SU(1,1) tomography

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Abstract. We present a tomographic reconstruction procedure that exploits the symmetry of the $SU(1,1)$ group existent in some physical systems. The tomographic algorithm is derived analytically and the convergence of the procedure is tested on Monte Carlo simulated data.

I. INTRODUCTION.

Optical homodyne tomography is a well-established method for measuring the quantum state of the radiation field [1–4], and, more generally, for estimating the expectation value of arbitrary observables of the field [5], also for any number of field modes [6]. The success of optical homodyne tomography has stimulated the design of state-reconstruction procedures in other fields, such as in atomic [7], molecular [8], and ion-trap [9] physics. Homodyne tomography allows the estimation of the ensemble average of any field operator—including the matrix elements of the quantum state— by averaging special functions of the field quadratures with varying phase. These are the analogue of all linear combinations of position and momentum of a harmonic oscillator. The set of field quadratures is an example of a *quorum* of observables, namely a "complete" set of noncommuting observables which are sufficient for determining the quantum state of the system. With the only exception of the method of Ref. [10]—in which the tomographic reconstruction is achieved through the measurement of the photon number probability of the "displaced" state—no other quorum, different from the set of field quadratures, has never been considered in the optical domain (also the "selfhomodyne" tomography [11] is based on quadrature measurements, although not via homodyning). Indeed, a non-quadrature-based tomographic method would be very interesting in comparison with homodyne tomography, in order to compare the efficiency of different quorums, and to analyze the role played by different kinds of quorums in the tomographic reconstruction.

In this paper a totally different tomographic reconstruction procedure is proposed. It is based on the general theory given in Refs. [12–14]. This reconstruction exploits the $SU(1,1)$ group symmetry [15] and its infinite dimensional unitary irreducible representations in order to derive the pattern functions. Once these have been obtained, it is immediate to derive the tomographic procedure, since it simply consists in averaging the pattern functions over the experimental results. As a byproduct of the derivation, a generalization of the Jacobi polynomials orthogonality relations is also obtained.

To test the method, we present numerical results for a set of computer-simulated state reconstruction experiments, corresponding to the reconstruction of different quantum states. As we will see, there is always an excellent quantitative agreement between theoretical and experimental matrix elements, with unbiased statistical errors that can be arbitrarily reduced for increasing number of data.

For the sake of illustrating the method, two examples of physical systems with $SU(1,1)$ symmetry are presented and the procedure for performing the tomographic reconstruction experiment is given. These two systems, based on conventional photodetection preceded by parametric amplification, correspond to the manifolds of the well known single-mode and two-mode squeezed states. No actual feasibility analysis has been performed for either of the two idealized experimental situations.

After a short review of the general quantum tomographic approach in Sect. II, we present the $SU(1,1)$ tomographic procedure in Sect. III: first we give the tomographic protocol and abstract experimental scheme expressed in terms of operators of the $SU(1,1)$ algebra, then we discuss the result of simulated experiments and we give the analytic derivation of the method. Two physical examples of systems with the required symmetry are also given. Sect. IV concludes the paper with a summary of the main results.

II. GENERAL TOMOGRAPHY.

In this section we briefly review the general quantum tomography method of Refs. [13,14]. The purpose of quantum tomography is to reconstruct the expectation value of any operator A acting on the system Hilbert space $\mathcal{H}$, using only measurement outcomes of a set, called *quorum*, of observables $C(x)$ (i.e. $C(x)$ for fixed x has spectral orthonormal resolution). By measuring $C(x)$ for various x, it is possible to reconstruct the expectation value of A starting from the following resolution of the identity on the operator space:

$$A = \int_{\mathcal{X}} d\mu(x) \ \mathrm{Tr}[AB^\dagger(x)]C(x) \ , \tag{1}$$

where the set $B(x)$ is the dual set of $C(x)$ under the Hilbert-Schmidt scalar product, and $\mu(x)$ denotes a suitable measure. In general $x = (x_1, x_2, \ldots)$ denotes a vector of parameters, with x_i either continuous or discrete, and the notation $\int d\mu(x)$ denotes multiple integrals/sums.

The dual couple $B(x)$ and $C(x)$ satisfy the orthogonality relation

$$\delta_{mk}\delta_{nl} = \int_{\mathcal{X}} d\mu(x)\ \langle m|B^\dagger(x)|n\rangle\langle l|C(x)|k\rangle\ , \tag{2}$$

which, for $\{|m\rangle\}$ denoting an orthonormal basis of $\mathcal{H}$, is just a restatement of the tomography identity (1). Alternatively, if $\{|m\rangle\}$ only spans a subspace of $\mathcal{H}$, then Eq. (1) will hold only for operators restricted to such subspace. As we will see, this is the case of the $SU(1,1)$ tomography in some of its physical realizations.

Once a dual couple $B(x)$ and $C(x)$ has been found, one can obtain the expectation value of an arbitrary operator A, by taking the expectation of both members of Eq. (1), *i.e.*

$$\langle A\rangle = \sum_n \int_{\mathcal{X}} d\mu(x)\ p_n(x)\ \lambda_n(x)\mathrm{Tr}[B^\dagger(x)A]\ , \tag{3}$$

where $p_n(x)$ is the probability of obtaining the n-th eigenvalue $\lambda_n(x)$ when measuring $C(x)$, *i.e.* $p_n(x) = \langle n|\varrho|n\rangle$ (with ϱ density matrix of the system). The quantity $\lambda_n(x)\,\mathrm{Tr}[B^\dagger(x)A]$, defined as pattern function, can be analytically evaluated, while the probability $p_n(x)$ must be determined experimentally. Notice that often (such as in the cases described in this paper) it is possible to obtain the whole quorum through a unitary operator $U(x)$ depending on the parameter x acting on a single observable O, i.e.

$$C(x) = U^\dagger(x)\ O\ U(x)\ . \tag{4}$$

In this case, by calculating $\mathrm{Tr}[C(x)\varrho]$ on a basis of eigenstates $|n\rangle$ of O, one finds that

$$\langle A\rangle = \sum_n \int_{\mathcal{X}} d\mu(x)\ p_n(x)\ o_n \mathrm{Tr}[B^\dagger(x)A]\ , \tag{5}$$

where here $p_n(x) = \langle n|U(x)\ \varrho\ U^\dagger(x)|n\rangle$ is the probability of obtaining the eigenvalue o_n, when measuring the observable O on the state ϱ evolved by the unitary operator $U(x)$.

III. $SU(1,1)$ TOMOGRAPHY.

A. Tomography protocol

In this subsection we give the tomographic reconstruction procedure. Some simulated tomographic experiments are presented and discussed. The mathematical derivation of the reconstruction algorithm (7) will be postponed to the following subsection (III B). The system on which the procedure is to be applied is described in terms of the abstract operators of the $SU(1,1)$ algebra defined by Eq. (6). Its realization in practice can change completely from case to case, depending on the physical realization of the $SU(1,1)$ algebra. One, then, only has to substitute the appropriate system operators and measurement results in the formulae given here in order to obtain the reconstruction procedure suited to his particular system. For the sake of illustration, a couple of simple idealized physical systems are given at the end of the section.

The Lie algebra $su(1,1)$ of the $SU(1,1)$ group is spanned by the operators K_+, K_-, K_z which are defined by the commutation relations

$$\begin{aligned} &[K_+,K_-] = -2K_z\ ;\ [K_z,K_\pm] = \pm K_\pm\ ; \\ &K_x \doteq \frac{1}{2}(K_+ + K_-),\ K_y \doteq -\frac{i}{2}(K_+ - K_-)\ . \end{aligned} \tag{6}$$

The Casimir invariant operator which labels all the unitary irreducible representations of the group is given by $(K_z)^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \kappa(\kappa-1)$, where the eigenvalue κ is also called Bargmann index.

The general $SU(1,1)$ tomographic identity corresponding to Eq. (5), for the reconstruction of the expectation value of the operator A on an ensemble in the state ϱ, will be throughly derived in the following subsection. It is given by

$$\begin{aligned} &\langle A\rangle = \frac{1}{\pi}\sum_{j=0}^{\infty}\int_0^{2\pi} d\varphi \int_0^\infty d\theta\ \mathrm{th}(\theta)\ \times \\ &\langle j|e^{i\frac{\theta}{2}(e^{-i\varphi}K_+ - e^{i\varphi}K_-)}\varrho\ e^{-i\frac{\theta}{2}(e^{-i\varphi}K_+ - e^{i\varphi}K_-)}|j\rangle\ \times \\ &\mathrm{Tr}[A\ \{(-1)^{K_z}\ e^{\theta(e^{i\varphi}K_+ - e^{-i\varphi}K_-)}, K_z\}_+](-1)^{\kappa+j}\ , \end{aligned} \tag{7}$$

where $\{|j\rangle\}$ is the orthonormal basis of eigenvectors of K_z corresponding to eigenvalue $\kappa + j$, and $\{\cdot,\cdot\}_+$ denotes the anticommutator. In Eq. (7) the quantity $\langle j|e^{i\frac{\theta}{2}(e^{-i\varphi}K_+ - e^{i\varphi}K_-)}\varrho\ e^{-i\frac{\theta}{2}(e^{-i\varphi}K_+ - e^{i\varphi}K_-)}|j\rangle$ is the probability distribution of the eigenvalue $j+\kappa$ obtained when measuring the observable K_z on the input state ϱ that has undergone an evolution described by the unitary operator

$$U(\theta,\varphi) \doteq e^{i\frac{\theta}{2}(e^{-i\varphi}K_+ - e^{i\varphi}K_-)}\ . \tag{8}$$

The experimenter must repeatedly measure the observable K_z on different ensemble elements after evolving each of them with the unitary operator $U(\theta,\varphi)$, varying the parameters θ and φ at each measurement. [As we will see in the following for the example of the electromagnetic field $SU(1,1)$ reconstruction scheme, this resorts to measuring the number of photons after having squeezed the radiation for varying values of the squeezing amplitude and phase.]

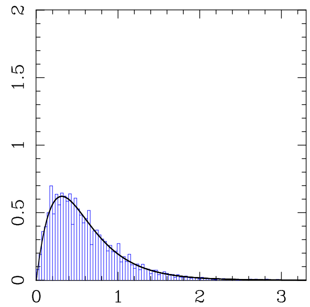


FIG. 1. Histogram for the squeezing amplitude parameter θ used in the reconstruction of Fig. 2. The full superimposed line is the theoretical probability distribution $p(\theta) = \frac{1}{l}\exp(-w\,\theta)\mathrm{th}(\theta)$. Here $w = 1.25$.

In practice, the unbounded real amplitude $\theta \in [0, \infty)$ must be chosen randomly weighted with a cut-off distribution $p(\theta) = \frac{1}{l}\exp(-w\,\theta)\mathrm{th}(\theta)$, where w is a constant and l is a normalization (an example is given in Fig. 1), while φ can be chosen randomly in $[0, 2\pi]$ with uniform probability. The parameters θ and φ thus obtained are used to tune the experimental apparatus so that the input state is evolved by the operator $U(\theta, \varphi)$. Then the measurement of K_z is performed, yielding the i-th experimental result $m_i(\theta_i, \varphi_i)$ for $K_z - \kappa$. The expectation (7) is estimated through the average on N measurements

$$\langle A \rangle = \frac{1}{N}\sum_{i=0}^{N}(-1)^{\kappa + m_i(\theta_i,\varphi_i)} 2l \exp(w\,\theta_i) \times \quad (9)$$
$$\mathrm{Tr}[A\{(-1)^{K_z}\, e^{\theta_i(e^{i\varphi_i}K_+ - e^{-i\varphi_i}K_-)}, K_z\}_+] \,,$$

where the function $l\exp(w\,\theta)$ is introduced in order to compensate the cut-off weighting. The parameter w must be chosen sufficiently large to obtain a distribution of the squeezing amplitude θ in the allowed experimental range, and at the same time not too large, otherwise only too small θ values will result, and a too large number of experimental data would be needed for the tomographic reconstruction. Notice that any normalized decaying weight function can be used in place of the negative exponential used here.

In Figs. 2 and 3 we show some Monte Carlo simulations of the proposed method, where the measurement results $m_i(\theta, \varphi)$ have been simulated from the theoretical probability distribution of the input state evolved by $U(\theta, \varphi)$. The simulation of Fig. 2 refers to the state reconstruction of a system in a Perelomov coherent state [16] defined as

$$|\alpha\rangle \doteq \frac{(1 - |\alpha|^2)^\kappa}{\sqrt{(2\kappa - 1)!}} \sum_{n=0}^{\infty} \sqrt{\frac{(2\kappa - 1 + n)!}{n!}}\alpha^n |n\rangle \,. \quad (10)$$

In Fig. 3 the simulation of the state reconstruction of an ensemble in the state $\frac{1}{\sqrt{2}}(|q\rangle + |r\rangle)$, *i.e.* in a coherent superposition of two eigenstates of K_z, is given and a comparison is made between experiments carried out with different weight parameters w. Notice how, with the same number of data, a better reconstruction is performed with lower w, which corresponds to higher available squeezing amplitudes θ.

The procedure which has been described is unbiased, since no systematic errors are introduced. Only statistical errors are present and can be made arbitrarily small by increasing the number of experimental data.

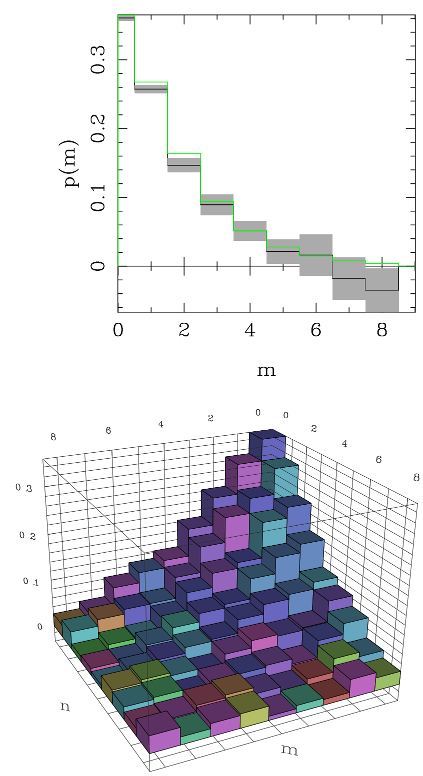
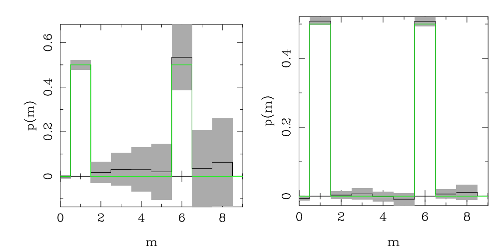


FIG. 2. Monte Carlo simulation of the proposed experiment for the reconstruction of the operator $A = |m\rangle\langle n|$, namely the measurement of the system density matrix elements ϱ_{nm}. The system is in a Perelomov coherent state $|\alpha\rangle$, with $\alpha = 0.7$. In this simulation $\kappa = \frac{3}{4}$ and $2 \cdot 10^5$ measurements of K_z divided into 20 statistical blocks have been simulated. In the upper graph the diagonal $p(m)$ of the density matrix is plotted with the statistical error bars and compared to its theoretical value. In the lower graph, the absolute values of the density matrix elements $|\varrho_{nm}|$ are plotted.



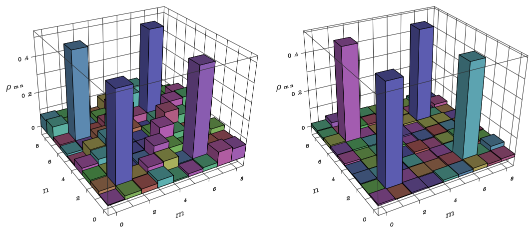


FIG. 3. Comparison of identical simulated experiments for the measurement of the density matrix elements ϱ_{nm}, carried out using different weight parameter w in the probability distribution for the squeezing amplitude θ. The system is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|q\rangle + |r\rangle)$, with $q = 1$ and $r = 6$. Here $\kappa = \frac{4}{5}$ and, as in the previous example, $2 \cdot 10^5$ measurements of K_z divided into 20 statistical blocks have been simulated. The upper graphs show the diagonal matrix elements with their theoretical value, and the lower graphs show the absolute values of all matrix elements. The simulation on the left is obtained with a weight $w = 2.8$ (corresponding to θ varying in the interval $[0, \sim 1.2]$), while the simulation on the right is obtained with $w = 0.75$ (corresponding to θ varying in $[0, \sim 4]$.

We now analyze two experimental setups in order to illustrate how, starting from the tomographic formulae given here, one may obtain the tomography suited to a particular system. They employ the $SU(1,1)$ reconstruction formula (7) for the tomographic reconstruction of a two–mode radiation field and of a single–mode radiation field respectively and are based on parametric amplifiers and photodetectors. Homodyne tomography has proved very successful on reconstructing the quantum state of the radiation field and these setups do not seem to achieve any big practical advantage over such a well established procedure. For this reason, only a rough description of the idealized experiments will be presented here as examples of the application of the $SU(1,1)$ tomography to actual physical systems without any feasibility analysis of the experimental situation where one will encounter the presence of loss, imperfect photodetection, non-monochromatic, depleting and non perfectly phase-matched pump, imprecisions in the determination of the squeezing parameters, *etc.*

The first setup is a two mode setup, that employs the following realization of $SU(1,1)$ through the radiation operators,

$$K_+ \doteq a^\dagger b^\dagger \ ; \ K_- \doteq K_+^\dagger \ ; \ K_z \doteq \frac{1}{2}(a^\dagger a + b^\dagger b + 1) \ , \quad (11)$$

where a and b are the annihilation operators for the two radiation modes. The Casimir invariant operator is given by $\frac{1}{2}\left(|a^\dagger a - b^\dagger b| + 1\right)$. The set $\{|j\rangle\}$ of eigenvectors of K_z spans only the subspace of the two–mode Hilbert space in which the photon difference in the two modes is fixed and is given by $2\kappa - 1$. Thus, only the operators A acting on this subspace can be reconstructed with such setup, and one can reconstruct the radiation state only if the difference between the photons in the two modes is fixed. Here K_z is the observable "sum of photons" in the two modes and can be measured by two photodetectors. The operator $U(\theta, \varphi)$ describes a non-degenerate parametric amplifier (or phase insensitive amplifier), where θ and φ are the squeezing amplitude and phase respectively, i.e. $U(\theta,\varphi) = \exp\left[\frac{1}{2}\theta(a^\dagger b^\dagger e^{i\varphi} - ab\, e^{-i\varphi})\right]$. In Fig. 4 the block diagram for this experiment is presented.

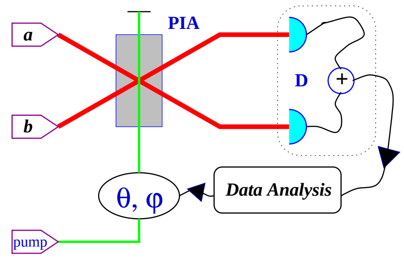


FIG. 4. Block diagram for the two–mode setup for the tomographic reconstruction of $SU(1,1)$ states. The sum–photodetection stage is denoted by D, while the non–degenerate parametric amplifier is labeled PIA.

The second setup, whose block diagram is given in Fig. 5, is a single mode setup that employs the following $SU(1,1)$ realization:

$$K_+ \doteq \frac{1}{2}(a^\dagger)^2 \ ; \ K_- \doteq K_+^\dagger \ ; \ K_z \doteq \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \ , \quad (12)$$

where a is the annihilation operator of the mode, and $\kappa = \frac{1}{4}, \ \frac{3}{4}$. Notice that this realization is actually a projective representation of $SU(1,1)$. It allows to reconstruct only the operators that act on the subspace of the radiation space constituted by the states with fixed (even or odd) parity in the number of photons. In this case, the observable K_z is related to the number of photons in the mode, while the operator $U(\theta,\varphi)$ describes a degenerate parametric amplifier (or phase sensitive amplifier), i.e. $U(\theta,\varphi) = \exp\left[\frac{1}{4}\theta({a^\dagger}^2 e^{-i\varphi} - a^2 e^{i\varphi})\right]$.

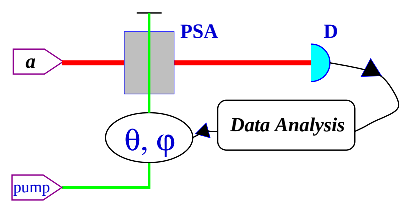


FIG. 5. Block diagram of the single–mode setup for the tomographic reconstruction. Here PSA denotes the degenerate parametric amplifier and D the photodetector.

B. Tomographic reconstruction procedure

In this subsection we derive the reconstruction procedure (7) that allows the tomography protocol described in the previous subsection. The derivation is obtained in the framework of the general tomography theory [13,14], which has been briefly reviewed in Sect. II.

As we will see, a resolution of the identity of the form (2) can be obtained by making use of the measure

$$d\mu(\gamma) \doteq \frac{d^2\gamma}{\pi}\,\frac{\mathrm{th}|\gamma|}{|\gamma|} \tag{13}$$

(where γ varies over the whole complex plane), and by using the operators

$$\begin{aligned} C(\gamma) &\doteq \exp[\gamma K_+ - \bar\gamma K_-]\ , \\ B(\gamma) &\doteq \{C(\gamma), K_z\}_+ = C(\gamma)K_z + K_z C(\gamma)\ . \end{aligned} \tag{14}$$

These operators are defined so that, by introducing a basis $\{|n\rangle\}$ of eigenstates of the operator K_z (i.e. $K_z|n\rangle = (\kappa+n)|n\rangle$, $n = 0,1,\cdots,\infty$), we have

$$\langle n|C(\gamma)|m\rangle = \mathcal{N}\langle n|B(\gamma)|m\rangle = \langle n|\exp[\gamma K_+ - \bar\gamma K_-]|m\rangle \tag{15}$$

with $\mathcal{N} = (2\kappa+m+n)^{-1}$.

We now show that the resolution of the identity of the form (2) based on the operators $C(\gamma)$ and $B(\gamma)$ defined in (14) follows from the orthogonality of some polynomials obtained from their matrix elements. These polynomials are defined by

$$\begin{aligned} &\langle m|e^{\gamma K_+ - \bar\gamma K_-}|n\rangle = \langle m|e^{\zeta K_+}(1-|\zeta|^2)^{K_z}e^{-\bar\zeta K_-}|n\rangle \doteq \\ &(-1)^n\zeta^{m-n}(1-|\zeta|^2)^{\kappa}\sqrt{\frac{n!\,\Gamma(2\kappa+m)}{m!\,\Gamma(2\kappa+n)}}\times \\ &S_n^{(2\kappa-1,m-n)}(|\zeta|^2)\ , \end{aligned} \tag{16}$$

where $\zeta \doteq \frac{\gamma}{|\gamma|}\mathrm{th}|\gamma|$. The n-degree polynomial $S_n^{(\alpha,d)}(\rho)$ is given by

$$S_n^{(\alpha,d)}(\rho) \doteq \sum_k (-1)^{k+n}\binom{\alpha+n+d+k}{k}\binom{n+d}{k+d}\rho^k\ , \tag{17}$$

where k takes all the values such that the binomials are non-zero, i.e. $\max(0,-d) \le k \le n$. Notice that $S_n^{(\alpha,d)}(\rho)$ is a generalization of the Jacobi polynomials, since by using the property

$$S_n^{(\alpha,m-n)}(\rho) = \rho^{n-m}\frac{m!\,\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+m+1)}S_m^{(\alpha,n-m)}(\rho)\ , \tag{18}$$

we can write

$$S_n^{(\alpha,m-n)}(\rho) = \begin{cases} P_n^{(\alpha,m-n)}(2\rho-1) & \text{for } m \ge n \\ P_m^{(\alpha,n-m)}(2\rho-1)\,\rho^{n-m}\,\frac{m!\,\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+m+1)} & \text{for } m \le n \end{cases}\ , \tag{19}$$

where $P_n^{(\alpha,\beta)}(x)$, with $\alpha > -1$ and $\beta > -1$ is the Jacobi polynomial of degree n [17].

The orthogonality relation for the polynomials $S_n^{(\alpha,d)}(\rho)$ can be either obtained from the orthogonality relations of the Jacobi polynomials through Eq. (19) or from the necessary condition (22) for the orthogonal polynomials. Thus we obtain

$$\frac{1}{\mathcal{N}'}\int_0^1 d\rho\,(1-\rho)^\alpha\rho^d\,S_n^{(\alpha,d)}(\rho)\,S_l^{(\alpha,d)}(\rho) = \delta_{nl}\ , \tag{20}$$

with the normalization

$$\mathcal{N}' = \frac{(n+d)!\,\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+d+n+1)(\alpha+d+2n+1)} \tag{21}$$

Incidentally, notice that a useful recursion may be obtained from the necessary condition [17] for orthogonal polynomials. Namely, a system of n-degree polynomials $p_n(\rho)$ satisfying the recursion

$$(A_n\rho + B_n)p_n(\rho) - C_n p_{n-1}(\rho) = p_{n+1}(\rho) \tag{22}$$

with $A_n, C_n \ge 0$ is an orthogonal system. One can verify that the recursion (22) is satisfied for $p_n(\rho) = S_n^{(\alpha,d)}(\rho)$, by choosing

$$A_n = \frac{(\alpha+2n+d+1)(\alpha+2n+d+2)}{(n+1)(\alpha+d+n+1)} \tag{23}$$

$$B_n = -\frac{(\alpha+2n+d+1)[\alpha(2n+d+1)+2n(n+d+1)+d+d^2]}{(\alpha+d+n+1)(n+1)(\alpha+d+2n)} \tag{24}$$

$$C_n = \frac{(d+n)(n+\alpha)(\alpha+2n+d+2)}{(n+1)(\alpha+n+d+1)(\alpha+2n+d)}\ . \tag{25}$$

By making use of the above relations, we may derive the orthogonality condition (2) for $SU(1,1)$. In fact, we see that

$$\begin{aligned} &\frac{1}{\mathcal{N}}\int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\,\frac{\mathrm{th}|\gamma|}{|\gamma|}\langle m|e^{\gamma K_+ - \bar\gamma K_-}|n\rangle\langle l|e^{-\gamma K_+ + \bar\gamma K_-}|k\rangle = \\ &\frac{(-1)^{n+l}}{\mathcal{N}}\sqrt{\frac{n!k!\,\Gamma(\alpha+m+1)\,\Gamma(\alpha+l+1)}{m!l!\,\Gamma(\alpha+n+1)\,\Gamma(\alpha+k+1)}}\int\frac{d^2\zeta}{\pi}\times \\ &(1-|\zeta|^2)^\alpha\zeta^{m-n+l-k}S_n^{(\alpha,m-n)}(|\zeta|^2)S_k^{(\alpha,l-k)}(|\zeta|^2) = \\ &\frac{(-1)^{n+l}}{2\pi\mathcal{N}}\sqrt{\frac{n!l!\,\Gamma(\alpha+m+1)\,\Gamma(\alpha+k+1)}{m!k!\,\Gamma(\alpha+n+1)\,\Gamma(\alpha+l+1)}}\int_0^{2\pi}d\varphi\times \\ &e^{i\varphi(m-n+l-k)}\int_0^1 d\rho(1-\rho)^\alpha\rho^{\frac{m-n+k-l}{2}}\times \\ &S_n^{(\alpha,m-n)}(\rho)S_l^{(\alpha,k-l)}(\rho) = \delta_{mk}\delta_{nl}\ , \end{aligned} \tag{26}$$

with $\alpha \doteq 2\kappa-1$, $\zeta \doteq \frac{\gamma}{|\gamma|}\mathrm{th}|\gamma|$, $\gamma \doteq \rho^{\frac12}e^{i\varphi}$ and

$$\mathcal{N} = \frac{n!\,\Gamma(\alpha+m+1)\,\mathcal{N}'}{m!\,\Gamma(\alpha+n+1)} = \frac{1}{\alpha+m+n+1}\ , \tag{27}$$

and where the double integral on $d^2\zeta$ is to be performed on the unit circle. Eq. (26) guarantees the existence of a tomographic identity of the form (1), with $B(\gamma)$ and $C(\gamma)$ given by (14).

Using the identity resolution (26), we can now obtain the $SU(1,1)$ tomographic reconstruction procedure as given in (3), namely

$$\langle A\rangle = \int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\,\frac{\mathrm{th}|\gamma|}{|\gamma|}\,\mathrm{Tr}[A\,B^\dagger(\gamma)]\mathrm{Tr}[\varrho\,C(\gamma)]\;, \tag{28}$$

where ϱ is the system density matrix and A is an arbitrary system operator.

How can this identity be used to obtain the $SU(1,1)$ tomography? The obvious way to obtain a reconstruction procedure of the form (3) would be to devise a measurement procedure that measures the Hermitian operator $i(\gamma K_+ - \bar\gamma K_-)$. However, in the physical realizations of systems with $SU(1,1)$ symmetry shown in the two examples given in the previous subsection, this is often practically impossible, or at least very challenging. It would be much better to measure the operator K_z, that, as we have shown, is obtained both in the single mode and in the two mode systems through a photodetection. We will now show how the tomographic identity we found can be employed requiring the measurement of K_z on a state that has been evolved by the unitary operator $\exp(\gamma K_+ - \bar\gamma K_-)$, i.e. we use a reconstruction procedure of the form (5). For example, in the single–mode and two–mode realizations given above, this resorts to squeezing the input state and then performing a photodetection. [Of course, in any physical situation in which $i(\gamma K_+ - \bar\gamma K_-)$ *is* easily measurable, then Eq. (28) can be directly employed for the $SU(1,1)$ tomography].

To this aim, let us regard K_x, K_y and K_z as elements of $su(1,1)$, that is as 2×2 complex matrices, and the exponential map exp as a function from $su(1,1)$ to $SU(1,1)$. In doing so

$$K_x = -\frac{i}{2}\sigma_1 \quad K_y = -\frac{i}{2}\sigma_2 \quad K_z = \frac{1}{2}\sigma_3\;, \tag{29}$$

where σ_i are the Pauli matrices. Denote $\vec K \doteq (K_x, K_y, K_z)$, $\vec n \doteq (\mathrm{sh}\theta\cos\varphi, \mathrm{sh}\theta\sin\varphi, \mathrm{ch}\theta)$ and $\vec n_\perp \doteq (\sin\varphi, -\cos\varphi, 0)$, then one can easily check that

$$\exp[2i(bK_x + aK_y)](-1)^{K_z} = (-1)^{\vec n\cdot\vec K} = \tag{30}$$

$$\exp(-i\theta\vec n_\perp\cdot\vec K)(-1)^{K_z}\exp(i\theta\vec n_\perp\cdot\vec K), \tag{31}$$

for all $\gamma \doteq a + ib \doteq \theta e^{i(\pi-\varphi)}$ in the complex plane.

Let U be the unitary representation of $SU(1,1)$ with Bargmann index κ. As a consequence of the Stone theorem, one has $e^{i\eta K_s} = U_{\exp(\eta K_s)}$, with $s = x, y, z$, and $C(\gamma) = U_{\exp[2i(bK_x + aK_y)]}$, where, with slight abuse of notation we denote by the same symbol K_s both the 2×2 matrix and the selfadjoint operator acting on the Hilbert space of the system. Hence, since Eqs. (30) and (31) hold at group level and U is a representation, one has

$$\mathrm{Tr}[C(\gamma)\;(-1)^{K_z}\varrho] \equiv \sum_{j=0}^{\infty}\langle j|e^{i\theta\vec n_\perp\cdot\vec K}\varrho e^{-i\theta\vec n_\perp\cdot\vec K}|j\rangle(-1)^{(\kappa+j)}\;, \tag{32}$$

where $\langle j|e^{i\theta\vec n_\perp\cdot\vec K}\varrho e^{-i\theta\vec n_\perp\cdot\vec K}|j\rangle$ is the probability of obtaining the result $j+\kappa$, when measuring the operator K_z on the input state ϱ evolved by the unitary operator $U(\theta,\varphi) = e^{i\theta\vec n_\perp\cdot\vec K}$, i.e. the state $e^{i\theta\vec n_\perp\cdot\vec K}\varrho\, e^{-i\theta\vec n_\perp\cdot\vec K}$.

Moreover, it is immediate to see that the orthogonality relation (26) is valid also if we take, in place of $C(\gamma)$ and $B(\gamma)$, the operators $C'(\gamma)$ and $B'(\gamma)$ defined as

$$C'(\gamma) \doteq e^{(\gamma K_+ - \bar\gamma K_-)}(-1)^{K_z}\;, \qquad B'(\gamma) \doteq \{C'(\gamma), K_z\}_+\;. \tag{33}$$

Thus, rewriting the tomographic identity (28) with $C'(\gamma)$ and $B'(\gamma)$, and using Eq. (32), we find the $SU(1,1)$ tomography reconstruction procedure, *i.e.*

$$\langle A\rangle = \frac{1}{\pi}\sum_{j=0}^{\infty}\int_0^{2\pi} d\varphi\int_0^\infty d\theta\;\mathrm{th}(\theta)\times \langle j|e^{i\theta\vec n_\perp\cdot\vec K}\varrho\, e^{-i\theta\vec n_\perp\cdot\vec K}|j\rangle\;\mathrm{Tr}[A\;B'^\dagger(\theta,\varphi)](-1)^{\kappa+j}\;. \tag{34}$$

IV. CONCLUSIONS.

In this paper we have introduced, derived and numerically tested a quantum tomographic reconstruction procedure based on $SU(1,1)$ symmetry. The tomographic algorithm has been derived using the general method of Refs. [13,14], and the orthogonality relation of the $SU(1,1)$ tomography turns out to be a generalization of the Jacobi polynomials orthogonality identity. The method has been tested on a set of computer-simulated experiments, corresponding to the reconstruction of different quantum states. An excellent agreement between theoretical and experimental-simulated matrix elements was found, within perfectly unbiased statistical errors. Two examples of physical systems to illustrate this reconstruction procedure have been given: the setups are based on the $SU(1,1)$ symmetry of parametric amplification of the two-mode and single-mode electromagnetic field.

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