SQUARE-INTEGRABILITY OF INDUCED REPRESENTATIONS OF SEMIDIRECT PRODUCTS

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Abstract. We consider a semidirect product $A \times' H$, with A abelian, and its unitary representations of the form $\mathrm{Ind}_{G_0}^{G}(x_0 m)$ where x_0 is in the dual group of A, G_0 is the stability group of x_0 and m is an irreducible unitary representation of $G_0 \cap H$. We give a new selfcontained proof of the following result: the induced representation $\mathrm{Ind}_{G_0}^{G}(x_0 m)$ is square-integrable if and only if the orbit $G[x_0]$ has nonzero Haar measure and m is square-integrable. Moreover we give an explicit form for the formal degree of $\mathrm{Ind}_{G_0}^{G}(x_0 m)$.

1. Introduction

In the paper [6], Kleppner and Lipsman describe the Plancherel measure for a locally compact group with a closed normal Type I subgroup. As a consequence, they characterise the square-integrability of unitary representations of such groups.

The study of Plancherel measure for nonunimodular groups has been continued by Duflo and Moore, [3]. They use as a fundamental tool for their investigation the notion of operator of formal degree for square-integrable representations.

In our paper we use this notion and some of its most elementary properties to obtain a new selfcontained and more accessible proof of the characterisation of square-integrable representations of semidirect products with an abelian normal factor.

More precisely, we consider a semidirect product G of a *locally compact second countable* (lcsc) topological abelian group A and a lcsc topological group H. We study the representations of the form $\mathrm{Ind}_{G_0}^{G}(x_0 m)$ where Ind is the Mackey unitary induction, x_0 is an element in $\hat{A}$, the dual group of A, G_0 is the stability group of x_0 with respect to the dual action of G on $\hat{A}$ and m is an irreducible unitary representation of $G_0 \cap H$. We prove that $\mathrm{Ind}_{G_0}^{G}(x_0 m)$ is square-integrable if and only if the orbit $G[x_0]$ has nonzero measure with respect to the Haar measure on $\hat{A}$ and m is square-integrable. Moreover, we give an explicit description of the formal degree of $\mathrm{Ind}_{G_0}^{G}(x_0 m)$.

The need of an elementary proof of this result comes from the role that square-integrable representations have in the construction of generalised coherent states, [1], and in wavelet analysis, [2], [8] (see the final section for

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a brief review of these applications). Indeed, quite recently, some authors, [2], [4], have considered the case of subrepresentations of the quasiregular representation on $L^2(\mathbb{R}^n)$ for semidirect products of the form $\mathbb{R}^n \times' H$ where H is a closed subgroup of $GL_n(\mathbb{R})$, obtaining, with an elementary proof, some of the results in [6] (nevertheless, it seems that these authors are not aware of this paper).

2. Preliminaries

Let G be a locally compact topological group. We denote by $\mathcal{K}(G)$ the vector space of continuous functions of compact support on G and by $\mathcal{B}(G)$ the Borel σ-algebra of G. By Borel measure on G we mean a positive measure defined on $\mathcal{B}(G)$ and finite on the compact sets. Let μ_G be a left Haar measure on G and Δ_G its modular function.

In this paper we will use the word *representation* to mean a continuous irreducible unitary representation of G acting in a complex separable Hilbert space. A representation U of G in $\mathcal{H}$ is *square-integrable* if there exist two nonzero vectors $\phi, \psi \in \mathcal{H}$ such that the map $c_{\phi,\psi}$ from G to $\mathbb{C}$

$$c_{\phi,\psi}(g) = \langle \phi, U_g \psi \rangle$$

is in $L^2(G, \mu_G)$.

In the following we will use this result of Duflo and Moore (Theorem 3 of [3]).

Theorem 1 (Duflo and Moore)**.** *Let U be a square-integrable representation of G acting in $\mathcal{H}$. There exists a unique positive selfadjoint operator K_U in $\mathcal{H}$ such that*

1. *if $\phi, \psi \in \mathcal{H}$, $\phi \neq 0$ the map $c_{\phi,\psi}$ is in $L^2(G, \mu_G)$ if and only if $\psi \in \mathrm{D}om\, K_U^{-1/2}$;*
2. *if $\phi, \phi' \in \mathcal{H}$ and if $\psi, \psi' \in \mathrm{D}om\, K_U^{-1/2}$, then*

$$\left\langle c_{\phi,\psi}, c_{\phi',\psi'} \right\rangle_{L^2(G,\mu_G)} = \left\langle \phi, \phi' \right\rangle_{\mathcal{H}} \left\langle K_U^{-1/2}\psi', K_U^{-1/2}\psi \right\rangle_{\mathcal{H}}.$$

The operator K_U is called the *formal degree* of the representation.

From now on we let G be a lcsc topological group which is semidirect product of a closed abelian normal subgroup A and a closed subgroup H

$$G = A \times' H.$$

The inner action of G on A is

$$g[a] = gag^{-1}, \qquad a \in A,\ g \in G.$$

We denote by $\hat{A}$ the dual group of A, which is in a natural way an abelian lcsc topological group; G acts continuously on it by

$$g[x](a) = x(g^{-1}ag), \qquad a \in A,\ x \in \hat{A},\ g \in G.$$

We fix an element x_0 of $\hat{A}$ and we denote by X the corresponding orbit

$$X = \{x \in \hat{A} \ : \ x = g[x_0], \ g \in G\}$$

and by G_0 the stability subgroup of x_0,

$$G_0 = \{g \in G \ : \ g[x_0] = x_0\}.$$

We recall some properties of X as a measurable space.

1. X is Borel set of $\hat{A}$ and we endow it with the σ-algebra
$$\mathcal{A} := \{E \in \mathcal{B}(\hat{A}) \ : \ E \subset X\}.$$
X is a standard measurable space with respect to this σ-algebra and it is a transitive H-space with respect to the restriction of the action of G.
2. there exists a σ-finite H-quasi-invariant measure ν on $(X, \mathcal{A})$, that is,
$$\nu(h[E]) \quad = \quad \int_E \lambda_\nu(h,x) d\nu(x) \qquad h \in H, \ E \in \mathcal{A}$$
where λ_ν is a measurable function from $H \times X$ to $(0, \infty)$ (see, for example, Theorem 5.19, of [9]);
3. there exists a measurable map $q : X \to H$ such that
$$\begin{aligned} q(x_0) &= e \qquad \text{where } e \text{ is the identity}, \\ q(x)[x_0] &= x, \qquad x \in X, \end{aligned}$$
and, for any compact set K in H, the set
$$\{q(h[x_0]) \ : \ h \in K\}$$
has compact closure in H (see, for example, Theorem 5.11, of [9]).

Let $H_0 = G_0 \cap H$, so that $G_0 = A \times' H_0$. We choose a representation m of H_0 acting in a complex separable Hilbert space $\mathcal{K}$ and we consider the representation $x_0 m$ of G_0 in $\mathcal{K}$

$$(x_0 m)(ah) := x_0(a) m(h) \qquad a \in A, h \in H_0.$$

We denote by $\mathrm{I}nd_{G_0}^{G}(x_0 m)$ the representation unitarily induced by $x_0 m$ from G_0 to G, [7]. If the semidirect product is *regular*, any irreducible unitary representation of G is equivalent to a representation of the previous form.

3. Main results

The main result of the paper is the following theorem (compare with Corollary 11.1 of [6] and Proposition 9 of [3]).

Theorem 2. *Let G be a lcsc topological group, semidirect product of an abelian normal closed subgroup A and a closed subgroup H. Let $x_0 \in \hat{A}$, X be the orbit $G[x_0]$, G_0 be the stability subgroup of x_0 and m a representation of $G_0 \cap H$. Then the representation $\mathrm{Ind}_{G_0}^{G}(x_0 m)$ is square-integrable if and only if $\mu_{\hat{A}}(X) \neq 0$ and m is square-integrable.*

The results obtained in [2] and [4] follow easily from

Corollary 1. *The representation* $\mathrm{Ind}_{G_0}^{G}(x_0 I)$*, where* I *is the trivial representation of* $G_0 \cap H$ *on* $\mathbb{C}$*, is square-integrable if and only if* $\mu_{\hat{A}}(X) \neq 0$ *and* $G_0 \cap H$ *is compact.*

We notice that, under the hypotheses of the previous corollary, the representation $\mathrm{Ind}_{G_0}^{G}(x_0 I)$ is equivalent to a subrepresentation of the *quasi-regular* representation of G in $L^2(A, \mu_A)$, regarding A as the quotient space G/H.

The representation $\mathrm{I}nd_{G_0}^{G}(x_0 m)$ can be realised in various unitarily equivalent ways. Since the square-integrability depends only on the equivalence class of the representation, we can freely choose the explicit form of $\mathrm{I}nd_{G_0}^{G}(x_0 m)$.

Fixed a H-quasi-invariant measure ν and a measurable map $q : X \to H$ as in points 2 and 3 of section 2, a suitable choice for $\mathrm{I}nd_{G_0}^{G}(x_0 m)$ is the representation U acting in $L^2(X, \nu, \mathcal{K})$ as

$$(U_{ah}\phi)(x) = (\lambda_\nu(h^{-1}, x))^{1/2} x(a)\, m(q(x)^{-1} h q(h^{-1}[x]))\phi(h^{-1}[x]) \tag{1}$$

where $x \in X$, $ah \in G$ and $\phi \in L^2(X, \nu, \mathcal{K})$. Taking into account that the Hilbert space $L^2(X, \nu, \mathcal{K})$ can be identified with $L^2(X, \nu) \otimes \mathcal{K}$, we have

Corollary 2. *If* U *is square-integrable, its formal degree is*

$$K_U = K_0 \otimes K_m$$

where K_m *is the formal degree of* m *and* K_0 *is the selfadjoint operator in* $L^2(X, \nu)$

$$(K_0\phi)(x) = \Delta_G(q(x))\phi(x), \qquad \phi \in L^2(X, \nu)$$

on its natural domain.

4. Proof of the theorem

In this section, we collect the proofs of the theorem and of its corollaries. We begin with two technical lemmas on the Haar measures on A, $\hat{A}$ and G.

Lemma 1. *For any* $h \in H$ *and* $E \in \mathcal{B}(A)$ *we have*

$$\mu_A(h[E]) = \rho(h)\mu_A(E)$$

where $\rho : H \to (0, \infty)$ *is a continuous group homomorphism. Moreover the Haar measure* μ_G *is*

$$\mu_G = \mu_A \otimes \frac{1}{\rho}\mu_H$$

and the modular function of G *satisfies*

$$\Delta_G(ah) = \frac{\Delta_H(h)}{\rho(h)}, \qquad a \in A,\ h \in H.$$

Proof. Let $h \in H$. Since $a \mapsto h[a]$ is a topological isomorphism, then

$$\mathcal{B}(A) \ni E \mapsto \mu_A(h[E]) \in [0, \infty]$$

is a Haar measure on A so that there exists a strictly positive constant $\rho(h)$ such that

$$\mu_A(h[E]) = \rho(h)\mu_A(E)$$

and the map $h \mapsto \rho(h)$ is obviously a group homomorphism. Let $f \in \mathcal{K}(A)$ such that $\int_A f(a)\, d\mu_A(a) = 1$, by the previous formula it follows that

$$\int_A f(h^{-1}[a])\, d\mu_A(a) = \rho(h) \int_A f(a)\, d\mu_A(a) = \rho(h),$$

so that ρ is continuous since the map $h \mapsto \int_A f(h^{-1}[a])\, d\mu_A(a)$ is continuous by the dominated convergence theorem.

Since G is a Hausdorff lcsc topological space, $\mu_A \otimes \frac{1}{\rho}\mu_H$ is a Borel measure. We show that it is left invariant. In fact, if $f \in \mathcal{K}(G)$ and $g = a'h' \in G$, then

$$\begin{aligned}
\int f(gah) d\mu_A(a) \otimes \frac{1}{\rho(h)} d\mu_H(h) &= \int f(a'h'[a]h'h) \frac{1}{\rho(h)} d\mu_A(a) \otimes d\mu_H(h) \\
&= \int f(a'h'[a]h) \frac{\rho(h')}{\rho(h)} d\mu_A(a) \otimes d\mu_H(h) \\
&= \int f(a'ah) \frac{1}{\rho(h)} d\mu_A(a) \otimes d\mu_H(h) \\
&= \int f(ah) \frac{1}{\rho(h)} d\mu_A(a) \otimes d\mu_H(h).
\end{aligned}$$

With a similar computation we show that $\mu_A \otimes \frac{1}{\Delta_H}\mu_H$ is the right Haar measure on G, hence

$$\Delta_G(ah) = \frac{\Delta_H(h)}{\rho(h)}, \qquad a \in A,\ h \in H.$$

This ends the proof. □

Let $M(\hat{A})$ be the Banach space of the complex measures on $\mathcal{B}(\hat{A})$ and $\mathcal{F}$ the Fourier transform on $M(\hat{A})$ defined as $(\mathcal{F}\nu)(a) = \int_{\hat{A}} \overline{x(a)} d\nu(x)$. The same symbol $\mathcal{F}$ denotes the Fourier-Plancherel operator on $L^2(\hat{A}, \mu_{\hat{A}})$. We choose the Haar measure on $\hat{A}$ is such a way that $\mathcal{F}$ is unitary from $L^2(\hat{A}, \mu_{\hat{A}})$ onto $L^2(A, \mu_A)$.

Lemma 2. *For any $h \in H$ and $E \in \mathcal{B}(\hat{A})$ we have*

$$\mu_{\hat{A}}(h[E]) = \rho(h^{-1})\mu_{\hat{A}}(E)$$

Proof. Arguing as in Lemma 1, we conclude that there exists a continuous homomorphism $\alpha : H \to (0, \infty)$ such that

$$\mu_{\hat{A}}(h[E]) = \alpha(h)\mu_{\hat{A}}(E).$$

Let now $f \in L^1(\hat{A}, \mu_{\hat{A}})$ such that $\mathcal{F}(f) \in L^1(A, \mu_A)$. For all $h \in H$, using the Fourier inversion theorem on $L^1(\hat{A}, \mu_{\hat{A}})$, we have

$$\begin{aligned}
f^h(x) := f(h[x]) &= \int_A h[x](a)(\mathcal{F}f)(a)\, d\mu_A(a) \\
&= \int_A x(h^{-1}[a])(\mathcal{F}f)(a)\, d\mu_A(a) \\
&= \int_A x(a)(\mathcal{F}f)(h[a])\rho(h)\, d\mu_A(a) \\
&= \rho(h) \int_A x(a) \int_{\hat{A}} \overline{x(h[a])} f(x)\, d\mu_{\hat{A}}(x)\, d\mu_A(a) \\
&= \rho(h) \int_A x(a) \int_{\hat{A}} \overline{h^{-1}[x](a)} f(x)\, d\mu_{\hat{A}}(x)\, d\mu_A(a) \\
&= \alpha(h)\rho(h) \int_A x(a) \int_{\hat{A}} \overline{x(a)} f(h[x])\, d\mu_{\hat{A}}(x)\, d\mu_A(a) \\
&= \alpha(h)\rho(h) f^h(x),
\end{aligned}$$

where in the last equality we have used the fact that $f^h \in L^1(\hat{A}, \mu_{\hat{A}})$ and $\mathcal{F}(f^h) \in L^1(A, \mu_A)$. Hence $\alpha(h) = \rho(h^{-1})$ for all $h \in H$. □

We now turn to the proof of Theorem 2. From now on let the notations be as in Theorem 2 and U be the realisation of $\mathrm{I}nd_{G_0}^G(x_0 m)$ given by formula (1). We split the proof in various lemmas.

Lemma 3. *If U is square-integrable, then $\mu_{\hat{A}}(X) \neq 0$.*

Proof. If U is square-integrable there exist two nonzero vectors $\phi, \psi \in \mathcal{H}$ such that the map $c_{\phi,\psi}$ is in $L^2(G, \mu_G)$. By the theorem of Fubini this implies that

$$\text{for } \mu_H\text{-almost all } h \qquad A \ni a \mapsto \langle \phi, U_a U_h \psi \rangle_{\mathcal{H}} \in L^2(A, \mu_A)$$

and we have

$$\langle \phi, U_a U_h \psi \rangle_{\mathcal{H}} = \int_X \overline{x(a)}\, \langle \phi(x), (U_h\psi)(x) \rangle_{\mathcal{K}}\, d\nu(x).$$

Since the function $X \ni x \mapsto \langle \phi(x), (U_h\psi)(x) \rangle_{\mathcal{K}}$ is ν-integrable, we can define the complex measure ν_h on X which has density $\langle \phi(x), (U_h\psi)(x) \rangle_{\mathcal{K}}$ with respect to ν. Suppose that $|\nu_h|(X) = 0$ (where $|\nu_h|$ is the total variation of ν_h) for μ_H-almost any h, then the map

$$X \ni x \mapsto \langle \phi(x), (U_h\psi)(x) \rangle_{\mathcal{K}}$$

is ν-negligible, hence $c_{\phi,\psi}(ah) = 0$ for all $a \in A$ and for μ_H-almost all h, that is, $c_{\phi,\psi} = 0$ in $L^2(G, \mu_G)$ and this is a contradiction, since U is irreducible and $\phi, \psi \neq 0$.

Hence there exists at least one $h \in H$ such that $|\nu_h|(X) \neq 0$ and

$$A \ni a \mapsto \langle \phi, U_a U_h \psi \rangle_{\mathcal{H}} \in L^2(A, \mu_A). \tag{2}$$

Fix this h.

The measure ν_h has a canonical extension to a complex measure on $\hat{A}$ which we still denote by ν_h. For all $a \in A$

$$\begin{aligned} c_{\phi,\psi}(ah) &= \int_{\hat{A}} \overline{x(a)} d\nu_h(x) \\ &= (\mathcal{F}\nu_h)(a). \end{aligned}$$

Due to condition (2), $\mathcal{F}\nu_h \in L^2(\hat{A}, \mu_{\hat{A}})$. Using a standard result on Fourier transform (see, for example, Theorem 31.33 of [5]) ν_h is absolutely continuous with respect to $\mu_{\hat{A}}$. Since $|\nu_h|(X) \neq 0$, it follows that $\mu_{\hat{A}}(X) \neq 0$ as claimed. □

If $\mu_{\hat{A}}(X) \neq 0$, the restriction μ_X of the measure $\mu_{\hat{A}}$ to X is a H-quasi-invariant measure, due to Lemma 2. In this case we can choose μ_X as the quasi-invariant measure ν in the definition (1) of U. With this choice, $\lambda_{\mu_X}(h,x) = \rho(h)^{-1}$ and U acts on $L^2(X, \mu_X, \mathcal{K})$ as

$$(U_{ah}\phi)(x) = \rho(h)^{1/2} x(a)\, m(q(x)^{-1} h q(h^{-1}[x]))\phi(h^{-1}[x]).$$

Moreover, since for all $s \in H_0$ the map ρ satisfies $\rho(s^{-1}) = \frac{\Delta_{H_0}(s)}{\Delta_H(s)}$ (see for example Theorem 5.30, of [9]), we have

$$\Delta_{H_0}(s) = \Delta_G(s) \qquad \forall s \in H_0. \tag{3}$$

Lemma 4. *Let G be as in Theorem 2. Suppose $\mu_{\hat{A}}(X) \neq 0$ and U be the representation described above. Let ϕ, ψ be two nonzero vectors in $L^2(X, \mu_X, \mathcal{K})$, then the condition*

$$c_{\phi,\psi} \in L^2(G, \mu_G) \tag{4}$$

holds if and only if both conditions

(5)

$$\text{for } \mu_X\text{-almost all } x \quad \left(h \mapsto \Delta_G(h^{-1})\, |\langle \phi(x), (U_{h^{-1}}\psi)(x_0)\rangle_{\mathcal{K}}|^2\right) \in L^1(H, \mu_H)$$

and

$$\left(x \mapsto \int_H \Delta_G(h^{-1})\, |\langle \phi(x), (U_{h^{-1}}\psi)(x_0)\rangle_{\mathcal{K}}|^2\, d\mu_H(h)\right) \in L^1(X, \mu_X) \tag{6}$$

are satisfied. In this case we have

(7)

$$\|c_{\phi,\psi}\|^2_{L^2(G,\mu_G)} = \int_X \int_H \Delta_G(h^{-1})\, |\langle \phi(x), (U_{h^{-1}}\psi)(x_0)\rangle_{\mathcal{K}}|^2\, d\mu_H(h)\, d\mu_X(x).$$

Proof. According to Lemma 1 we have that $\mu_G = \mu_A \otimes \frac{1}{\rho}\mu_H$. Using the theorem of Fubini, the unitarity of the Fourier transform on $L^2(\hat{A}, \mu_{\hat{A}})$ and again the theorem of Fubini, we conclude that the condition (4) is equivalent to the following two conditions

– For μ_X-almost all $x \in X$ the function
$$H \ni h \mapsto \rho(h)^{-1} \left|\langle \phi(x), (U_h\psi)(x)\rangle_{\mathcal{K}}\right|^2$$
is in $L^1(H, \mu_H)$;

– the function
$$X \ni x \mapsto \int_H \rho(h)^{-1} \left|\langle \phi(x), (U_h\psi)(x)\rangle_{\mathcal{K}}\right|^2 d\mu_H(h)$$
is in $L^1(X, \mu_X)$.

First perform the change of variables $h \mapsto (q(x)h)$ in the integrals on H and observe that $(U_{q(x)}U_h\psi)(x) = \rho(q(x))^{1/2}(U_h\psi)(x_0)$. Next send h to h^{-1} and use Lemma 1. In doing so we obtain that the two previous conditions are equivalent to (5) and (6). Equation (7) is now consequence of the theorem of Fubini. □

We need a technical lemma that allows us to compute the integrals on H as integrals on $X \times H_0$. Let $\beta : X \times H_0 \to H$ be the map
$$\beta(x, s) = q(x)s \quad x \in X,\ s \in H_0.$$

Lemma 5. *The map β is an isomorphism of measurable spaces. By suitably choosing the normalisation of μ_S, the image measure of*
$$(\rho \circ \beta)\mu_X \otimes \mu_S,$$
under the map β, is the Haar measure μ_H.

Proof. Since q is measurable, β is an isomorphism of measurable spaces. Hence, with the use of β we can regard $X \times H_0$ as an H-space, explicitly
$$h[(x, s)] = (h[x], q(h[x])^{-1}hq(x)s).$$
Fix a Haar measure μ_S on H_0. We claim that $(\rho\circ\beta)\mu_X \otimes \mu_S$ is a σ-finite H-invariant measure on $X \times H_0$. Since ρ is a continuous group homomorphism, $\rho\mu_S$ is a σ-finite measure on H_0. Moreover, let $(K_n)_{n\geq 1}$ be a sequence of compacts sets of H such that $\bigcup_n K_n = H$ and define
$$E_n = \{h[x_0] \ :\ h \in K_n\}.$$
Then, as recalled in Section 2, $q(E_n)$ has compact closure, so that $\rho \circ q$ is bounded on E_n. Since μ_X is σ-finite, the same holds for $(\rho\circ q)\mu_X$. Moreover if $h \in H$ and $E \in \mathcal{A} \times \mathcal{B}(S)$, we have
$$\begin{aligned}
&(\rho \circ \beta)\, \mu_X \otimes \mu_S(h[E]) = \\
&\quad = \int_X \int_S \rho(q(x)s)\chi_E(h^{-1}[x], q(h^{-1}[x])^{-1}h^{-1}q(x)s)\, d\mu_S(s)\, d\mu_X(x) \\
&\quad = \int_X \int_S \rho(hq(h^{-1}[x])s)\chi_E(h^{-1}[x], s)\, d\mu_S(s)\, d\mu_X(x) \\
&\quad = \int_X \int_S \rho(h^{-1})\rho(hq(x)s)\chi_E(x, s)\, d\mu_S(s)\, d\mu_X(x) \\
&\quad = (\rho \circ \beta)\, \mu_X \otimes \mu_S(E).
\end{aligned}$$

The image measure of $(\rho \circ \beta)\mu_X \otimes \mu_S$ with respect to β is a σ-finite H-invariant measure on H, hence it is the Haar measure μ_H, up to a constant factor that can be put equal to 1 by redefining μ_S. □

Lemma 6. *With the assumptions of Lemma 4, the conditions* (5) *and* (6) *hold if and only if the following ones are satisfied*

a) *m is a square-integrable representation of H_0 (we denote by K_m its formal degree);*

b) *$\psi(x) \in \mathrm{Dom}\, K_m^{-1/2}$ for μ_X-almost all $x \in X$;*

c) *the function*

$$X \ni x \mapsto \Delta_G(q(x))^{-1} \left\| K_m^{-1/2}\psi(x) \right\|_{\mathcal{K}}^2$$

is in $L^1(X, \mu_X)$.

In this case, we have that

$$\|c_{\phi,\psi}\|^2_{L^2(G,\mu_G)} = \|\phi\|^2_{\mathcal{H}} \int_X \Delta_G(q(x))^{-1} \left\| K_m^{-1/2}\psi(x) \right\|_{\mathcal{K}}^2 d\mu_X(x). \tag{8}$$

Proof. By Lemma 5 and the theorem of Fubini, taking into account that

$$(U_{(q(x')s)^{-1}}\psi)(x_0) = (\rho(q(x')s))^{-1/2} m(s^{-1})\psi(x'),$$

the condition (5) is equivalent to the two conditions

(5') for μ_X-almost all $x' \in X$ the function

$$H_0 \ni s \mapsto \Delta_G(q(x')s)^{-1} \left| \langle \phi(x), m(s^{-1})\psi(x') \rangle_{\mathcal{K}} \right|^2$$

is in $L^1(H_0, \mu_S)$;

(5") the function

$$X \ni x' \mapsto \int_S \Delta_G(q(x')s)^{-1} \left| \langle \phi(x), m(s^{-1})\psi(x') \rangle_{\mathcal{K}} \right|^2 d\mu_S(s)$$

is in $L^1(X, \mu_X)$.

Performing the change of variables $s \mapsto s^{-1}$ and using Equation (3), condition (5') becomes

$$H_0 \ni s \mapsto \Delta_G(q(x'))^{-1} \left| \langle \phi(x), m(s)\psi(x') \rangle_{\mathcal{K}} \right|^2$$

is in $L^1(H_0, \mu_S)$. Since $\psi \neq 0$, this last relation is precisely the condition that m is square-integrable and that

$$\psi(x') \in \mathrm{Dom}\, K_m^{-1/2}$$

for μ_X-almost all $x' \in X$. According to Theorem 1,

$$\int_S \Delta_G(q(x'))^{-1} \left| \langle \phi(x), m(s)\psi(x') \rangle_{\mathcal{K}} \right|^2 d\mu_S(s) =$$
$$= \Delta_G(q(x'))^{-1} \|\phi(x)\|^2_{\mathcal{K}} \left\| K_m^{-1/2}\psi(x') \right\|_{\mathcal{K}}^2.$$

Using this equation, the conditions (5") and (6) turn out to be equivalent to c) and to the fact that ϕ is in $L^2(X, \mu_X, \mathcal{K})$. Equation (8) is now evident. □

We are now in the position to prove the results stated in section 3.

Proof of Theorem 2. By lemma 3 we have $\mu_{\hat{A}}(X) \neq 0$. Since the square-integrability depends only on the equivalence class of U, we choose $\nu = \mu_X$ and we explicitly realise U acting on $L^2(X, \mu_X, \mathcal{K})$.

"if": By definition of square-integrability of U there exist two nonzero vectors $\phi, \psi \in L^2(X, \mu_X, \mathcal{K})$ such that $c_{\phi,\psi} \in L^2(G, \mu_G)$. Applying Lemmas 4 and 6, we obtain that m is square-integrable.

"only if": Due to Theorem 1, the domain of $K_m^{-1/2}$ is dense in $\mathcal{K}$. Moreover the domain of the multiplicative operator by the function $x \mapsto \Delta_G(q(x))^{1/2}$ is dense in $L^2(X, \mu_X, \mathcal{K})$, hence the conditions b) and c) of lemma 6 are satisfied in a dense subspace of $L^2(X, \mu_X, \mathcal{K})$. Fix a nonzero vector ψ in this dense subspace and a nonzero ϕ in $L^2(X, \mu_X, \mathcal{K})$. Applying Lemmas 4 and 6, we have that $c_{\phi,\psi} \in L^2(G, \mu_G)$. □

Proof of Corollary 1. It is an immediate consequence of the fact that the Haar measure of a topological group is finite if and only if the group is compact. □

Proof of Corollary 2. If U is the representation (1), with the choice $\nu = \mu_X$, apply Lemmas 4 and 6, taking into account the equation

$$\|c_{\phi,\psi}\|_{L^2(G,\mu_G)} = \|\phi\|_{\mathcal{H}} \left\| K_U^{-1/2}\psi \right\|_{\mathcal{H}}.$$

If ν is any σ-finite quasi-invariant measure on X, then it is equivalent to μ_X (see for example Theorem 5.19, of [9]). The thesis follows. □

5. A glance to some applications

In this final section we summarise the role of square-integrable representations in the construction of wavelet transforms and coherent states.

Let G be any lcsc topological group and U be a square-integrable representation of G acting in a Hilbert space $\mathcal{H}$. Fixed a vector ψ in the domain of $K_U^{-\frac{1}{2}}$, normalised such that $\left\| K_U^{-\frac{1}{2}}\psi \right\| = 1$, the isometry

$$\mathcal{H} \ni \phi \mapsto \mathcal{W}_\psi \phi \in \mathrm{L}^2(G, \mu_G)$$

defined by

$$\mathcal{W}_\psi \phi(g) = c_{\phi,\psi}(g) = \langle \phi, U_g \psi \rangle_{\mathcal{H}}$$

is called, in the framework of signal analysis, the *wavelet transform* defined by the *mother wavelet* ψ. The elements in the domain of $K_U^{-\frac{1}{2}}$ are called *admissible vectors*.

In many concrete cases considered in the literature, G is a semidirect product and U is an irreducible subrepresentation of the representation V acting in $L^2(A, \mu_A)$ as

$$(V_{(a,h)}\phi)(b) = \rho(h)^{-\frac{1}{2}} \phi(h^{-1}[ba^{-1}]).$$

The Fourier-Plancherel operator $\mathcal{F}$ from $L^2(A,\mu_A)$ onto $L^2(\hat{A},\mu_{\hat{A}})$ intertwines U with a representation of the form considered in Corollary 1, so that the square-integrability condition turns out to be equivalent to the fact that the orbit X has positive measure and the stabilizer is compact. In this case the wavelet transform is an isometry from the closed subspace of $L^2(A,\mu_A)$

$$\{\phi \in L^2(A,\mu_A) \ : \ \mathrm{supp}(\mathcal{F}\phi) \subset X\}$$

to $L^2(G,\mu_G)$ explicitly given by

$$\begin{aligned}(\mathcal{W}_\psi\phi) &= \int_A \rho(h)^{-\frac{1}{2}}\phi(b)\overline{\psi(h^{-1}[ba^{-1}])}\,d\mu_A(b)\\ &= \int_{\hat{A}} \rho(h)^{\frac{1}{2}}\overline{x(a)}(\mathcal{F}\phi)(x)\overline{(\mathcal{F}\psi)(h^{-1}[x])}\,d\mu_{\hat{A}}(x).\end{aligned}$$

The condition that ψ is admissible turns out to be

$$\begin{aligned}\mathrm{supp}(\mathcal{F}\psi) &\subset X\\ \left(x \mapsto \Delta_G(c(x))^{-\frac{1}{2}}(\mathcal{F}\psi)(x)\right) &\in L^2(\hat{A},\mu_{\hat{A}}).\end{aligned}$$

We notice that, since H_0 is compact, then the restriction of Δ_G to G_0 is 1, so that $\Delta_G(c(x))$ is in fact independent of the choice of the section c.

The above explicit formula for $\mathcal{W}\psi$ is the starting point for any further investigation, including numerical treatment of signals, and there is a huge amount of papers on these topics, dealing mainly with the case of the group "$ax+b$" or its n-dimensional generalisation, see for example [5] and references therein.

Theorem 2 suggests the possibility of considering also wavelet transforms on spaces of vector valued functions.

In the framework of quantum mechanics the square-integrable representations are relevant in order to describe the so called coherent states. Roughly speaking, given a representation U of a group G in $\mathcal{H}$ and a vector $\psi \in \mathcal{H}$, if the following *integral decomposition* (or *resolution of the identity*) holds

$$\phi = \int_G \langle \phi, U_g\psi\rangle\, U_g\psi\, d\mu_G(g) \qquad \phi \in \mathcal{H},$$

then the set

$$\{U_g\psi \ : \ g \in G\}$$

is called a set of coherent states for $\mathcal{H}$. This intuitive idea can be made precise assuming that U is a square-integrable representation and ψ is admissible (and suitably normalised). Indeed, with these assumptions, the previous formula has to be interpreted in weak sense and, in doing so, it is just the relation stated in Theorem 1. Also in this case, the notion of square-integrability is at the root of an extremely useful construction, see for example [1] and references therein.

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