GROUP THEORETICAL QUANTUM TOMOGRAPHY

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ABSTRACT. The paper is devoted to the mathematical foundation of the quantum tomography using the theory of square-integrable representations of unimodular Lie groups.

1. INTRODUCTION

In Quantum Mechanics a physical system is associated with a Hilbert space $\mathcal{H}$: the states are described by positive trace-class trace-one operators T on $\mathcal{H}$, the physical quantities by self-adjoint operators A on $\mathcal{H}$ and the physical content of the theory is given by the expectation values $\mathrm{Tr}\,(AT)$. The state T is completely determined by $\mathrm{Tr}\,(Q_nT)$ for Q_n running on a suitable set $\{Q_n\}$ of observables and, for arbitrary operator A, $\mathrm{Tr}\,(AT)$ can be computed in terms of $\mathrm{Tr}\,(Q_nT)$. In order to implement this scheme one has to estimate $\mathrm{Tr}\,(Q_nT)$ experimentally, facing the problems arising from statistical errors and instrumental noise. Moreover, the number of experimental data is clearly finite, while A and T are operators on an infinite dimensional Hilbert space and the set $\{Q_n\}$ is infinite.

The problem of determining the state of a quantum system entered the realm of experiments in the last decade, in the domain of quantum optics. Many authors, see for example [1, 2, 3, 4], proposed and used various techniques to reconstruct the density operator of a single mode of the e.m. field from the probability distributions of its quadratures. These methods were originally based on the use of the Radon transform, as in medical tomographic imaging. Due to this analogy the name *quantum tomography* is currently used to refer to these techniques. Their common feature, for a review see [5], is the use of a set of observables $\{Q_n \,:\, n \in X\}$, called *quorum*, parametrised by a space X endowed with a probability measure μ. The fundamental property of the quorum is that any observable A can be expressed as *integral transform* on the space X

$$A = \int_X \mathcal{E}[A](n)\; d\mu(n)$$

in such a way that, for all $n \in X$, the operator $\mathcal{E}[A](n)$ is a function of Q_n in the sense of the functional calculus. Then, if T is the state, one has that

$$\mathrm{Tr}\,(AT) = \int_{X\times\mathbb{R}} \sigma(A)(n,\lambda)\; \omega(n,\lambda)\; d\mu(n)d\lambda, \tag{1}$$

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where $\lambda \mapsto \omega(n,\lambda)$ is the probability density of Q_n in the state T, *i.e.*

$$\mathrm{Tr}\,(TQ_n) = \int_{\mathbb{R}} \lambda\ \omega(n,\lambda)\ d\lambda,$$

and $\lambda \mapsto \sigma(A)(n,\lambda)$ is the function defined by $\mathcal{E}[A](n)$ using the functional calculus, *i.e.*

$$\mathrm{Tr}\,(T\mathcal{E}[A](n)) = \int_{\mathbb{R}} \sigma(A)(n,\lambda)\omega(n,\lambda)\ d\lambda$$

(in the above formulas we assumed for simplicity that each Q_n has a continuous spectrum). Selecting randomly Q_n in the quorum according to the probability measure μ and measuring it, the outcome probability of obtaining the value λ is given by $\omega(n,\lambda)d\mu(n)d\lambda$. Then, by means of Eq. (1), the expectation value $\mathrm{Tr}\,(AT)$ can be reconstructed, by averaging the function $\sigma(A)$ over $X \times \mathbb{R}$ endowed with the probability measure $\omega d\mu d\lambda$. We notice that the function $\sigma(A)$, called the *estimator* of A, does not depend on T, and that the same set of data can be used to estimate all the expectation values $\mathrm{Tr}\,(AT)$.

In [6] and [7] a general method has been proposed to realize a quorum and define estimators in terms of suitable unitary representations of Lie groups (for a self contained synthetic exposition see [8] and [9]). The present paper is devoted to the mathematical foundation of this method using the theory of square-integrable representations of unimodular Lie groups. In Section 2 we present the mathematical theory and in Section 3 we apply it to two examples: the homodyne tomography related to the Weyl-Heisenberg group and the angular momentum tomography associated with the rotation group.

2. Group-dynamical quorum

In this section we define a quorum associated with a square-integrable representation of a Lie group.

Let G be a unimodular connected Lie group G and K a central closed subgroup. The quotient space $H = G/K$ is a unimodular connected Lie group. We denote by $\mathfrak{H}$ its Lie algebra, by $m+1$ the (real) dimension of $\mathfrak{H}$ as a vector space, by dv the Lebesgue measure on $\mathfrak{H}$ and by dh the Haar measure of H, uniquely defined up to a positive constant, which will be fixed in the following.

Denoted by exp the exponential map from $\mathfrak{H}$ to H, we assume that there is an open subset V of $\mathfrak{H}$ such that $\exp(V)$ is open in H, its complement has zero measure with respect to dh and exp is a diffeomorphism from V onto $\exp(V)$. This hypothesis implies that, given $f \in L^1(H, dh)$,

$$\int_H f(h)\, dh = D \int_{\mathfrak{H}} f(\exp(v)) |\det\,(d(\exp)_v)| \chi_V(v)\, dv, \tag{2}$$

where $d(\exp)_v$ is the differential of the exponential map at $v \in \mathfrak{H}$, *i.e.*

$$d(\exp)_v(w) = \left(\frac{d}{dt} \exp(-v) \exp(v + tw) \right)_{t=0} \qquad w \in \mathfrak{H},$$

$\det(\cdot)$ is the determinant and D is a positive constant, see for example Th. 1.14, Ch. I of [10]. We normalize the Haar measure dh of H in such a way that $D = 1$.

Remark 1. The density $\det(d(\exp)_v)$ can be easily computed observing that, if $\lambda_1, ..., \lambda_{m+1}$ are the (possibly repeated) eigenvalues of $d(\exp)_v$, viewed as linear operator on $\mathfrak{H}$, then

$$\det(d(\exp)_v) = \frac{1 - e^{-\lambda_1}}{\lambda_1} \cdots \frac{1 - e^{-\lambda_{m+1}}}{\lambda_{m+1}},$$

with $\frac{1-e^{-0}}{0} = 1$, see for example Th. 1.7, Ch. I of [11].

Let U be an irreducible continuous unitary representation of G. We denote by $\mathcal{H}$ the (complex separable) Hilbert space where the representation acts and by $\langle \cdot, \cdot \rangle$ the scalar product, linear in the second argument.

We assume that the representation U is square-integrable modulo K, *i.e.* there is a non-zero vector $v \in \mathcal{H}$ such that

$$\int_H |\left\langle U_{c(h)}v, v \right\rangle|^2 \, dh < \infty, \tag{3}$$

where c is a section from H to G, *i.e.* a measurable map $c : H \to G$ such that

$$\begin{aligned} c(e_H) &= e_G \\ \pi(c(h)) &= h \qquad h \in H, \end{aligned}$$

with π being the canonical projection from G to H. Notice that the value of the integral in Eq. (3) is independent of the choice of the section and that Eq. (3) implies that the function $h \mapsto \left\langle U_{c(h)}u, w \right\rangle$ is square integrable for all $u, w \in \mathcal{H}$, [12].

Remark 2. In many examples K is trivial, *i.e.* $K = e_G$, so that $H = G$ and Eq. (3) reduces to the usual notion of square-integrability. Nevertheless, there are cases, as the Weyl-Heisenberg group, that require the full theory. Moreover, in this framework one can easily consider projective representations. Indeed, let $\widehat{U}$ be a projective representation of a Lie group $\widehat{H}$ with multiplier m. Define G as the central extension of the torus K by $\widehat{H}$ associated with m. Then K is a central closed subgroup of G, H is canonically isomorphic with $\widehat{H}$ and there is a unitary representation U of G such that

$$\widehat{U}_{\pi(g)} = U_g \qquad g \in G.$$

Clearly, the fact that U is square-integrable modulo K is equivalent to the fact that $\widehat{U}$ is a square-integrable projective representation of $\widehat{H}$.

Being U square-integrable modulo K, one can prove [12] that there is a constant $d > 0$, called *the formal degree* of U, such that, for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$,

$$\int_H \overline{\left\langle U_{c(h)}v_1, u_1 \right\rangle} \left\langle U_{c(h)}v_2, u_2 \right\rangle dh = \frac{1}{d} \langle u_1, u_2 \rangle \langle v_2, v_1 \rangle . \tag{4}$$

Using the above relation we can represent the Hilbert-Schmidt operators as square integrable functions on H. Indeed, let $\mathcal{L}^2(\mathcal{H})$ be the Hilbert space of the Hilbert-Schmidt operators with the scalar product

$$(A, B) \mapsto \mathrm{Tr}\,(A^* B),$$

where $\mathrm{Tr}\,(\cdot)$ denotes the trace and A^* is the adjoint operator of A. If $u, v \in \mathcal{H}$, let $u \otimes v^*$ be the operator in $\mathcal{L}^2(\mathcal{H})$

$$(u \otimes v^*)(w) = \langle v, w \rangle\, u \qquad w \in \mathcal{H}.$$

Given a section c, we define $\Sigma(u \otimes v^*)$ as the function from H to $\mathbb{C}$ given by

$$\Sigma(u \otimes v^*)(h) = \left\langle U_{c(h)} v, u \right\rangle \qquad h \in H.$$

From Eq. (4), it follows that $\Sigma(u \otimes v^*)$ is square-integrable with respect to dh and

$$\left\| \Sigma(u \otimes v^*) \right\|^2_{L^2(H,dh)} = \frac{1}{d} \left\| u \right\|^2 \left\| v \right\|^2 = \frac{1}{d} \left\| u \otimes v^* \right\|^2_{\mathcal{L}^2(\mathcal{H})}.$$

Taking into account that the set $\{u \otimes v^* \ :\ u, v \in \mathcal{H}\}$ is total in $\mathcal{L}^2(\mathcal{H})$, it follows that Σ is defined uniquely by continuity on $\mathcal{L}^2(\mathcal{H})$ and, if $A, B \in \mathcal{L}^2(\mathcal{H})$,

$$\mathrm{Tr}\,(A^* B) = d \left\langle \Sigma(A), \Sigma(B) \right\rangle. \tag{5}$$

Moreover, if A is of trace-class, then for almost all $h \in H$

$$\Sigma(A)(h) = \mathrm{Tr}\,(U_{c(h)^{-1}} A). \tag{6}$$

Indeed, let

$$A = \sum_i \lambda_i e_i \otimes f_i^*$$

be the canonical decomposition of A, where (e_i) and (f_i) are orthonormal sequences in $\mathcal{H}$, (λ_i) is an ℓ_1-sequence and the series converges in trace-norm and, hence, in the Hilbert-Schmidt norm. Since Σ is continuous, then

$$\Sigma(A) = \sum_i \lambda_i \Sigma(e_i \otimes f_i^*),$$

where the series converges in $L^2(H, dh)$. On the other hand, fixed $h \in H$, since A is of trace class, so is $U_{c(h)^{-1}} A$, hence

$$\begin{aligned}
\mathrm{Tr}\,(U_{c(h)^{-1}} A) &= \sum_i \left\langle f_i, U_{c(h)^{-1}} A e_i \right\rangle \\
&= \sum_i \lambda_i \left\langle U_{c(h)} f_i, e_i \right\rangle \\
&= \sum_i \lambda_i \Sigma(e_i \otimes f_i^*)(h),
\end{aligned}$$

where the series converges pointwise. The claim is now clear.

We are now ready to define a quorum associated with the square-integrable (modulo K) representation U of G.

Let T be a state of $\mathcal{H}$, *i.e.* a positive trace-class operator of trace one, and A a Hilbert-Schmidt operator on $\mathcal{H}$. Taking into account Eq. (5) and Eq. (6),

$$\begin{aligned}\mathrm{Tr}\,(TA) &= d\,\langle \Sigma(T), \Sigma(A)\rangle_{L^2(H,dh)} \\ &= d \int_H \overline{\mathrm{Tr}\,(U_{c(h)^{-1}}T)}\,\Sigma(A)(h)\,dh,\end{aligned}$$

so that

$$\mathrm{Tr}\,(AT) = d \int_H \Sigma(A)(h)\,\mathrm{Tr}\,(TU_{c(h)})\,dh.$$

By means of Eq. (2), the above equation becomes

$$\mathrm{Tr}\,(AT) = d \int_{\mathfrak{H}} \Sigma(A)(\exp v)\,\mathrm{Tr}\,(TU_{c(\exp v)})\chi_V(v)|\det\,(d(\exp)_v)|\,dv.$$

Let S^m be the sphere in $\mathfrak{H}$. Then, for all $n \in S^m$, the map

$$t \mapsto U_{c(\exp\,(tn))}$$

is a projective representation of $\mathbb{R}$. Since all the multipliers of $\mathbb{R}$ are equivalent to an exact one, there is a selfadjoint unbounded operator Q_n and a measurable complex function α_n with modulo 1 such that, for all $t \in \mathbb{R}$,

$$U_{c(\exp\,(tn))} = \alpha_n(t) e^{itQ_n}. \tag{7}$$

Using polar coordinates in the above equation, one has that

$$\begin{aligned}\mathrm{Tr}\,(AT) &= dC_m \int_{S^m} d\Omega(n) \int_0^\infty dt\; t^m \Sigma(A)(\exp\,(tn))\alpha_n(t) \\ &\qquad \mathrm{Tr}\,(Te^{itQ_n})\chi_V(tn)|\det\,(d(\exp)_{tn})|,\end{aligned} \tag{8}$$

where $d\Omega$ is the normalized measure on the sphere S^m, C_m is the volume of S^m and dt is the Lebesgue measure on the real line. The set of self-adjoint operators $\{Q_n \,: n \in S^m\}$, labelled by the probability space $(S^m, d\Omega)$, is called the *quorum* defined by the representation U. We notice that Eq. (7) defines Q_n uniquely up to an additive constant, see, also, Remark 3 below.

Since Q_n is selfadjoint, by means of the spectral theorem, there is a projection valued measure $E \mapsto P_n(E)$ defined on $\mathbb{R}$ such that

$$\mathrm{Tr}\,(TQ_n) = \int_{\mathbb{R}} \lambda\; d\mathrm{Tr}\,(TP_n(\lambda)),$$

where $d\mathrm{Tr}\,(TP_n(\lambda))$ denotes the positive bounded measure on $\mathbb{R}$

$$E \mapsto \mathrm{Tr}\,(TP_n(E)).$$

Using this equation, one obtains that

$$\begin{aligned}\mathrm{Tr}\,(AT) &= dC_m \int_{S^m} d\Omega(n) \int_0^\infty dt \int_{\mathbb{R}} d\mathrm{Tr}\,(TP_n(\lambda)) \\ &\qquad e^{i\lambda t}\Sigma(A)(\exp\,(tn))\alpha_n(t)\chi_V(tn)|\det\,(d(\exp)_{tn})|t^m.\end{aligned} \tag{9}$$

In order to obtain a reconstruction formula for $\mathrm{Tr}\,(AT)$, we would like to interchange the integrals in dt and in $d\mathrm{Tr}\,(TP_n(\lambda))$.

We consider first the case when $\Sigma(A)$, which is only square-integrable, is in fact integrable with respect to dh, *i.e.*

$$\int_H |\Sigma(A)(h)|\, dh < \infty. \tag{10}$$

By means of Fubini theorem, this condition implies that, for almost all $n \in S^m$, the map $t \mapsto \Sigma(A)(\exp(tn))$ is integrable with respect to the measure

$$dt_n = \chi_V(tn)|\det(d(\exp)_{tn})|t^m dt. \tag{11}$$

Then the map from $S^m \times \mathbb{R}$ to $\mathbb{C}$

(12)

$$\sigma(A)(n,\lambda) = dC_m \int_0^\infty e^{i\lambda t}\Sigma(A)(\exp(tn))\alpha_n(t)\chi_V(tn)|\det(d(\exp)_{tn})|t^m\, dt,$$

is well-defined and it is called the *estimator* of the observable A. We notice that the estimator does not depend on T and, given the representation U, can be computed analytically.

Since the measure $d\mathrm{Tr}\,(TP_n(\lambda))$ is bounded, by means of Fubini theorem, one can interchange the integrals in Eq. (9) obtaining

$$\mathrm{Tr}\,(AT) = \int_{S^m} d\Omega(n) \int_0^\infty d\mathrm{Tr}\,(TP_n(\lambda))\ \sigma(A)(n,\lambda). \tag{13}$$

The above integral transform is the core of the *quantum tomography* and is a concrete realization of the scheme proposed in the introduction, compare with Eq. (1). Indeed, $d\Omega(n)d\mathrm{Tr}\,(TP_n(\lambda))$ is the probability to obtain the value λ when the observable Q_n, chosen randomly in the quorum according to $d\Omega$, is measured. Moreover, by means of Eq. (13), the expectation value $\mathrm{Tr}\,(AT)$ can be reconstructed as average of the estimator $\sigma(A)$ over many random measures of the observables Q_n in the quorum.

Remark 3. There is a choice for the section that simplifies the expression of the estimator. Indeed, denoted by $\mathfrak{G}$ the Lie algebra of G, since the differential $d\pi$ of π is a surjective linear map from $\mathfrak{G}$ onto $\mathfrak{H}$, there is an injective linear map j from $\mathfrak{H}$ to $\mathfrak{G}$ such that $d\pi(j(v)) = v$ for all $v \in \mathfrak{H}$. Since exp is a diffeomorphism from V onto $\exp(V)$, it is well defined a smooth map $\hat{c}$ from $\exp(V)$ to G such that

$$\hat{c}(\exp(v)) = \exp(j(v)) \qquad v \in \mathfrak{H}.$$

Clearly $\hat{c}$ is a section and the relation $U_{\hat{c}(\exp(tn))} = U_{\exp(tj(n))}$ shows that one can always choose $\alpha_n(t) = 1$ in Eq. (7). Hence $U_{\hat{c}(\exp(tn))} = e^{itQ_n}$.

One can easily prove that, if one changes $j \mapsto j + l$ in such a way that $d\pi(j(v) + l(v)) = v$, then the quorum transforms according to $Q_n \mapsto Q_n + q_n I$. However, in most of the cases, there is a *natural* choice for the map j, so that the quorum Q_n is, in fact, defined uniquely by the representation U.

Remark 4. Once the quorum $\{Q_n\}$ is fixed, Eq. (12) is independent of the choice of the section c. Indeed if c' is another section, then, for all $h \in H$, $c'(h) = k(h)c(h)$ and $k(h) \in K$. Since K is central in G and U is irreducible, then $U_{k(h)} = \beta(h)I$, where $\beta(h)$ is a complex number of modulo one. Hence, with obvious notations, for almost all $h \in H$ and for all $t \in \mathbb{R}$,

$$\begin{aligned} \Sigma'(A)(h) &= \overline{\beta(h)}\Sigma(A)(h) \\ {\alpha'}_n(t) &= \beta(h)\alpha_n(t), \end{aligned}$$

so that $\sigma(A)$ is invariant with respect to the change $c \mapsto c'$.

Remark 5. If A is of trace class and satisfies Eq. (10), using Eq. (6) one obtains a more explicit formula for the estimator of A

$$\sigma(A)(n,\lambda) = dC_m \int_0^\infty e^{i\lambda t}\mathrm{Tr}\,(Ae^{-itQ_n})\chi_V(tn)|\det(d(\exp)_{tn})|t^m\, dt.$$

Moreover, in most examples the set V is sufficiently nice so that the map $n \mapsto \chi_V(tn)$ is continuous for almost all $t \in \mathbb{R}$. In this case, if one chooses the section $\hat{c}$ as in Remark 3, taking into account that the function $g \mapsto \mathrm{Tr}\,(TU_g)$ is continuous (since the ultra-weak operator topology is equivalent to the weak operator topology on the unit ball of $\mathcal{L}(\mathcal{H})$), it follows that the estimator $\sigma(A)$ is continuous on $S^m \times \mathbb{R}$. This property is important in order to approximate the integral of Eq. (13) by a finite sum.

Remark 6. We notice that this procedure is *unbiased* since the observables Q_n are chosen randomly and the integral given by Eq. (13) can be approximated by a finite sum since $d\Omega(n)d\mathrm{Tr}\,(TP_n(\lambda))$ is a probability measure. This means that this approach is not affected by the systematic errors that were present in the first tomographic scheme [1], [2] due to the cutoff needed in the inversion of the Radon transform, see [3].

Remark 7. If H is compact then dh is finite and any irreducible representation is square-integrable. Since the Hilbert space $\mathcal{H}$ where the representation acts is finite dimensional, $\mathcal{L}^2(\mathcal{H})$ coincides with the space of the all operators. Moreover, since $L^2(H,dh) \subset L^1(H,dh)$, Eq. (10) holds for every operator.

Remark 8. If U is an integrable representation (modulo K), there exists a dense set S in $\mathcal{H}$ such that, if $u, v \in S$, then $\Sigma(u \otimes v^*)$ satisfies Eq. (10).

If condition (10) does not hold, it may happen that, for a non-negligible set of $n \in S^m$, the map $t \mapsto \Sigma(A)(\exp(tn))$ is not integrable with respect to the measure dt_n defined by Eq. (11) (it is only square-integrable), so that the estimator $\sigma(A)$ given by Eq. (12) is not well defined.

In these cases, in order to define the estimator one has to use a suitable regularization procedure. For example, fixed $L > O$ let, for all $n \in S^m$ and $\lambda \in \mathbb{R}$,

(14)

$$\sigma_L(A)(n,\lambda) = dC_m \int_0^L e^{i\lambda t}\Sigma(A)(\exp(tn))\alpha_n(t)\chi_V(tn)|\det(d(\exp)_{tn})|t^m\, dt.$$

It may be the case that there exists a function $\sigma(A)$ such that

$$\lim_{L\to\infty}\int_{S^m} d\Omega(n)\int_{\mathbb{R}} d\mathrm{Tr}\,(TP_n(\lambda))\ \sigma_L(A)(n,\lambda) = \int_{S^m} d\Omega(n)\int_{\mathbb{R}} d\mathrm{Tr}\,(TP_n(\lambda))\ \sigma(A)(n,\lambda).$$

Then, as an easy consequence of dominated convergence theorem, one has that

$$\mathrm{Tr}\,(AT) = \int_{S^m} d\Omega(n)\int_{\mathbb{R}} d\mathrm{Tr}\,(TP_n(\lambda))\ \sigma(A)(n,\lambda).$$

Analogous regularization procedures could be used to extend $\Sigma(A)$ to non-Hilbert-Schmidt operators. Although this problem is physically relevant (many observables of interest are unbounded) it is out of the scope of the present paper.

3. EXAMPLES

3.1. **The Weyl-Heisenberg group.**

Let G be the Weyl-Heisenberg group, *i.e.* $G=\mathbb{R}^3$ with the composition law

$$(\eta_1,a_1,b_1)(\eta_2,a_2,b_2) = (\eta_1+\eta_2+\frac{b_1a_2-a_1b_2}{2},a_1+a_2,b_1+b_2).$$

It is known that G is a connected simply-connected nilpotent unimodular Lie group.

The set $K=\{(\eta,0,0)\ :\ \eta\in\mathbb{R}\}$ is clearly a central closed subgroup of G and the quotient group $H=G/K$ can be identified with the vector group $\mathbb{R}^2$. One has the following facts.

1. The canonical projection π is given by $\pi(\eta,a,b)=(a,b)$.
2. A smooth section c is given by $c(a,b)=(0,a,b)$.
3. A Haar measure on H is the Lebesgue measure $da\,db$ of $\mathbb{R}^2$.
4. The Lie algebra $\mathfrak{H}$ of H can be identified with $\mathbb{R}^2$ so that the exponential map is the identity and, for all $v\in\mathfrak{H}$, $\det\,(d(\exp)_v)=1$.
5. The constant D in Eq. (2) is equal to 1.

It follows that the choice $V=\mathfrak{H}$ satisfies the assumptions of the previous section.

Let U be the representation of G acting in $\mathcal{H}=L^2(\mathbb{R},dx)$ as

$$(U_{(\eta,a,b)}u)(x) = e^{i(\eta+\frac{ab}{2})}e^{ixa}u(x+b)$$

where $x\in\mathbb{R}$, $u\in L^2(\mathbb{R},dx)$ and $(\eta,a,b)\in G$. It is known that U is a unitary continuous irreducible representation of G, called the *Schrödinger representation.*

We prove that U is square-integrable modulo K. Given $u\in\mathcal{H}$, the map

$$(x,b)\mapsto \overline{u(x+b)}u(x)$$

is measurable and

$$\int_{\mathbb{R}} dx\int_{\mathbb{R}} db|\overline{u(x+b)}u(x)|^2 = \|u\|^4<\infty.$$

By Fubini theorem, for almost all $b \in \mathbb{R}$, the map

$$x \mapsto \overline{u(x+b)}u(x)$$

is square-integrable and, since both $x \mapsto u(x+b)$ and $x \mapsto u(x)$ are square-integrable, it is integrable. Then, the map

$$a \mapsto \int_{\mathbb{R}} dx e^{-iax} \overline{u(x+b)}u(x)$$

is well defined and square-integrable with respect to da for almost all $b \in \mathbb{R}$. Moreover, the map

$$(a,b) \mapsto \int_{\mathbb{R}} dx e^{-iax} \overline{u(x+b)}u(x)$$

is measurable and, by means of the isometry of the Fourier transform,

$$\begin{aligned} \int_{\mathbb{R}} db \int_{\mathbb{R}} da \left| \int_{\mathbb{R}} dx e^{-iax} \overline{u(x+b)}u(x) \right|^2 &= 2\pi \int_{\mathbb{R}} db \int_{\mathbb{R}} dx |\overline{u(x+b)}u(x)|^2 \\ &= 2\pi \left\|u\right\|^4 . \end{aligned}$$

Since

$$\left\langle U_{c(a,b)}u, u \right\rangle = \int_{\mathbb{R}} dx e^{-iax} \overline{u(x+b)}u(x),$$

by Fubini theorem one has that U is square-integrable and the formal degree is $d = \frac{1}{2\pi}$.

Since U is square-integrable modulo K, according to Section 2, it defines a quorum. In order to explicit it, we observe that, with the notation of the previous section,

$$S^m = \{n_\Phi := (\cos(\Phi), \sin(\Phi)) \ : \ \Phi \in [0, 2\pi]\},$$

$m = 1$, $C_1 = 2\pi$ and $d\Omega = \frac{d\Phi}{2\pi}$. Moreover, since $t \mapsto U_{c(tn_\Phi)}$ is a one parameter subgroup, then

$$U_{c(tn_\Phi)} = e^{itY_\Phi}$$

where Y_Φ is a selfadjoint operator (in this example $\alpha_{n_\Phi}(t) = 1$). If u is a Schwartz function, one has that

$$Y_\Phi u = \cos(\Phi)Qu + \sin(\Phi)Pu,$$

where Q is the multiplicative operator by x, *i.e.* the position operator, and P is $-i$ times the weak derivative operator, *i.e.* the momentum operator. Hence the quorum defined by U is given by the set of self-adjoint operators

$$\{Y_\Phi \ : \ \Phi \in [0, 2\pi]\}$$

labelled by the space $[0, 2\pi]$ with the uniform measure $\frac{d\Phi}{2\pi}$.

The above quorum has the following property. For all $\Phi \in [0, 2\pi]$, there is a unitary operator W_Φ such that

$$Y_\Phi = W_\Phi Q W_\Phi^{-1}. \tag{15}$$

To prove it, given $\Phi \in [0, 2\pi]$, let f_Φ from G to G

$$f_\Phi(\eta, a, b) = (\eta, \cos(\Phi)a - \sin(\Phi)b, \sin(\Phi)a + \cos(\Phi)b) .$$

One can easily check that f_Φ is a continuous group automorphism of G, so that $g \mapsto U^{f_\Phi}$ is a unitary irreducible continuous representation of G and the restriction to K is the character $\eta \mapsto e^{i\eta}$. From the unicity of the Schrödinger representation, it follows that there exists a unitary operator W_Φ such that

$$U^{f_\Phi} = W_\Phi U W_\Phi^{-1}.$$

Then

$$U_{c(tn_\Phi)} = U^{f_\Phi}_{(0,t,0)} = W_\Phi U_{(0,t,0)} W_\Phi^{-1},$$

and Eq. (15) follows by Stone theorem.

Let now T be a state of $\mathcal{H}$. Recalling that the spectral measure P_Q of Q is the one given by the multiplicative operators by characteristic functions, then, by means of Eq. (15), for each $\Phi \in [0, 2\pi]$ there is a $L^1(\mathbb{R}, d\lambda)$-function $\lambda \mapsto \omega(\Phi, \lambda)$ such that

$$\mathrm{Tr}\,(TP_\Phi(E)) = \mathrm{Tr}\,(W_\Phi^{-1} T W_\Phi P_Q(E)) = \int_E \omega(\Phi, \lambda) d\lambda,$$

where $E \mapsto P_\Phi(E)$ is the spectral measure associated with Y_Φ. The map ω can always be chosen measurable as a function on $[0, 2\pi] \times \mathbb{R}$ and, in doing so, it is a probability density on $[0, 2\pi] \times \mathbb{R}$ with respect to the measure $\frac{d\Phi}{2\pi} d\lambda$.

Finally, fix a Hilbert-Schmidt operator A in $\mathcal{H}$ such that $\Sigma(A)$ is integrable with respect to $da\, db$. According to Eq. (12), the estimator of A is, if $\Phi \in [0, 2\pi]$ and $\lambda \in \mathbb{R}$,

$$\sigma(A)(\Phi, \lambda) = \int_0^\infty t\Sigma(A)(t\cos(\Phi), t\sin(\Phi)) e^{i\lambda t}\; dt,$$

and the reconstruction formula Eq. (13) is explicitly given by

$$\mathrm{Tr}\,(AT) = \int_0^{2\pi} \int_{\mathbb{R}} \sigma(A)(\Phi, \lambda)\omega(\Phi, \lambda)\; \frac{d\Phi}{2\pi} d\lambda.$$

The representation U is actually integrable and, if (u_n) is the basis of eigenvectors of the number operator, then $\Sigma(u_n \otimes u^*_{n+l}) \in L^1(H, dh)$ and one has the explicit formula

$$\sigma(u_n \otimes u^*_{n+l})(\Phi, \lambda) = \frac{(-i)^l}{2^{\frac{l}{2}}} \sqrt{\frac{n!}{(n+l)!}} e^{il\Phi} \int_0^\infty t^{l+1} L_n^l(\frac{t^2}{2}) e^{-\frac{t^2}{4} + i\lambda t}\; dt, \tag{16}$$

where L^k_m are the associated Laguerre polynomials. The statistical reliability of Eq. (16) has been verified in [3].

This example is physically realized by the homodyne tomography, [5]. The quantum system is the harmonic oscillator representing a single mode of the e.m. field with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. In terms of such operators, one has the following *translation table*

$$\begin{aligned} Q &= \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}} \\ P &= \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i} \\ U_{(\eta,a,b)} &= e^{i\eta} e^{\left(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}\right)} \\ Y_\Phi &= \sqrt{2}\frac{\hat{a}^\dagger e^{i\Phi} + \hat{a}e^{-i\Phi}}{2} =: \sqrt{2}X_\Phi \end{aligned}$$

where $\alpha = \frac{-b+ia}{\sqrt{2}} \in \mathbb{C}$, $e^{\left(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}\right)}$ is *displacement group* and X_Φ is the quadrature with phase $\Phi \in [0, 2\pi]$.

The measuring apparatus is a homodyne detector with tunable phase with respect to the local oscillator. The function $\sqrt{2}\omega(\Phi, \sqrt{2}\lambda)$ is the probability density (with respect to $d\lambda$) to obtain the value λ measuring the quadrature X_Φ, chosen randomly according to the measure $\frac{d\Phi}{2\pi}$. Moreover, the explicit form of the estimator of A, being A of trace class, is

$$\sigma(A)(\Phi, \sqrt{2}\lambda) = \frac{1}{2}\int_0^\infty \mathrm{Tr}\,(Ae^{-it(X_\Phi-\lambda)})tdt.$$

One could consult [14] for an example of an experimental realization of the above tomographic method.

Remark 9. In this example one is able to obtain an estimator also for for monomials in $\hat{a}$ and $\hat{a}^\dagger$, [13, 6]. For example, one has that

$$\sigma(\hat{a}^\dagger\hat{a})(\Phi, \sqrt{2}\lambda) = 2\lambda^2 - \frac{1}{2}.$$

3.2. **The group $SU(2)$.** Let $SU(2)$ be the group of the unitary 2×2 complex matrices with determinant 1. It is a unimodular connected simply-connected compact Lie group. The corresponding Lie algebra is

$$su(2) = \{\frac{i}{2}(x\sigma_1 + y\sigma_2 + z\sigma_3) \ : \ x, y, z \in \mathbb{R}\}$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the following we identify $su(2)$ with $\mathbb{R}^3$ using the basis $(\frac{i\sigma_k}{2})_{k=1}^3$. Let $V = \{(x, y, z) \in \mathbb{R}^3 \ : \ \sqrt{(x^2+y^2+z^2)} < 2\pi\}$, it is known that V is an open neighborhood of 0 such that the exponential map restricted to V is a diffeomorphism from V onto the open set $\exp(V)$ and the complement of $\exp(V)$ is negligible with respect to the Haar measure of $SU(2)$. Moreover one can check that

$$d(exp)_{(x,y,z)} = 4\frac{\sin^2(\frac{\sqrt{x^2+y^2+z^2}}{2})}{x^2+y^2+z^2}.$$

If we choose the Haar measure on $SU(2)$ in such a way that the constant D in Eq. (2) is 1 one has that

$$\int_H 1\, dh = \int_V |d(exp)_{(x,y,z)}| dxdydz = 16\pi^2, \tag{17}$$

(usually the Haar measure on compact groups is normalized to 1).

Given j such that $2j \in \mathbb{N}$, let D^j be the irreducible representation of $SU(2)$ acting on $\mathcal{H} = \mathbb{C}^{2j+1}$. Since the group is compact, D^j is square-integrable and the space of the Hilbert-Schmidt operators coincides with the space of all operators $\mathcal{L}(\mathbb{C}^{2j+1})$.

Since the measure of $SU(2)$ is normalized according to Eq. (17), it is well known that the formal degree is $d = \frac{2j+1}{16\pi^2}$, see for example [12].

For all $n \in S^2$, define J_n as the hermitian matrix such that

$$D^j(\exp(tn)) = e^{itJ_n} \qquad t \in \mathbb{R}.$$

Then, the quorum defined by D^j is the set of spin operators $\{J_n \ : \ n \in S^2\}$ labelled by the space S^2 with the measure $\frac{dn}{4\pi}$, being dn the area element of the sphere. It is known that the (simple) eigenvalues of each J_n are $\lambda = -j, \cdots, j$ and there exists a unitary operator W_n, unique up to a phase, such that

$$Q_n = {W_n}^{-1} J_z W_n,$$

where $J_z = J_{(0,0,1)}$.

Let now $A \in \mathcal{L}(\mathbb{C}^{2j+1})$, then, according to Eq. (12) and taking into account that $C_2 = 4\pi$, the corresponding estimator is

$$\sigma(A)(n,\lambda) = \frac{2j+1}{\pi} \int_0^{2\pi} e^{i\lambda t} \mathrm{Tr}\,(Ae^{-itJ_n}) \sin^2(\frac{t}{2}) dt,$$

where $n \in S^2$ and $\lambda = -j, \cdots, j$. Equation (13) becomes

$$\mathrm{Tr}\,(TA) = \sum_{\lambda=-j}^{j} \int_{S^2} \sigma(A)(n,\lambda) |\langle W_n e_\lambda, TW_n e_\lambda \rangle|^2 \frac{dn}{4\pi},$$

where $(e_\lambda)_{\lambda=-j}^{j}$ is a basis of eigenvectors of J_z.

This example is realized experimentally by a Stern-Gerlach machine. The quantum system is the spin degree of freedom of an elementary particle with spin j and the number $|\langle W_n e_\lambda, TW_n e_\lambda \rangle|^2$ is the probability to obtain the value λ measuring the spin along the axis n, chosen randomly according to the measure $\frac{dn}{4\pi}$.

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