\ &\exists i = 1, \dots, N \qquad\qquad \text{such that} \qquad\qquad D_i P \neq 0. \end{aligned} \tag{19}$$

Let now $t \in \hat{G}$. By Lemma 4 there is $P \in \mathrm{Hom}_G(\mathcal{H}_t; \mathcal{D}(A; V')')$, $P \neq 0$, if and only if m_t is a subrepresentation of L restricted to H_{p_t}. Hence we can assume, without loss of generality, that $\mathcal{K}_t$ is finite dimensional and, due to the hypotheses in the statement of the theorem, we can apply Lemma 3 to describe $\mathrm{Hom}_G(\mathcal{H}_t; \mathcal{D}(A; V')')$. In doing so, conditions (18) and (19) can be reformulated as

$$\begin{aligned} &\exists B \in \mathrm{Hom}_{H_{p_s}}(\mathcal{K}_s; V) \quad B \neq 0 \text{ unique up to a constant} \\ &\qquad\qquad \text{such that} \qquad D_i T^B = 0 \ \ \forall i = 1, \dots, N \end{aligned}$$

$$\begin{aligned} &\forall t \in \hat{G},\ t \neq s \quad \forall B \in \mathrm{Hom}_{H_{p_t}}(\mathcal{K}_t; V) \quad B \neq 0 \\ &\exists i = 1, \dots, N \qquad\qquad \text{such that} \qquad\qquad D_i T^B \neq 0. \end{aligned}$$

Using the explicit relation between the wave operators D_i and the functions M_i given by Eq. (17), we conclude that, for each $\psi \in {}^{p_t \otimes m_t}C^\infty(G; \mathcal{K}_t)$ and $f \in \mathcal{S}(\hat{A}; V')$, one has either

$$\langle\, D_i T^B \psi, \mathcal{F}^{-1} f \,\rangle = \int_{G/G_{p_t}} \langle\, M_i(p_t) B \psi(0, h), h^{-1}.f(h.p_t) \,\rangle d\nu(q(0, h)),$$

if D_i is invariant, or

$$\langle\, D_i T^B \psi, \mathcal{F}^{-1} f \,\rangle = \int_{G/G_{p_t}} \langle\, M_i(p_t) B \psi(0,h), (h^*)^{-1}.f(h.p_t) \,\rangle d\nu(q(0,h)),$$

if D_i is $*$-invariant. These relations show that $D_i T^B = T^{M_i(p_t)B}$, and, due to the injectivity of the map $B \mapsto T^B$, $D_i T^B = 0$ if and only if $M_i(p_t)B = 0$. The thesis is now evident. □

Remark 8. If one assumes that only the invariant measure on $G.p_s$ is tempered and L is at most of polynomial growth only on $G.p_s$, one can easily prove that the conditions on the functions $(M_i)_1^N$ given in Th. 2 are still sufficient (in general, not necessary) to assure that $(D_i)_1^N$, defined by Eq. (17), is a wave equation.

Remark 9. Let $\mathcal{H}$ be a subspace of $\mathcal{D}(A;V')'$ satisfying condition (a) of item 3 of Def. 1, U the corresponding unitary representation. In the hypotheses of Lemma 4 (with t denoting the equivalence class of U), one has that $\mathcal{H}$ is in fact a subspace of $\mathcal{S}(A;V')'$ and U is equivalent to a representation induced from a finite dimensional representation.

The above theorem reduces the study of the wave equations to a problem on the intertwining operators among finite-dimensional representations of the stability subgroups of H with respect to its action on $\hat{A}$. As a further consequence, one has that the wave operators defining the wave equations are finite order differential operators with constant coefficients. From the physical point of view, there is an interest to find representation L of H such that the wave operators are given by *first order* differential operators. We stress that it is always possible to reduce the order of any given differential equation by adding auxiliary degrees of freedom, nevertheless, in general, one obtains a wave equation that does not satisfy Eq. (5) or Eq. (6). This observation motivates

Definition 2. *An L-invariant wave equation $(D_i)_{i=1}^N$ such that each D_i is a first order differential operator is called a wave equation of Dirac type.*

4. The Galilei elementary particles

In this section we apply the results obtained in the previous section to the Galilei group in $3+1$ and $2+1$ dimensions. In both cases we assume that the structure of the Galilei group is known, we describe, without proofs, the structure of the universal central extension of the component of the Galilei group connected to the identity and we classify its unitary irreducible representations. Then, we consider the problem of finding Galilei invariant wave equations of Dirac type. Finally we introduce on these wave equations the interaction with an external electromagnetic field.

4.1. **The universal central extension: the $3+1$ case.** Since the Lie algebra of the Galilei group has a non-trivial cohomology, its universal central extension is not the universal covering group.

Let $SU(2)$ be the two-dimensional special unitary group, $SO(3)$ be the three-dimensional special orthogonal group and δ be the covering homomorphism from $SU(2)$ onto $SO(3)$. Let $\mathcal{V} := \mathbb{R}^3$ be the three-dimensional vector group, then $SU(2)$ acts on $\mathcal{V}$ in a natural way by means of δ. We denote by H the semidirect product $\mathcal{V} \times' SO(3)$. The elements of H are denoted by $(\vec{v}, h)$. The group H is a connected Lie group.

Let $A := \mathbb{R}^5$ be the five-dimensional vector group, its element are denoted by $(c, \vec{a}, b)$ with $c, b \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$. The group H acts on A as

$$(\vec{v}, h)[(c, \vec{a}, b)] = (c + \vec{v} \cdot \delta(h)\vec{a} + \tfrac{1}{2} b\vec{v}^2, \delta(h)\vec{a} + b\vec{v}, b).$$

We define G as the semidirect product $A \times' H$ with respect to the above action. The group G is a connected Lie group. We have the following result, see [2] for the proof.

Theorem 3. *The group G defined above is a central extension of $\mathbb{R} \times \mathbb{Z}_2$ by the component of the Galilei group connected to the identity and G is the universal central extension of this latter group.*

We identify G/H with A in the natural way and we denote its element as $(\xi, \vec{x}, t)$ with $\xi, t \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^3$. Moreover we identify the dual group $\hat{A}$ of A with $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ by the pairing

$$\prec (m, \vec{p}, E), (c, \vec{a}, b) \succ = mc - \vec{p} \cdot \vec{a} + Eb.$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^3$. With these identifications, the action of G on A and $\hat{A}$ defined in Sec. 3 (see Eq. (refinner), (3) and below) are explicitly given by

$$\begin{aligned} g.(\xi, \vec{x}, t) &= (\xi + \vec{v} \cdot \delta(h)\vec{x} + \frac{1}{2} t\vec{v}^2 + c, \delta(h)\vec{x} + t\vec{v} + \vec{a}, t + b) \\ g.(m, \vec{p}, E) &= (m, \delta(h)\vec{p} + m\vec{v}, E + \tfrac{1}{2} m\vec{v}^2 + \vec{v} \cdot \delta(h)\vec{p}), \end{aligned}$$

with $g = (c, \vec{a}, b, \vec{v}, h) \in G$. Moreover, with respect to the Fourier transform, one has the following correspondence between the multiplicative operators on $\mathcal{D}(\hat{A}; V')$ and the differential operators on $\mathcal{D}(A; V')$,

$$\begin{aligned} c &\leftrightarrow -i\frac{\partial}{\partial \xi} \\ \vec{p} &\leftrightarrow i\vec{\nabla} \\ E &\leftrightarrow -i\frac{\partial}{\partial t}. \end{aligned}$$

We now describe the unitary irreducible representations of G in terms of the orbits $G.p$ in $\hat{A}$ and of the irreducible unitary representations of the corresponding stability subgroups H_p. Taking into account Remark 9, we consider only finite dimensional representations of H_p. In doing so, one has three families of representations corresponding to three kinds of G-orbits in $\hat{A}$.

1. For each $\varepsilon \in \mathbb{R}, m \in \mathbb{R}, m \neq 0$, let $p_{\varepsilon,m} = (m, \vec{0}, \varepsilon)$

$$\begin{aligned} G.p_{\varepsilon,m} &= \{(m, \vec{p}, \varepsilon + \frac{\vec{p}^2}{2m}) \,:\, \vec{p} \in \mathbb{R}^3\} \\ G_{p_{\varepsilon,m}} &= A \times' SU(2) \end{aligned}$$

   The irreducible unitary representations of $SU(2)$ are all finite dimensional, the dual group of $SU(2)$ is parametrised by an index j such that $2j$ is a positive integer and, for each j, as usual, we denote by $\mathbb{D}^j$ a representative of the equivalence class. The orbits $G.p_{\varepsilon,m}$ are naturally isomorphic to $\mathbb{R}^3$ and, with this identification, the invariant measure is the Lebesgue measure. Hence, one obtains the family of irreducible unitary representations of G

$$U^{\varepsilon,m,j} = {}^u\mathrm{Ind}^G_{A\times' SU(2)}(p_{\varepsilon,m} \otimes \mathbb{D}^j),$$

   parametrised by $\varepsilon, m \in \mathbb{R}$, $m \neq 0$, and $j \in \frac{1}{2}\mathbb{N}$.
2. For each $r \in \mathbb{R}, r > 0$, let $p_{1,r} = (0, \vec{p}_r, 0)$ where $\vec{p}_r = (0, 0, r)$

$$\begin{aligned} G.p_{1,r} &= \{(0, \vec{p}, E) \,:\, \vec{p} \in \mathbb{R}^3, \vec{p}^2 = r^2, E \in \mathbb{R}\} \\ G_{p_{1,r}} &= A \times' E(2)^* \end{aligned}$$

   where $E(2)^* = \{(x, y, 0; \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}) \,:\, x, y \in \mathbb{R},\ z \in \mathbf{T}\}$. The finite dimensional irreducible unitary representations of $E(2)^*$ are the characters z^n of the torus lifted to $E(2)^*$ in a trivial way. The orbits $G.p_{1,r}$ are naturally isomorphic to $S^2 \times \mathbb{R}$ and, with this identification, the invariant measure is the product of the Haar measure on the sphere and the Lebesgue measure on $\mathbb{R}$. Now one obtains the family

$$U^{r,n} = {}^u\mathrm{Ind}^G_{A\times' E(2)^*}(p_{1,r} \otimes z^n),$$

   parametrised by $r \in \mathbb{R}$, $r > 0$, and $n \in \mathbb{Z}$.
3. For each $\varepsilon \in \mathbb{R}$, let $p_{2,\varepsilon} = (0, \vec{0}, \varepsilon)$

$$\begin{aligned} G.p_{2,\varepsilon} &= \{p_{2,\varepsilon}\} \\ G_{p_{2,\varepsilon}} &= G \end{aligned}$$

   The only finite dimensional irreducible unitary representation of H is the trivial one. The corresponding family of representations of G is explicitly given by

$$U^{\varepsilon}_{(c,\vec{a},b;\vec{v},h)} = e^{i\varepsilon b},$$

   with $\varepsilon \in \mathbb{R}$.

From the previous analysis, one deduces that the orbits are closed in $\hat{A}$ (hence the semidirect product is regular), each stability subgroup is unimodular and the invariant measures on the orbits are tempered. Hence, to study the Galilei invariant wave equations we can apply Th. 2, provided that the condition on the polynomial growth of the representation L is satisfied.

4.2. **A wave equation of Dirac type: the $3+1$ case.** The aim of the present section is to find a Dirac type wave equation for an elementary particle invariant with respect to the Galilei group.

The elementary Galilei invariant systems are associated with the irreducible unitary representations of G. Nevertheless, only the representations of the form $U^{0,m,j}$ with $m>0$ and $2j\in\mathbb{N}$ describe a physical system, namely an elementary particle with mass m and spin j.

Hence, fixed $m_0\in\mathbb{R}$, $m_0>0$ and $2j\in\mathbb{N}$, one has to select a finite-dimensional representation L of H such that there is an L-invariant wave equation of Dirac type associated with (m_0,j). In the appendix, following the results of George and Lévy-Nahas, [4], we briefly present a method to obtain finite-dimensional representations of H.

Remark 10. If we do not require that the wave equation is of Dirac type, the Schrödinger equation is the only wave equation with the property that the vector space carrying the representation L has minimal dimension. Indeed, let $\mathbb{D}^j$ be the irreducible representation of $SU(2)$ acting on $\mathbb{C}^{2j+1}$, extend it to a representation L^j of H in a trivial way, *i.e.* $L^j(\vec{v},h)=\mathbb{D}^j(h)$. Define M_1 and M_2 from $\hat{A}$ into the space of $(2j+1)\times(2j+1)$ matrices as

$$\begin{aligned} M_1(m,\vec{p},E) &= (m-m_0)I_{2j+1} \\ M_2(m,\vec{p},E) &= (2mE-\vec{p}^2)I_{2j+1}. \end{aligned}$$

One can easily check that L^j is at most of polynomial growth (since it is unitary) and that M_1 and M_2 satisfy all the conditions of Th. 2. Using relation (17), one obtains the explicit form of the corresponding wave equation (D_1,D_2):

$$\begin{aligned} D_1 &= (i\frac{\partial}{\partial\xi}-m_0)I_{2j+1} \\ D_2 &= \left(-2\frac{\partial^2}{\partial\xi\partial t}+\vec{\nabla}^2\right)I_{2j+1}. \end{aligned}$$

To show that these equations are the usual Schrödinger equation, consider $\phi\in C^\infty(A,\mathbb{C}^{2j+1})$ such that $\phi\in\mathcal{D}(A;\mathbb{C}^{2j+1})'$ and $D_1\phi=D_2\phi=0$ in the sense of distribution and, hence, in the classical one (see Remark 6). Then one easily checks that

$$\phi(\xi,\vec{x},t)=e^{-im_0\xi}\psi(\vec{x},t)$$

where ψ is in $C^\infty(\mathbb{R}^4,\mathbb{C}^{2j+1})$ and satisfies the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2m_0}\vec{\nabla}^2\psi.$$

One has to consider two differential operators in order to assure the correct invariant property of the wave equation (it is well known that the

Schrödinger equation is not Galilei invariant). Moreover it is clear that the dimension of $\mathbb{C}^{2j+1}$ is minimal.

We come back to the problem of finding a Dirac type wave equation. We consider first the case $j \geq \frac{1}{2}$. Let L^j be the representation of H, acting on

$$V^j = \mathbb{C}^{2j-1} \oplus \mathbb{C}^{2j+1} \oplus \mathbb{C}^{2j+1},$$

given by

$$L^j(\vec{v}, h) = \begin{pmatrix} \mathbb{D}^{j-1}(h) & 0 & \vec{v} \cdot \vec{T}^{(j-1)j}\mathbb{D}^j(h) \\ 0 & \mathbb{D}^j(h) & \vec{v} \cdot \vec{H}^j\mathbb{D}^j(h) \\ 0 & 0 & \mathbb{D}^j(h) \end{pmatrix} \tag{20}$$

with $\vec{v} \in \mathcal{V}$, $h \in SU(2)$ and $T^{(j-1)j}$ and H^j introduced in the appendix.

With the notations of Remark 6, let D_1 and D_2 be the differential operators

$$\begin{aligned} D_1 &= (i\frac{\partial}{\partial \xi} - m_0)I_{6j+1} \\ D_2 &= i\begin{pmatrix} I_{2j-1}\frac{\partial}{\partial \xi} & 0 & \vec{T}^{(j-1)j} \cdot \vec{\nabla} \\ 0 & I_{2j+1}\frac{\partial}{\partial \xi} & \vec{H}^j \cdot \vec{\nabla} \\ (\vec{T}^{(j-1)j})^* \cdot \vec{\nabla} & \vec{H}^j \cdot \vec{\nabla} & 2j^2 I_{2j+1}\frac{\partial}{\partial t}. \end{pmatrix} \end{aligned}$$

Proposition 2. *Let $m_0 > 0$ and $2j \in \mathbb{N}$, $j \geq \frac{1}{2}$, then the set (D_1, D_2) is an L^j-invariant wave equation of Dirac type associated with the Galilean elementary particle of mass m_0 and spin j.*

Proof. Denote by $s \in \hat{G}$ the equivalence class of ${}^u\mathrm{Ind}_{A \times' SU(2)}^G(p_{0,m_0} \otimes \mathbb{D}^j)$, so that $p_s = p_{0,m_0}$. Moreover, notice that L^j is at most of polynomial growth on the orbit $G.p_{0,m_0}$. Indeed, using the section

$$G.p_{0,m_0} \ni (m, \vec{p}, E) \mapsto (\frac{\vec{p}}{m_0}, e) \in H$$

one has that $\left\| L^j(\frac{p}{m_0}, e) \right\| \leq (1 + C|\vec{p}|^2)$, where C is a suitable constant.

Now we define two functions M_1 and M_2 from $\hat{A}$ to $\mathcal{L}(V)$ as

$$\begin{aligned} M_1(m, \vec{p}, E) &= (m - m_0)I_{6j+1} \\ M_2(m, \vec{p}, E) &= \begin{pmatrix} mI_{2j-1} & 0 & -\vec{p} \cdot \vec{T}^{(j-1)j} \\ 0 & mI_{2j+1} & -\vec{p} \cdot \vec{H}^j \\ -\vec{p} \cdot (\vec{T}^{(j-1)j})^* & -\vec{p} \cdot \vec{H}^j & 2j^2 EI_{2j+1}. \end{pmatrix} \end{aligned}$$

One can easily check that, for all $T \in \mathcal{D}(A; (V^j)')'$ and $f \in \mathcal{D}(A; (V^j)')$

$$\langle\, D_i T, f \,\rangle = \langle\, T, \mathcal{F}^{-1} M_i^t \mathcal{F} f \,\rangle, \qquad i = 1, 2.$$

Hence to show the thesis we can use Th. 2 with Remark 8.

Obviously the maps M_1 and M_2 are polynomials with maximal degree 1 and M_1 satisfies Eq. (13). We now show that M_2 satisfies Eq. (14). Indeed, it is enough to verify it on the set

$$\Omega = \{(m, \vec{p}, E) \in \hat{A} \ : \ m \neq 0\},$$

which is dense in $\hat{A}$. By means of an explicit computation, one has that for all $(m, \vec{p}, E) \in \Omega$,

$$(m, \vec{p}, E) = (\frac{\vec{p}}{m}, e).(m, \vec{0}, E - \frac{\vec{p}^2}{2m}).$$

Moreover, define for all $a, b \in \mathbb{R}$ the matrix

$$\Delta(a,b) := \left(\begin{array}{ccc} aI_{2j-1} & 0 & 0 \\ 0 & aI_{2j+1} & 0 \\ 0 & 0 & 2j^2 bI_{2j+1} \end{array} \right),$$

then $\Delta(a,b)$ is in the commuting ring of the restriction of L^j to $SU(2)$ (observe that if $h \in SU(2)$, then $(L^j)^*(h) = L^j(h)$). Finally, for all $(m, \vec{p}, E) \in \Omega$

$$M_2(m, \vec{p}, E) = (L^j)^*((\frac{\vec{p}}{m}, e))\Delta(m, E - \frac{\vec{p}^2}{2m})(L^j)((-\frac{\vec{p}}{m}, e))$$

where we use Eq. (32) to linearise the term proportional to $\frac{\vec{p}^2}{2m}$.

We now show that conditions (15) and (16) are satisfied. Consider first condition (15). Let $B \in \mathrm{Hom}_{SU(2)}(\mathbb{C}^{2j+1}; V^j)$, then B is of the form

$$B(w) = (0, \beta w, \alpha w) \in \mathbb{C}^{2j-1} \oplus \mathbb{C}^{2j+1} \oplus \mathbb{C}^{2j+1},$$

where $\alpha, \beta \in \mathbb{C}$. Since $M_1(p_s) = 0$, obviously $M_1(p_s)B = 0$, whereas, by an explicit computation, the condition $M_2(p_s)B = 0$ is equivalent to $\beta = 0$. Hence, there is a unique $B \neq 0$, up to a constant, satisfying condition (15).

Consider now condition (16). Let $t \in \hat{G}$, $t \neq s$, then $p_t = (m_t, \vec{p_t}, E_t)$.

1. if $m_t \neq m_0$, then $M_1(p_t) = (m_t - m_0)I_{6j+1}$ is clearly invertible, so that if $B \in \mathrm{Hom}_{H_{p_t}}(\mathcal{K}_t; V^j)$, $B \neq 0$, then $M_1(p_t)B \neq 0$
2. if $m_t = m_0 \neq 0$, then $p_t = (m_0, \vec{0}, \varepsilon_t)$, hence
   (a) if $\varepsilon_t \neq 0$,

   $$M_2(p_t) = \left(\begin{array}{ccc} m_0 I_{2j-1} & 0 & 0 \\ 0 & m_0 I_{2j+1} & 0 \\ 0 & 0 & 2j^2 \varepsilon_t I_{2j+1}, \end{array} \right)$$

   which is invertible, and, as above, we can conclude that relation (16) holds with $i = 2$.

   (b) if $\varepsilon_t = 0$, the representation U_t is of the form

   $${}^u\mathrm{Ind}^G_{A \times' SU(2)}(p_{0,m_0} \otimes \mathbb{D}^{j_t}),$$

   so that condition $t \neq s$ implies that $j_t \neq j$. Moreover, condition $\mathrm{Hom}_{H_{SU(2)}}(\mathcal{K}_t; V^j) \neq \{0\}$, implies that $j_t = j - 1$ and that the range of B is contained in the subspace $\mathbb{C}^{2j-1} \oplus \{0\} \oplus \{0\}$. Hence $M_2(p_t)B \neq 0$ if $B \neq 0$.

□

Remark 11. In the case $j = \frac{1}{2}$, the vector space $\mathbb{C}^{2j-1}$ reduces to the zero vector and the matrices $\vec{T}^{(j-1)j}$ are the zero matrices, so that the wave operator D_2 is given by

$$D_2 = i \begin{pmatrix} I_2 \frac{\partial}{\partial \xi} & \vec{H}^{\frac{1}{2}} \cdot \vec{\nabla} \\ \vec{H}^{\frac{1}{2}} \cdot \vec{\nabla} & \frac{1}{2} I_2 \frac{\partial}{\partial t} \end{pmatrix}$$

and it was found for the first time by Lévy-Leblond, [7]. The main difference with the case $j \geq 1$ is that D_2 is both *-invariant and invariant. From the mathematical point of view this phenomenon is consequence of the fact that the matrices $\vec{H}^{\frac{1}{2}}$, which are proportional to the Pauli matrices, satisfy also the anti-commutation relations

$$H_i^{\frac{1}{2}} H_j^{\frac{1}{2}} + H_j^{\frac{1}{2}} H_i^{\frac{1}{2}} = \frac{1}{2} \delta_{ij}.$$

Remark 12. The case $j = 0$ requires small modifications. Let L^0 be the representation of H acting on $\mathbb{C}^3 \oplus \mathbb{C}$ given by

$$L^0(v, h) = \begin{pmatrix} \delta(h) & \vec{v} \\ 0 & 1 \end{pmatrix}.$$

where $v \in \mathcal{V}$ and $h \in SU(2)$. Let D_1 and D_2 be the differential operators

$$\begin{aligned} D_1 &= (i \frac{\partial}{\partial \xi} - m_0) I_4 \\ D_2 &= i \begin{pmatrix} I_3 \frac{\partial}{\partial \xi} & \vec{\nabla} \\ \vec{\nabla} & 2 \frac{\partial}{\partial t} \end{pmatrix}. \end{aligned}$$

The proof that (D_1, D_2) is a Dirac type wave equation for the spinless particle of mass m_0 is the same of Prop. 2. Observe that in this case D_2 is a *-invariant wave operator and the dimension of the vector space carrying the representation is 4.

4.3. **The gyromagnetic ratio: the $3+1$ case.** In the previous section we proved that with each elementary particle of mass m_0 and spin j is associated an L-invariant wave equation (D_1, D_2) of Dirac type.

We now introduce the interaction of the particle with a given external electromagnetic field $(\vec{E}, \vec{B})$. We stress that $(\vec{E}, \vec{B})$ are not dynamical variables and, obviously, can destroy the Galilei invariance. The interaction will be introduced on the *free* wave equation by means of the minimal coupling principle. Taking into account that the variable ξ has not physical meaning, the prescription of the minimal coupling is

$$\begin{aligned} i \frac{\partial}{\partial \xi} &\mapsto i\mathcal{D}_0 := i \frac{\partial}{\partial \xi} \\ i \frac{\partial}{\partial t} &\mapsto i\mathcal{D}_4 := i \frac{\partial}{\partial t} - qV(\vec{x}, t) \\ -i\vec{\nabla} &\mapsto -i\vec{\mathcal{D}} := -i\vec{\nabla} - q\vec{A}(\vec{x}, t), \end{aligned}$$

where V and $\vec{A}$ are potentials defined by $(\vec{E},\vec{B})$ and q is the electric charge. In doing so, the wave equation becomes, if $j>0$,

$$\begin{aligned} D_1^{int} &= (i\mathcal{D}_0 - m_0)I_{6j+1} \\ D_2^{int} &= \begin{pmatrix} iI_{2j-1}\mathcal{D}_0 & 0 & i\vec{T}^{(j-1)j}\cdot\vec{\mathcal{D}} \\ 0 & iI_{2j+1}\mathcal{D}_0 & i\vec{H}^j\cdot\vec{\mathcal{D}} \\ (\vec{T}^{(j-1)j})^*\cdot\vec{\mathcal{D}} & i\vec{H}^j\cdot\vec{\mathcal{D}} & 2j^2 iI_{2j+1}\mathcal{D}_4. \end{pmatrix} \end{aligned}$$

To explain the meaning of the above wave equation, let us suppose that ψ is a smooth function from A to $\mathbb{C}^{6j+1}$ defining an element in $\mathcal{D}(A;\mathbb{C}^{6j+1})'$, see Remark 6, and satisfying

$$D_1^{int}\psi = 0 \qquad D_2^{int}\psi = 0.$$

Writing $\psi = (\omega,\chi,\phi)$ with ϕ and χ taking values in $\mathbb{C}^{2j+1}$ and ω in $\mathbb{C}^{2j-1}$, one obtains

$$\begin{aligned} i\mathcal{D}_0\psi &= m_0\psi \\ i\mathcal{D}_0\omega + i\vec{T}^{(j-1)j}\cdot\vec{\mathcal{D}}\phi &= 0 \\ i\mathcal{D}_0\chi + i\vec{H}^j\cdot\vec{\mathcal{D}}\phi &= 0 \\ i(\vec{T}^{(j-1)j})^*\cdot\vec{\mathcal{D}}\omega + i\vec{H}^j\cdot\vec{\mathcal{D}}\chi + 2j^2 i\mathcal{D}_4\phi &= 0. \end{aligned}$$

Solving these four equations in terms of ϕ, using Eq. (32) to compute the expression $H_aH_b + (T_a^{(j-1)j})^* T_b^{(j-1)j}$ and the commutation relations $[\mathcal{D}_a,\mathcal{D}_b] = -iq\varepsilon_{abc}B_c$ $(a,b,c=1,\dots,3)$, one has

$$\left(i\frac{\partial}{\partial t} - qV - \frac{1}{2m_0}\left(i\vec{\nabla} + q\vec{A}\right)^2 + \frac{q}{2jm_0}\vec{H}^j\cdot\vec{B}\right)\phi = 0. \tag{21}$$

The physical interpretation of this last equation is the fact that a quantum particle with spin j interacting with an electromagnetic field acquires an intrinsic magnetic momentum with gyromagnetic ratio $g=\frac{1}{j}$, as deduced by Hurley, [6]. Although the form of the interaction between the particle and the electromagnetic field has its root in the relativistic framework, Eq. (21) follows from Galilean covariance requirements, as soon as one assumes the minimal coupling principle and the fact that the wave equation is of Dirac type.

Remark 13. The case $j=0$ can be treated with similar calculation. The result is that the wave equation for the scalar component is the usual wave equation for a spinless particle in the electromagnetic field.

4.4. **The universal central extension: the $2+1$ case.** The Galilei group in $2+1$ dimensions has acquired some interest a few years ago, when it has become possible to produce some genuine examples of planar systems in solid state physics. The mathematical structure of this group is different from the one of the corresponding group in $3+1$ dimensions, so some new phenomena take place.

Let A be the vector group $A = \mathbb{R}^4$ and H the covering group of the *diamond group*, *i.e.* the Lie group $H = \mathcal{V} \times \mathbb{R} \times \mathbb{R}$ with $\mathcal{V} = \mathbb{R}^2$ and product law

$$(\vec{v}, r, x)(\vec{v}', r', x') = (\vec{v} + \delta(r)\vec{v}', r + r', x + x' + \vec{v} \wedge \delta(r)\vec{v}').$$

Here $(\vec{v}, r, x), (\vec{v}', r', x') \in H$, $\vec{v} \wedge \vec{v}'$ is the determinant of $\vec{v}$ and $\vec{v}'$, and δ is the universal covering homomorphism of $\mathbb{R}$ onto $SO(2)$, the two-dimensional special orthogonal group.

As proved in [2], the universal central extension G of the connected component of the identity of the Galilei group in $2+1$ dimensions is the semidirect product of $A \times' H$ with respect to the action of H on A given by

$$(\vec{v}, r, x)[(c, \vec{a}, b)] = (c + \vec{v} \cdot \delta(r)\vec{a} + \frac{b\vec{v}^2}{2}, \delta(r)\vec{a} + b\vec{v}, b),$$

where the elements of A are denoted by $(c, \vec{a}, b)$, with $c, b \in \mathbb{R}$, $\vec{a} \in \mathbb{R}^2$, and the ones of H by $(\vec{v}, r, x)$, with $x, r \in \mathbb{R}$ and $\vec{v} \in \mathcal{V}$. As usual, we denote the elements of G as $(c, \vec{a}, b, \vec{v}, r, x)$. From a physical point of view, $\vec{a}$ are the spatial translations, b the time translations, $\vec{v}$ the Galilei boosts, $\delta(r)$ the rotations in the plane and c, x are the two parameters associated with the central extension.

As usual, we identify G/H with A (its element are denoted by $(\xi, \vec{x}, t)$ with $\xi, t \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$), and the dual group $\hat{A}$ of A with $\mathbb{R}^4$ using the pairing

$$\prec (m, \vec{p}, E), (c, \vec{a}, b) \succ = mc - \vec{p} \cdot \vec{a} + Eb,$$

where $\cdot$ is the scalar product in $\mathbb{R}^2$. The actions of G on A and $\hat{A}$ are respectively given by

$$\begin{aligned} g.(\xi, \vec{x}, t) &= (\xi + \vec{v} \cdot \delta(r)\vec{x} + \frac{t\vec{v}^2}{2} + c, \delta(r)\vec{x} + t\vec{v} + \vec{a}, t + b) \\ g.(m, \vec{p}, E) &= (m, \delta(r)\vec{p} + m\vec{v}, E + \delta(r)\vec{p} \cdot \vec{v} + \frac{1}{2}m\vec{v}^2), \end{aligned}$$

where $g = (c, \vec{a}, b, \vec{v}, r, x) \in G$.

We now describe the unitary irreducible representations of G in terms of the orbits $G.p$ in $\hat{A}$ and of the irreducible unitary representations of the corresponding stability subgroups H_p. Taking into account Remark 9, we consider only finite dimensional representations of H_p. In doing so, one has three families of representations corresponding to three kinds of G-orbits in $\hat{A}$.

1. For each $\varepsilon \in \mathbb{R}, m \in \mathbb{R}, m \neq 0$, let $p_{\varepsilon,m} = (m, \vec{0}, \varepsilon)$

$$\begin{aligned} G.p_{\varepsilon,m} &= \{(m, \vec{p}, \varepsilon + \frac{\vec{p}^2}{2m}) \ : \ \vec{p} \in \mathbb{R}^2\} \\ G_{p_{\varepsilon,m}} &= A \times' \{(\vec{0}, r, x) \in H \ : \ r, x \in \mathbb{R}\}. \end{aligned}$$

The irreducible unitary representations of $G_{p_{\varepsilon,m}} \cap H = \mathbb{R}^2$ are its characters $\chi^{\mu,\lambda}$ with $\mu, \lambda \in \mathbb{R}$. The orbits $G.p_{\varepsilon,m}$ are naturally isomorphic to $\mathbb{R}^2$ and, with this identification, the invariant measure is the

Lebesgue measure. Hence, one obtains the family of irreducible unitary representations of G

$$U^{\varepsilon,m,\mu,\lambda} = {}^u\mathrm{Ind}_{G_{p_{\varepsilon,m}}}^G (p_{\varepsilon,m} \otimes \chi^{\mu,\lambda}),$$

parametrised by $\varepsilon, m, \mu, \lambda \in \mathbb{R}$, $m \neq 0$.

2. For each $r \in \mathbb{R}, r > 0$, let $p_{1,r} = (0, \vec{p}_r, 0)$ where $\vec{p}_r = (0, r)$

$$\begin{aligned} G.p_{1,r} &= \{(0, \vec{p}, E) \,:\, \vec{p} \in \mathbb{R}^2, \vec{p}^2 = r^2, E \in \mathbb{R}\} \\ G_{p_{1,r}} &= A \times' \{((v, 0), 2\pi n, x) \,:\, v, x \in \mathbb{R},\ n \in \mathbb{Z}\}. \end{aligned}$$

The irreducible unitary representations of $G_{p_{1,r}} \cap H = \mathbb{R} \times \mathbb{Z} \times \mathbb{R}$ are its characters, $\chi^{\nu,z,\lambda}$ with $\nu, \lambda \in \mathbb{R}$ and $z \in \mathbf{T}$. The orbits $G.p_{1,r}$ are naturally isomorphic to $\mathbf{T} \times \mathbb{R}$ and, with this identification, the invariant measure is the product of the Haar measure on the torus and the Lebesgue measure on $\mathbb{R}$. Now one obtains the family

$$U^{r,\nu,z,\lambda} = {}^u\mathrm{Ind}_{G_{p_{1,r}}}^G (p_{1,r} \otimes \chi^{\nu,z,\lambda}),$$

parametrised by $r, \nu, \lambda \in \mathbb{R}$, $r > 0$, and $z \in \mathbf{T}$.

3. For each $\varepsilon \in \mathbb{R}$, let $p_{2,\varepsilon} = (0, \vec{0}, \varepsilon)$

$$\begin{aligned} G.p_{2,\varepsilon} &= \{p_{2,\varepsilon}\} \\ G_{p_{2,\varepsilon}} &= G \end{aligned}$$

The only finite dimensional irreducible unitary representations of H are the characters of H, *i.e*

$$\chi^\mu : H \ni (\vec{v}, r, x) \mapsto e^{i\mu r} \in \mathbf{T}. \tag{22}$$

The corresponding family of representations of G is explicitly given by

$$U^{\varepsilon,\mu}_{(c,\vec{a},b,\vec{v},r,x)} = e^{i(\varepsilon t + \mu r)},$$

with $\varepsilon, \mu \in \mathbb{R}$.

From the previous analysis, one deduces that the orbits are closed in $\hat{A}$ (hence the semidirect product is regular), each stability subgroup is unimodular and the invariant measures on the orbits are tempered, so that we can apply the results of Sec. 3.

4.5. **A *minimal* wave equation of Dirac type: the $2 + 1$ case.** The elementary particles covariant with respect the $2+1$-Galilei group are associated with irreducible unitary representations of G. From a physical point of view the representations of the form $U^{r,\nu,z,\lambda}$ and $U^{\varepsilon,\mu}$ have a trivial time evolutor, so that in the following we do not consider them.

Moreover, there are no wave equations associated with the equivalence class of $U^{\varepsilon,m,\mu,\lambda}$ if $\lambda \neq 0$. Indeed, from Lemma 4 it follows that the representation L of H, restricted to $\vec{0} \times \mathbb{R}^2$, has to contain $\chi^{\mu,\lambda}$ as a subrepresentation. Due to the following Lemma, this condition implies that $\lambda = 0$.

Lemma 5. *Let L be a finite dimensional representation of H, then for all $x \in \mathbb{R}$, $L(\vec{0}, 0, x)$ is a unipotent operator.*

Proof. Let $\mathrm{Lie}(H) = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}$ be the Lie algebra of H and (e_1, e_2, e_3, e_4) the canonical basis. Define $\widehat{L}$ as the differential of L, which is a Lie algebra representation of $\mathrm{Lie}(H)$. Let l be an eigenvalue of $\widehat{L}(e_4)$ and W the corresponding eigenspace. Since $[e_1, e_4] = [e_2, e_4] = 0$, it follows that $\widehat{L}(e_1)$ and $\widehat{L}(e_2)$ leave invariant W, but $[e_1, e_2] = e_4$, so that $\widehat{L}(e_4)$ restricted to W has null trace and, hence, $l = 0$. □

Hence, we are looking for wave equations of Dirac type associated with the equivalence class of $U^{\varepsilon,m,\mu,0}$.

Fix $\varepsilon_0, m_0, \mu \in \mathbb{R}$, $m_0 \neq 0$. To obtain concrete wave equations associated with $U^{\varepsilon_0,m_0,\mu,0}$, a finite dimensional representation L of the group H has to be fixed. Since H is not semisimple, there is not a complete classification of its finite dimensional representations.

We start from the one-dimensional representations of H increasing the dimension of the vector space V carrying L until we find a wave equation of Dirac type, which will be *minimal* with respect to the dimension of V.

If $V = \mathbb{C}$, the finite dimensional representations of H are its characters χ^μ given by Eq. (22). Let $(D_i)_1^N$ be a χ^μ-invariant wave equation of Dirac type and M_i the corresponding polynomials given by Eq. (17).

Then, conditions (13) and (14) are equivalent to the fact that M_i are invariant polynomials of first degree. This implies that

$$M_i(m, \vec{p}, E) = d_1 + d_2 m \qquad d_1, d_2 \in \mathbb{C}.$$

However, conditions (15) and (16) of Th. 2 cannot be satisfied so that there are no χ^μ-invariant wave equations of Dirac type (using these representations one obtains the Schrödinger equation in $2+1$ dimensions).

Let now assume that $V = \mathbb{C}^2$, then the finite dimensional representations of H are of two kinds. First, there are the two dimensional representations of $\mathbb{R} = \{(\vec{0}, r, 0)\}$ extended trivially to H. As above, this implies that M_i are invariant polynomials of first degree and there are not wave equations of Dirac type.

In addition, one has the representations of the form

$$L_{\pm}^{\gamma}(\vec{v}, r, x) = e^{i\gamma r} \begin{pmatrix} e^{\pm ir} & (v_x \pm i v_y) \\ 0 & 1 \end{pmatrix},$$

parametrised by $\gamma \in \mathbb{C}$.

As an easy consequence of Th. 2, the following result is obtained.

Corollary 1. *Let $\varepsilon_0, m_0, \mu \in \mathbb{R}$, $m_0 \neq 0$, and $\gamma \in \mathbb{C}$. There are $L_{\pm}^{\gamma}$-invariant wave equations of Dirac type associated with the equivalence class of $U^{\varepsilon_0,m_0,\mu,0}$ if and only if $\gamma = \mu$.*

The only[2] $L^{\mu}_{\pm}$*-wave equations of Dirac type is* (D_1, D_2) *where*

$$\begin{aligned} D_1 &= (i\frac{\partial}{\partial \xi} - m_0)I_2 \\ D_2 &= i\begin{pmatrix} \frac{\partial}{\partial \xi} & (\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}) \\ (\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial y}) & 2(\frac{\partial}{\partial t} - \varepsilon_0) \end{pmatrix}. \end{aligned}$$

Proof. Let $(D_i)_i^N$ be an $L^{\gamma}_{\pm}$-invariant wave equation of Dirac type and M_i the corresponding polynomials of first degree given by Eq. (17). We apply Th. 2.

The fact that $\mathrm{Hom}_{H_{(m_0,\vec{0},\varepsilon_0)}}(\mathbb{C};\mathbb{C}^2) \neq 0$ (see condition (15)) implies that either $\gamma = \mu$ or $\gamma = \mu \mp 1$. Moreover, by direct computation, the polynomials of first degree satisfying Eq. (13) are of the form

$$P(m, \vec{p}, E) = \begin{pmatrix} d_1 + d_2 m & (d_3 - d_2)(p_x \pm i p_y) \\ 0 & (d_1 + d_3 m) \end{pmatrix}$$

while the ones satisfying Eq. (14) are of the form

$$Q(m, \vec{p}, E) = \begin{pmatrix} d_4 m & -d_4(p_x \pm i p_y) \\ -d_4(p_x \mp i p_y) & 2d_4 E + d_5 + d_6 m, \end{pmatrix}$$

where $d_1, \dots, d_6 \in \mathbb{C}$.

To satisfy conditions (15) and (16), it follows that $N \geq 2$, $M_1 = P$ and $M_2 = Q$. We will show that $N = 2$ is enough. Indeed, condition (15) with $p_s = (m_0, \vec{0}, \varepsilon_0)$ implies that, if $\gamma = \mu$,

$$(d_1 + d_2 m_0)^2 + (d_4 m_0)^2 \neq 0 \tag{23}$$

$$(d_1 + d_3 m_0)^2 + (2d_4\varepsilon_0 + d_5 + d_6 m_0)^2 = 0 \tag{24}$$

whereas, if $\gamma = \mu \mp 1$,

$$(d_1 + d_2 m_0)^2 + (d_4 m_0)^2 = 0 \tag{25}$$

$$(d_1 + d_3 m_0)^2 + (2d_4\varepsilon_0 + d_5 + d_6 m_0)^2 \neq 0, \tag{26}$$

while condition (16) where $p_t = (m, \vec{0}, \varepsilon)$ with $(m, \varepsilon) \neq (m_0, \varepsilon_0)$ and $m \neq 0$,

$$(d_1 + d_2 m)^2 + (d_4 m)^2 \neq 0 \tag{27}$$

$$(d_1 + d_3 m)^2 + (2d_4\varepsilon + d_5 + d_6 m)^2 \neq 0. \tag{28}$$

Since Eq. (27) (with $m = m_0$ and $\varepsilon \neq \varepsilon_0$) is not compatible with Eq. (25), it follows that $\gamma = \mu$, and comparing Eq. (28) with Eq. (24), one has that $d_4 \neq 0$. From Eq. (24), $d_1 = -d_3 m_0$ and $d_5 = -2d_4\varepsilon_0 - d_6 m_0$.

Moreover, let $p_t = (0, (0, r), 0)$ with $r > 0$, condition (16) is satisfied by M_2, while, letting $p_t = (0, \vec{0}, \varepsilon)$, one has $d_3 \neq 0$.

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[2]In the sense that the other wave equations give rise to the same space of solutions.

Summarising, with the change $d_3 \to d_1$, $d_3 - d_2 \to d_2$, $d_4 \to d_3$ and $d_6 \to d_4$, one has

$$\begin{aligned} M_1(m, \vec{p}, E) &= \begin{pmatrix} d_1(m - m_0) - d_2 m & d_2(p_x \pm i p_y) \\ 0 & d_1(m - m_0) \end{pmatrix} \\ M_2(m, \vec{p}, E) &= \begin{pmatrix} d_3 m & -d_3(p_x \pm i p_y) \\ -d_3(p_x \mp i p_y) & 2d_3(E - \varepsilon_0) + d_4(m - m_0) \end{pmatrix}, \end{aligned}$$

with $d_1, d_2, d_3, d_4 \in \mathbb{C}$ and

$$d_1, d_3 \neq 0. \tag{29}$$

Finally, one can easily check that the space of the solutions of the wave equation does not depend on the parameters d_i (provided, of course, that (29) is satisfied!), so one can fix $d_1 = d_3 = 1$ and $d_2 = d_4 = 0$ and, in doing so, the thesis is obtained. □

Remark 14. If the interaction with a given external electromagnetic field $(\vec{E}, B_z)$ is introduced by means of the minimal coupling principle, with the same computations of Sec. 4.3, one obtains the equation

$$\left(i\frac{\partial}{\partial t} - qV - \varepsilon_0 - \frac{1}{2m_0}\left(i\vec{\nabla} + q\vec{A} \right)^2 \mp \frac{q}{2m_0} B_z \right) \phi = 0, \tag{30}$$

where ϕ is a smooth function from A to $\mathbb{C}$ defining an element in $\mathcal{D}(A; \mathbb{C})'$.

From a physical point of view, comparing Eq. (30) with the corresponding three-dimensional equation (21), we see that the two-dimensional elementary particles of mass m_0 and *spin* μ (recall that $\lambda = 0$) interact with the E.M. field as three-dimensional 1/2-spin particles with spin down (L_+-invariant wave equations) or spin up (L_--invariant wave equations). Obviously, this result depends on the minimal assumption on the dimension of the vector space carrying the representation L and it will have to be checked experimentally.

Moreover, observing that the *spin* operator is μI, where I is the identity operator, one has that the gyromagnetic ratio is $\frac{1}{\mu}$ (if $\mu \neq 0$). In our opinion, this result has not a clear physical meaning since for a free particle the *spin* μ is a constant and can not be measurable (as the parameter ε both in $3+1$ and $2+1$ dimensions). Only experiments on true $2+1$-dimensional systems can clarify this point.

Remark 15. Since the stability subgroup of H at $(m_0, \vec{0}, \varepsilon_0)$ is not compact, as it occurs in the relativistic case for the massless particles, there is the possibility to obtain non-trivial wave equations considering non-unitary representations (as the ones connected with the Gupta-Bleuler and the Landau gauge). These wave equations cannot be studied with the theory developed in Sec. 3, nevertheless we hope to obtain some results in this direction in the future.

4.6. **The Clifford algebra for the $2+1$-Galilei group.** In this final section we show that it is possible to obtain a wave equation of Dirac type for the 2+1-dimensional Galilean particles without knowing *a priori* the explicit form of the representation of H. Such a representation will be obtained by trying to linearise the Schrödinger equation in $2+1$ dimensions, as Dirac did in the relativistic framework. Quite surprisingly, this will provide a nice application of the Clifford algebra in $3+1$ relativistic space time.

The Schrödinger equation is

$$\begin{aligned} D_1 &= (i\frac{\partial}{\partial\xi} - m_0) \\ D_2 &= \left(-2\frac{\partial^2}{\partial\xi\partial t} + \vec{\nabla}^2\right). \end{aligned}$$

The wave operator D_1 is of first order, so only D_2 has to be linearised.

The action of H on $\hat{A}$ defines a representation ρ of H on the vector space underlying $\hat{A}$. Let $< \cdot, \cdot >$ be the symmetric non-degenerate bilinear form on $\hat{A}$

$$< (m, \vec{p}, E), (m', \vec{p}', E') >= mE' + m'E - \vec{p} \cdot \vec{p}'.$$

One can easily check that $< \cdot, \cdot >$ is invariant with respect to ρ and, in particular, the polynomial function P on $\hat{A}$

$$P(m, \vec{p}, E) :=< (m, \vec{p}, E), (m, \vec{p}, E) >= 2mE - \vec{p}^2.$$

is invariant too. We are going to linearise it.

The signature of $< \cdot, \cdot >$ is $(1, -1, -1, -1)$ and it is diagonal with respect to the basis

$$\begin{aligned} e_0 &= \frac{1}{\sqrt{2}}(1, 0, 0, 1) \\ e_1 &= (0, 1, 0, 0) \\ e_2 &= (0, 0, 1, 0) \\ e_3 &= \frac{1}{\sqrt{2}}(1, 0, 0, -1). \end{aligned}$$

Hence, for all $h \in H$, $\rho(h) \in SO(1,3)$, where $SO(1,3)$ is the connected component of the identity of the Lorentz group with respect to the basis (e_0, e_1, e_2, e_3). The group homomorphism ρ from H to $SO(1,3)$ defines a unique group homomorphism $\widehat{\rho}$ from H to $SL(2,\mathbb{C})$ such that

$$\rho(h) = \delta(\widehat{\rho}(h)),$$

where δ is the universal covering homomorphism from $SL(2,\mathbb{C})$ to $SO(1,3)$. After some calculation, taking into account that $(e_0, \cdots, e_3)$ is not the standard basis of $\hat{A}$, one obtains

$$\widehat{\rho}(\vec{v}, r, x) = \begin{pmatrix} e^{-i\frac{r}{2}} & e^{i\frac{r}{2}}\frac{v^*}{\sqrt{2}} \\ 0 & e^{i\frac{r}{2}} \end{pmatrix},$$

where $v = v_x + iv_y \in \mathbb{C}$, if $\vec{v} = (v_x, v_y) \in \mathcal{V}$.

Let now γ from $\hat{A}$ to $\mathbb{C}^4$ be the irreducible realisation of the Clifford algebra given by the Dirac matrices and S the spin representation of $SL(2,\mathbb{C})$ acting on $\mathbb{C}^4$, *i.e.*

$$\begin{aligned} \gamma(e_0) &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ \gamma(e_i) &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad i = 1,2,3 \\ S(B) &= \begin{pmatrix} B & 0 \\ 0 & (B^*)^{-1} \end{pmatrix} \end{aligned}$$

where σ_i are the Pauli matrices and $B \in SL(2,\mathbb{C})$. Because of the well known covariant condition on the Dirac matrices

$$S(B)\gamma(p)S(B)^{-1} = \gamma(\delta(B)p) \qquad B \in SL(2,\mathbb{C}),\ p \in \hat{A},$$

if we define $L(h) := S(\widehat{\rho}(h))$ for all $h \in H$, then

$$L(h)\gamma(p)L(h)^{-1} = \gamma(h.p) \qquad h \in H,\ p \in \hat{A}. \tag{31}$$

Hence $p \mapsto \gamma(p)$ fulfills Eq. (13) and can be used to define a wave operator.

Remark 16. Let $h \in H$, since $p \mapsto \gamma(h.p)$ is an irreducible realisation of the Clifford algebra, by Schur lemma, Eq. (31) defines $L(h)$ uniquely, up to a constant depending on h, so that the representation of H satisfying Eq. (31) is unique up to a character of H.

Let $\mathcal{D}_2$ be the wave operator corresponding to γ as in Eq. (17), *i.e.*

$$\mathcal{D}_2 = i\left(\frac{\gamma_0 + \gamma_3}{\sqrt{2}}\frac{\partial}{\partial\xi} - \vec{\gamma}\cdot\vec{\nabla} + \frac{\gamma_0 - \gamma_3}{\sqrt{2}}\frac{\partial}{\partial t}\right),$$

where $\vec{\gamma} = (\gamma_1, \gamma_2)$. Since

$$\gamma(m,\vec{0},0) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix},$$

whose kernel is the vector space W generated by $\{e_0, e_3\}$, it easily follows that the space of solutions of the L-invariant wave equation $(D_1, \mathcal{D}_2)$ contains the Hilbert space carrying the reducible representation $U^{0,m_0,1/2,0} \oplus U^{0,m_0,-1/2,0}$, so it is not a wave equation in the sense of Def. 1.

Remark 17. Taking into account the previous remark and the form of the characters of H given by Eq. (22), one has that the wave equation $(D_1, \mathcal{D}_2)$ is still invariant with respect to the representation $\chi^\mu \otimes L$ of H and the space of solutions contains the Hilbert space carrying the reducible representation $U^{0,m_0,\mu+1/2,0} \oplus U^{0,m_0,\mu-1/2,0}$ with $\mu \in \mathbb{R}$.

Appendix A. The representations of $\mathbb{R}^3 \times' SU(2)$

Let $H = \mathbb{R}^3 \times' SU(2)$ be the group define in Sec. 4.1. Since H is simply-connected, its finite dimensional representations are in one to one correspondence with the representations of its Lie algebra. One has that Lie(H) is the semidirect sum of the abelian Lie algebra $\mathbb{R}^3$ and the simple Lie algebra $su(2)$. Due to the fact that Lie(H) is neither compact nor semisimple, it is very difficult to obtain a complete classification of the representations of Lie(H). Nevertheless George and Lévy-Nahas, [4], reduce the problem to the one of solving a non-linear matrix equation, see relation (33) below. In the following we briefly describe their results.

Let $\vec{H} = (H_1, H_2, H_3)$ be the usual basis of $su(2)$ and denote the element of Lie(H) as $(\vec{v}, \vec{l})$, with $\vec{v}, \vec{l} \in \mathbb{R}^3$, instead of $(\vec{v}, \vec{l} \cdot \vec{H})$.

For each $j \in \frac{1}{2}\mathbb{N}$, let d^j be the irreducible representation of $su(2)$ labelled by j (d^j is the differential of the representation $\mathbb{D}^j$ introduced in Sec. 4) and let

$$\vec{H}^j = \big(id^j(H_1), id^j(H_2), id^j(H_3)\big)$$

be the corresponding Hermitian generators.

For all $j, k \in \frac{1}{2}\mathbb{N}$, the dimension of the vector space

$$W^{1,k,j} := \mathrm{Hom}_{SU(2)}(\mathbb{R}^3; \mathcal{L}(\mathbb{C}^{2j+1}, \mathbb{C}^{2k+1}))$$

is one if $|k-j| \leq 1$, whereas is zero in the other cases. Hence, let $T^{kj} \in W^{1,k,j}$ normalised in such way that,

$$\begin{aligned} \vec{T}^{jj} &= \vec{H}^j \\ \vec{T}^{j(j-1)} &= (\vec{T}^{(j-1)j})^* \end{aligned}$$

(32)

$$\vec{v}_1 \cdot \vec{H}^j \vec{v}_2 \cdot \vec{H}^j + \vec{v}_1 \cdot (\vec{T}^{j(j-1)})^* \vec{v}_2 \cdot \vec{T}^{(j-1)j} = i\, j(\vec{v}_1 \wedge \vec{v}_2) \cdot \vec{H}^j + j^2 \vec{v}_1 \cdot \vec{v}_2,$$

where $\vec{v} \cdot \vec{T}^{(j-1)j} := T^{(j-1)j}(\vec{v})$. The explicit form of the matrices $\vec{T}^{(j-1)j}$ can be found in [3]. The relation (32) is fundamental for the problem of finding a Dirac type wave equation.

Let ρ be a finite dimensional representation of Lie(H). One can prove, [4], that ρ can be written, in a suitable basis, in upper triangular block form of the type:

$$\rho(\vec{v}, \vec{l}) = \begin{pmatrix} \rho_1(\vec{l}) & \widetilde{f}_{12}(\vec{v}) & \dots & \widetilde{f}_{1n}(\vec{v}) \\ 0 & \rho_2(\vec{l}) & \dots & \widetilde{f}_{2n}(\vec{v}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_n(\vec{l}) \end{pmatrix}.$$

The blocks on the diagonal are given by

$$\rho_p(\vec{l}) = \oplus_j (I_{\widetilde{n}_{jp}} \otimes (\vec{l} \cdot \vec{H}^j))$$

where $\widetilde{n}_{jp}$ is the multiplicity of the j-representation in the p-block. The off-diagonal blocks are of the form, with $p < q$,

$$\widetilde{f}_{pq}(\vec{v}) = (\widetilde{f}_{ij,pq}(\vec{v}))_{ij} = (M_{ij,pq} \otimes \vec{v} \cdot \vec{T}^{ij})_{ij}.$$

where $M_{ij,pq}$ are $\widetilde{n}_{ip} \times \widetilde{n}_{jq}$ matrices. One has to determine these matrices in a such way that the following non linear equation holds

$$\sum_k \gamma_{ijk} M_{ik} M_{kj} = 0 \tag{33}$$

where $M_{ij} = (M_{ij,pq})_{pq}$ is a block matrix $(\sum_p \widetilde{n}_{ip} \times \sum_q \widetilde{n}_{jq})$ whose elements are the matrices $M_{ij,pq}$ ($M_{ij,pq} = 0$ if $p \geq q$), and γ_{ijk} are known coefficients that can be found in [4]. We notice that in the case $\widetilde{n}_{ip} = 1$ for all i, p, the matrices $M_{ij,pq}$ are reduced to scalars.

Applying this technique, one obtains the representation L^j introduced by relation (20), with $2j \in \mathbb{N}$, $j \geq \frac{1}{2}$.

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