CONTINUOUS AND DISCRETE FRAMES GENERATED BY THE EVOLUTION FLOW OF THE SCHRÖDINGER EQUATION

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Abstract. We study a family of coherent states, called Schrödingerlets, both in the continuous and discrete setting. They are defined in terms of the Schrödinger equation of a free quantum particle and some of its invariant transformations.

1. Introduction

In Quantum Mechanics, the time evolution of a d-dimensional free particle is described by the Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}f(x,t) = -\frac{1}{2\pi}\Delta f(x,t) \\ f(\cdot,0) = f_0, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator acting on the "space" variable $x \in \mathbb{R}^d$, and f_0 is a square-integrable function on $\mathbb{R}^d$ describing the state of the quantum particle at time zero (for the sake of simplicity, the mass is normalised so that the Laplacian has the simple factor $1/2\pi$).

The aim of this paper is to introduce a new family of coherent states (*i.e.* a frame) generated by the time evolution unitary operator defined by the Schrödinger equation; following [1, 2, 15], its elements are called *Schrödingerlets*.

Clearly, the time evolution operator $e^{i\frac{t}{2\pi}\Delta}$ is not enough to generate a frame for $L^2(\mathbb{R}^d)$, hence we need to add other unitary transformations. Observe that equation (1.1) is invariant both with respect to the rotations $R \in \mathrm{SO}(d)$, under the canonical action

$$f(x,t) \mapsto f(Rx,t),$$

and with respect to the dilations $a \in \mathbb{R}_+$, under the parabolic action

$$f(x,t) \mapsto a^{\frac{d}{4}} f(\sqrt{a}x, at),$$

where the factor $a^{\frac{d}{4}}$ ensures that the L^2-norm of $f(\cdot,t)$ is preserved. Thus, it is natural to consider the group $G = (\mathbb{R} \rtimes \mathbb{R}_+) \times \mathrm{SO}(d)$, *i.e.* the direct product of

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the identity component of the one-dimensional affine group and $\mathrm{SO}(d)$, and the corresponding unitary representation π acting on $L^2(\mathbb{R}^d)$ as

$$\pi(t,a,R)f = a^{-\frac{d}{4}} e^{i\frac{t}{2\pi}\Delta} f_{a,R}, \tag{1.2}$$

where $f_{a,R}(x) = f(a^{-\frac{1}{2}}R^{-1}x)$. It follows that the solution of (1.1) is given by

$$f(x,t) = \pi(t,1,\mathrm{I})f_0(x),$$

and for any rotation $R \in \mathrm{SO}(d)$

$$f(Rx,t) = \pi(t,1,R^{-1})f_0(x),$$

whereas for any dilation $a \in \mathbb{R}_+$

$$a^{\frac{d}{4}} f(\sqrt{a}x, at) = \pi(t,a^{-1},\mathrm{I})f_0(x).$$

Our goal is to study the properties of the corresponding family of coherent states $\{\pi(x)\eta\}_{x\in G}$ where η is a suitable "ground state", *i.e.* an admissible vector. In the context of signal analysis, this amounts to analyze the *voice transform*

$$f \mapsto \langle f, \pi(\cdot)\eta\rangle$$

as a map from $L^2(\mathbb{R}^d)$ into a suitable Banach space of functions on G. We restrict ourselves to the L^2-framework, both in the continuous and in the discrete setting. Our main contribution is twofold. First, we show that π is a reproducing representation of G and we characterize its admissible vectors. This result was already known for $d = 2$ [15], and here we extend the proof to arbitrary d. Furthermore, we construct a discrete Parseval frame of the form $\{\pi(x_i)\eta\}_{i\in I}$, where $\{x_i\}_{i\in I}$ is a suitable sampling of G.

In Section 2 we introduce the Schrödingerlets in two dimensions and we discuss the construction of a Parseval frame of two-dimensional Schrödingerlets. The purpose of this dimensionality restriction is twofold. Firstly, it allows to present the main ideas of this work in a simpler way, so that it may serve as a good introduction to the more involved general setting. Secondly, the two-dimensional case is somehow different from the higher dimensional cases, since when $d = 2$ the spherical harmonics on S^{d-1} correspond to the standard Fourier series; thus, a separate presentation allows to underline the peculiarities of the case $d = 2$.

Section 3 is devoted to studying the Schrödingerlets in any dimension. Proposition 3.3 shows that π is a reproducing representation and characterizes its admissible vectors. As a consequence, the Schrödingerlet voice transform permits to represent the quantum states as continuous functions on the parameter space $\mathbb{R} \times \mathbb{R}_+ \times \mathrm{SO}(d)$. Time evolution and rotations correspond to translations in the first and third variable, respectively, whereas dilations give rise to a multi-scale analysis of the original quantum state.

The main result of the paper is Theorem 3.4, which provides sufficient conditions in order to have a Parseval discrete frame.

We refer to [3, 19] for a general introduction to coherent states and reproducing formulæ associated with unitary representations. Schrödingerlets in dimension two were first introduced in [15] and further discussed in [1, 2], where G is regarded as a closed subgroup of the symplectic group and π is equivalent to the restriction to G of the metaplectic representation, whose role in signal analysis has been investigated in a series of papers [9, 10, 11, 12, 23]. We remark that the representation π is reducible and its reproducing kernel is not integrable. Hence, we cannot directly apply the classical theory of square-integrable representations by Duflo and Moore [16], nor the coorbit space theory developed by Feichtinger and Gröchenig [17, 18].

Another construction based on the covariance properties of a free quantum particle is given by the coherent states associated to the isochronous Galilei group (see [3, Chapter 8.4.2] and references therein). However, in this case, the dilations are not present and the frame does not depend on the time parameter. Indeed, in order to make the representation square-integrable it is necessary to reduce the Galilei group by taking the quotient modulo a group that contains the time translations.

The proof that π is a reproducing representation is based on the general theory developed in [15]. However, since π is the direct sum of a countable family of square-integrable representations π_i. A general approach to obtain a discrete frame without assuming that the kernel is in $L^1(G)$ has been developed in [4, 5, 6, 20], but it requires the boundedness of a suitable convolution operator (see condition (R3) of [4]), which is hard to prove in our setting. We follow here a different approach. Taking into account that $\pi = \bigoplus_i \pi_i$, the discretization is achieved by a slight generalization of a well known result on discrete wavelet frames in $L^2(\mathbb{R})$ [22, Theorem 1.6, Chapter 7], by Schur's orthogonality relations for finite groups and a technical lemma about Parseval frames (Lemma 3.6). Comparing with the approach taken in [4], we are able to provide only Hilbert frames; we hope to extend our results in future work and succeed in describing Banach frames related to the function spaces introduced in [5, 6, 13].

2. The main result in two dimensions

We state here the main result of this paper particularized for two-dimensional signals. In the first part of the section we introduce the continuous Schrödingerlets following [13].

2.1. The continuous Schrödingerlets in 2D. For $d = 2$ by identifying the abelian group SO(2) with the one dimensional torus $\mathcal{T} = \mathbb{R}/2\pi\mathbb{Z}$ as

$$\theta \longleftrightarrow R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

the group G is $(\mathbb{R} \rtimes \mathbb{R}_+) \times \mathcal{T}$ and its elements are denote by (b, a, θ), writing b instead of time variable t. In order to better visualize the action of π given by (1.2), it is worth rewriting it in an equivalent formulation by means of an intertwining operator S which we shall now define. We work in the Fourier domain with polar coordinates, and then perform a Fourier series with respect to the angular variable.

Below we write $\widehat{\mathbb{R}}^2$ for the dual space to $\mathbb{R}^2$ and $\mathrm{d}x$ and $\mathrm{d}\xi$ denote the corresponding Lebesgue measures, whereas $\mathrm{d}\theta$ is the Riemannian measure of $\mathcal{T}$ (so that $\int_{\mathcal{T}} \mathrm{d}\theta = 2\pi$). We let $\mathcal{F}\colon L^2(\mathbb{R}^2) \to L^2(\widehat{\mathbb{R}}^2)$ denote the Fourier transform given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i x\cdot\xi}\,\mathrm{d}x \qquad \xi \in \widehat{\mathbb{R}}^2$$

whenever $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $x \cdot \xi$ is the Euclidean scalar product.

Define the unitary operator $J\colon L^2(\widehat{\mathbb{R}}^2) \to L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$ by

$$J\hat{f}(\omega, \theta) = \hat{f}(\sqrt{\omega}\cos\theta, \sqrt{\omega}\sin\theta)/\sqrt{2} \qquad \hat{f} \in L^2(\widehat{\mathbb{R}}^2),\ \omega \in \widehat{\mathbb{R}}_+,\ \theta \in \mathcal{T}.$$

The unitarily equivalent representation $(J\mathcal{F})\pi(J\mathcal{F})^{-1}$ acting on $L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$ reads

$$(J\mathcal{F})\pi(J\mathcal{F})^{-1}(b, a, \phi)\hat{f}(\omega, \theta) = a^{1/2} e^{-2\pi i b\omega} \hat{f}(a\omega, \theta - \phi) \qquad \omega \in \widehat{\mathbb{R}}_+,\ \theta \in \mathcal{T} \tag{2.1}$$

for all $(b, a, \phi) \in G$ and $\hat{f} \in L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$. The action on the radial variable can be described by the representation of $\mathbb{R} \rtimes \mathbb{R}_+$ on $L^2(\widehat{\mathbb{R}}_+)$ given by

$$\widehat{W}^+(b, a)g(\omega) = a^{1/2} e^{-2\pi i b\omega} g(a\omega) \quad \omega \in \widehat{\mathbb{R}}_+,\ (b, a) \in \mathbb{R} \rtimes \mathbb{R}_+,\ g \in L^2(\widehat{\mathbb{R}}_+), \tag{2.2}$$

which is nothing else than the one-dimensional wavelet representation in the positive frequency domain. The action on the angular variable is simply given by a rotation $\rho(\phi)z(\theta) = z(\theta - \phi)$ for $z \in L^2(\mathcal{T})$. Therefore, the action of π on two-dimensional functions should be thought of as a classical one-dimensional wavelet representation on the radial component combined with rotations around the origin.

Consider now the Fourier series with respect to θ and define the unitary operator $S\colon L^2(\mathbb{R}^2) \to \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ by

$$(Sf)_n(\omega) = \int_0^{2\pi} (J\mathcal{F}f)(\omega, \theta) e^{-in\theta} \frac{\mathrm{d}\theta}{\sqrt{2\pi}} \qquad \omega \in \widehat{\mathbb{R}}_+,\ n \in \mathbb{Z},\ f \in L^2(\mathbb{R}^2).$$

From now on, we shall consider the equivalent representation $\pi' = S\pi S^{-1}$ of G acting on $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. In view of (2.1) and (2.2), the action of π' is given by

$$(\pi'(b, a, \phi)\hat{f})_n = e^{-in\phi}\,(\widehat{W}^+(b, a)\hat{f}_n) \qquad n \in \mathbb{Z},\ \hat{f} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+),\ (b, a, \phi) \in G.$$

Denoted by ρ_n the character $\phi \mapsto e^{-in\phi}$ of $\mathcal{T}$, the representation π' can be decomposed as

$$\pi' = \bigoplus_{n\in\mathbb{Z}} \rho_n \widehat{W}^+$$

where each component $\rho_n \widehat{W}^+$ acts irreducibly on $L^2(\widehat{\mathbb{R}}_+)$.

It was proven in [2, 13] that π', and therefore π, is reproducing, namely

$$\|\hat{f}\|^2_{\bigoplus_n L^2(\widehat{\mathbb{R}}_+)} = \int_G |\langle \pi'(b, a, \phi)\widehat{\eta}, \hat{f}\rangle|^2 \,\mathrm{d}b \frac{\mathrm{d}a}{a^2} \frac{\mathrm{d}\phi}{2\pi} \qquad \hat{f} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+) \tag{2.3}$$

for some admissible vector $\widehat{\eta} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. A vector $\widehat{\eta} = (\widehat{\eta}_n)_n \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ is admissible for π' if and only if

$$\int_0^{+\infty} |\widehat{\eta}_n(\omega)|^2 \frac{\mathrm{d}\omega}{\omega} = 1 \qquad n \in \mathbb{Z}, \tag{2.4}$$

namely, if and only if each component $\widehat{\eta}_n$ is a one-dimensional wavelet [14]. A simple way to construct admissible vectors in $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ satisfying (2.4) is to fix a one-dimensional wavelet $\widehat{\eta}_0 \in L^2(\widehat{\mathbb{R}}_+)$ satisfying (2.4) and then construct all the other components $\widehat{\eta}_n$ by dilating $\widehat{\eta}_0$. Since (2.4) is invariant under positive dilations, it is immediately satisfied for all n. More precisely, set for all $n \in \mathbb{Z}$

$$\widehat{\eta}_n(\omega) = \widehat{\eta}_0(\alpha_n^{-1}\omega) \qquad \omega \in \widehat{\mathbb{R}}_+, \tag{2.5}$$

for some weights $\alpha_n > 0$ that satisfy $\alpha_0 = 1$ and $\sum_n \alpha_n < \infty$. This last condition ensures that the resulting $\widehat{\eta}$ has finite norm in $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$, because

$$\|\widehat{\eta}\|^2 = \|\widehat{\eta}_0\|^2_{L^2(\widehat{\mathbb{R}}_+)} \sum_n \alpha_n.$$

2.2. **The discrete Schrödingerlets in 2D.** We now show how to construct a Parseval frame of $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ associated to π'. Then, by means of the intertwining operator S, this frame can be transformed into a Parseval frame of $L^2(\mathbb{R}^2)$ associated to π. Constructing a Parseval frame corresponds to a discretization of (2.3) of the form

$$\|\hat{f}\|^2_{\oplus_n L^2(\widehat{\mathbb{R}}_+)} = \sum_{i\in\mathbb{N}} |\langle \pi'(x_i)\widehat{\eta}, \hat{f}\rangle|^2 \qquad \hat{f} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+),$$

for suitable choices of the admissible vector $\widehat{\eta}$ and of a sampling $\{x_i\}_{i\in\mathbb{N}}$ of the group G.

Our approach is based on the fact that π' is the direct sum of one-dimensional wavelet representations. Thus, it is instructive to look first at the well known one-dimensional case, namely at the representation $\widehat{W}^+$ acting on $L^2(\widehat{\mathbb{R}}_+)$. Standard wavelet theory [22, Thm. 1.1, Chapter 7] gives that $\{\widehat{W}^+(2^j k, 2^j)\,\widehat{\eta}_0 : k, j \in \mathbb{Z}\}$ is a Parseval frame for $L^2(\widehat{\mathbb{R}}_+)$, namely

$$\|\hat{f}\|^2_{L^2(\widehat{\mathbb{R}}_+)} = \sum_{k,j\in\mathbb{Z}} |\langle \widehat{W}^+(2^j k, 2^j)\,\widehat{\eta}_0, \hat{f}\rangle|^2 \qquad \hat{f} \in L^2(\widehat{\mathbb{R}}_+),$$

provided that the conditions

$$\text{(2.6a)} \qquad \sum_{j\in\mathbb{Z}} |\widehat{\eta}_0(2^j\omega)|^2 = 1, \quad \text{for a.e. } \omega \in \widehat{\mathbb{R}}_+,$$

$$\text{(2.6b)} \qquad \sum_{j\in\mathbb{N}} \widehat{\eta}_0(2^j\omega)\overline{\widehat{\eta}_0(2^j(\omega + 2\pi m))} = 0, \quad \text{for a.e. } \omega \in \widehat{\mathbb{R}}_+,\ m \in 2\mathbb{Z}+1$$

hold true. Note that in this case the sampling of the group $\mathbb{R} \rtimes \mathbb{R}_+$ is the discrete set $\{(2^j k, 2^j) : k, j \in \mathbb{Z}\}$.

We now generalize this construction to the Schrödingerlets. In view of the above sampling of the affine group, it is natural to consider the discretization of G given by

$$\{x_{k,j,l} = (2^j k, 2^j, 2\pi l/L) : k, j \in \mathbb{Z}, l = 0, \ldots, L-1\},$$

for some $L \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Note that the angles $\phi_l = 2\pi l/L$ give a uniform sampling of $\mathcal{T}$ and form a finite cyclic subgroup of order L. Let us now discuss suitable assumptions on the admissible vector $\widehat{\eta}$, and therefore on $\widehat{\eta}_0$ and the weights α_n in the case where $\widehat{\eta}_n$ is given by (2.5), so that $\{\pi'(x_{k,j,l})\,\widehat{\eta} : k, j \in \mathbb{Z}, l = 0, \ldots, L-1\}$ is a Parseval frame for $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$.

We first observe that for every $n \in \mathbb{Z}$ it is necessary that each $\widehat{\eta}_n \in L^2(\widehat{\mathbb{R}}^+)$ give rise to a Parseval frame for the corresponding space $L^2(\widehat{\mathbb{R}}_+)$, *i.e.* that each $\widehat{\eta}_n$ satisfies (2.6) (suitably normalized):

$$\text{(2.7a)} \qquad \sum_{j\in\mathbb{Z}} |\widehat{\eta}_n(2^j\omega)|^2 = 1/L, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+,\ n \in \mathbb{Z},$$

$$\text{(2.7b)} \qquad \sum_{j\in\mathbb{N}} \widehat{\eta}_n(2^j\omega)\overline{\widehat{\eta}_n(2^j(\omega + 2\pi m))} = 0, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+,\ n \in \mathbb{Z},\ m \in 2\mathbb{Z}+1.$$

In the continuous setting, it is necessary and sufficient to assume that each $\widehat{\eta}_n$ is a one-dimensional wavelet, *i.e.* that (2.4) holds true for every n, in order to have the continuous reproducing formula (2.3). In the discrete case, however, assumptions (2.7) are not sufficient, and it is necessary to assume the following conditions:

(2.8a)
$$\sum_{j\in\mathbb{Z}} \widehat{\eta}_n(2^j\omega)\overline{\widehat{\eta}_{n+kL}(2^j\omega)} = 0, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+,\, n \in \mathbb{Z},\, k \in \mathbb{Z}^*,$$

(2.8b)
$$\sum_{j\in\mathbb{N}} \widehat{\eta}_n(2^j\omega)\overline{\widehat{\eta}_{n+kL}(2^j(\omega+2\pi m))} = 0, \text{ a.e. } \omega \in \widehat{\mathbb{R}}_+,\, n \in \mathbb{Z},\, k \in \mathbb{Z}^*,\, m \in 2\mathbb{Z}+1,$$

where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$. When (2.5) holds true, the above expressions can be simplified into conditions involving $\widehat{\eta}_0$ and the weights α_n.

These orthogonality relations do not contain all the cross terms between $\widehat{\eta}_n$ and $\widehat{\eta}_m$ for $n \neq m$, but only those corresponding to the cases when $m - n \in L\mathbb{Z}$. The reason for this simplification can be explained as follows. Two characters ρ_n and ρ_m restricted to the finite subgroup $\{2\pi l/L : l = 0, \dots, L-1\}$ are equivalent if and only if $m - n \in L\mathbb{Z}$. As a consequence, all the cross terms corresponding to m and n for which $m - n \notin L\mathbb{Z}$ are zero by Schur orthogonality relations for finite groups.

The following theorem shows that the above conditions are also sufficient.

Theorem 2.1. *Let $\widehat{\eta} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ be such that* (2.7) *and* (2.8) *hold true, and take $L \in \mathbb{N}^*$. Then $\{\pi'(x_{k,j,l})\,\widehat{\eta} : k, j \in \mathbb{Z}, l = 0, \dots, L-1\}$ is a Parseval frame for $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$, namely*

$$\|\hat{f}\|^2_{\oplus_n L^2(\widehat{\mathbb{R}}_+)} = \sum_{k,j,l} |\langle \pi'(x_{k,j,l})\widehat{\eta}, \hat{f}\rangle|^2 \qquad \hat{f} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+).$$

The proof will be given below since the above result is a special instance of our main result, see Theorem 3.4. We just exhibit functions $\widehat{\eta}$ satisfying the assumptions. Take $\widehat{\eta}_0 \in L^2(\widehat{\mathbb{R}}_+)$ such that (2.7a) is satisfied for $n = 0$ and such that $\operatorname{supp}\widehat{\eta}_0 \subseteq [0, 2\pi]$. Moreover, choose weights $\alpha_n \in (0, 1]$ such that $\alpha_0 = 1$, $\sum_n \alpha_n < \infty$ and

$$|\operatorname{supp}(\widehat{\eta}_0) \cap \alpha_n^{-1}\alpha_{n+kL}\operatorname{supp}(\widehat{\eta}_0)| = 0 \qquad n \in \mathbb{Z}, k \in \mathbb{Z}^*, \tag{2.9}$$

where $|\cdot|$ denotes Lebesgue measure. It is easy to see that the admissible vector $\widehat{\eta} \in \bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ defined by (2.5) satisfies (2.7) and (2.8). A simple choice valid for any L is $\widehat{\eta}_0 = L^{-1}\chi_{[1/2,1]}$ and

$$\alpha_n = \begin{cases} 2^{-2n} & \text{if } n \geq 0 \\ 2^{2n+1} & \text{if } n < 0. \end{cases}$$

We now comment on the role of the number of rotations L. The conclusion of Theorem 2.1 still holds true when $L = 1$, namely when no rotations are considered. However, the rotations do play a role in the choice of the admissible vector $\widehat{\eta}$. Indeed, condition (2.8), or (2.9) in the case when (2.5) holds true, becomes weaker as L increases. More precisely, if L_2 is a multiple of L_1 and $\widehat{\eta}$ satisfies (2.8) with $L = L_1$, then the same equalities hold true with $L = L_2$. Note that this is equivalent to saying that the two corresponding discrete subgroups of $\mathcal{T}$ are one contained into the other.

Note that for $L = 1$ a simple computation shows that $\|\widehat{\eta}\| = 1$, hence the frame obtained in Theorem 2.1 is in fact an orthonormal basis of $\bigoplus_{n\in\mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. Indeed, it is a standard general fact that a tight frame whose elements have norm (greater than or equal to) one is necessarily an orthonormal basis (see e.g. [22, Theorem 1.8, Ch. 7]).

3. The d-dimensional case

3.1. **The continuous setting.** We define $G = (\mathbb{R} \rtimes \mathbb{R}_+) \times \mathrm{SO}(d)$ as the direct product of the identity component of the one-dimensional affine group and $\mathrm{SO}(d)$. Clearly, the set

$$H = \{(0, a, R) \mid a \in \mathbb{R}_+,\, R \in \mathrm{SO}(d)\} \simeq \mathbb{R}_+ \times \mathrm{SO}(d)$$

is a closed unimodular subgroup of G and its Haar measure is $\mathrm{d}h = a^{-1}\mathrm{d}a\mathrm{d}R$, and the set

$$\{(b, 1, \mathrm{I}) \mid b \in \mathbb{R}\} \simeq \mathbb{R}$$

is a normal abelian closed subgroup of G, whose Haar measure is the Lebesgue measure $\mathrm{d}b$. Moreover, G is the semi-direct product of $\mathbb{R}$ and H with respect to the inner action of H on $\mathbb{R}$ given by

$$h[b] = ab \qquad b \in \mathbb{R},\ h = (a, R) \in H.$$

We set

$$\gamma(h) = \det\left(b \mapsto h[b]\right) = a. \tag{3.1}$$

The Schrödinger representation π of G acts on $L^2(\mathbb{R}^d)$ as

$$\pi(b, a, R) = U(b)V(a, R) \qquad (b, a, R) \in G. \tag{3.2a}$$

Here $V(a, R)$ is the unitary operator

$$V(a, R)f(x) = a^{-\frac{d}{4}} f(a^{-\frac{1}{2}} R^{-1} x) \qquad f \in L^2(\mathbb{R}^d),\, x \in \mathbb{R}^d,$$

and $b \mapsto U(b)$ is the one-parameter group of unitary operators on $L^2(\mathbb{R}^d)$ associated with the Laplacian by the spectral calculus, namely

$$U(b) = e^{i\frac{b}{2\pi}\Delta}. \tag{3.2b}$$

Thus

$$\mathcal{F}U(b)\mathcal{F}^{-1}\hat{f}(\xi) = e^{-2\pi i b\, \xi\cdot\xi}\hat{f}(\xi) \qquad \xi \in \widehat{\mathbb{R}}^d. \tag{3.2c}$$

Setting $\widehat{\pi} = \mathcal{F}\pi\mathcal{F}^{-1}$ we get

$$\widehat{\pi}(b, a, R)\hat{f}(\xi) = a^{\frac{d}{4}} e^{-2\pi i b\, \xi\cdot\xi} \hat{f}(a^{\frac{1}{2}} R^{-1}\xi) \qquad \hat{f} \in L^2(\widehat{\mathbb{R}}^d),\, \xi \in \widehat{\mathbb{R}}^d. \tag{3.2d}$$

We now prove that π is a reproducing representation.

Proposition 3.1. *The Schrödinger representation π of G is a reproducing representation.*

Proof. It is enough to prove the result for $\widehat{\pi}$, which belongs to the family of mock-metaplectic representations introduced in [15], regarding G as semi-direct product of $\mathbb{R}$ and H. Indeed, H acts on the dual group $\hat{\mathbb{R}}$ of $\mathbb{R}$ by the contra-gradient action

$${}^t h[\omega] = a^{-1}\omega \qquad \omega \in \widehat{\mathbb{R}},\, h = (a, R) \in H.$$

The group H acts on $\mathbb{R}^d$ as well as on the dual space $\widehat{\mathbb{R}}^d$ by means of

$$\begin{aligned} h.x &= a^{\frac{1}{2}} R x \\ {}^t h.\xi &= a^{-\frac{1}{2}} R \xi \end{aligned} \qquad x \in \mathbb{R}^d,\, \xi \in \widehat{\mathbb{R}}^d,\, h = (a, R) \in H.$$

We set

$$\beta(h) = \det\big(\xi \mapsto {}^t h.\xi\big) = a^{-\frac{d}{2}}.$$

The map

$$\Phi : \widehat{\mathbb{R}}^d \longrightarrow \widehat{\mathbb{R}}, \qquad \Phi(\xi) = \xi \cdot \xi \tag{3.3}$$

is easily seen to satisfy the following properties:

i) Φ is a smooth map whose gradient is $\nabla\Phi(\xi) = 2\xi$;
ii) the set of critical points of Φ reduces to the origin, which is a Lebesgue negligible set, and $\Phi(\widehat{\mathbb{R}}^d \setminus \{0\}) = \widehat{\mathbb{R}}_+$;
iii) $\Phi({}^t h.\xi) = {}^t h[\Phi(\xi)]$ for all $\xi \in \widehat{\mathbb{R}}^d$ and $h \in H$;
iv) the action of H on $\widehat{\mathbb{R}}_+$ is transitive, the stability subgroup at $1 \in \mathbb{R}_+$ is the compact group $\mathrm{SO}(d)$, and $q : (0, +\infty) \to H$, $q(\omega) = \omega^{-1}$, is a smooth section, namely
$$^t q(\omega)[1] = \omega \qquad \omega \in \mathbb{R}_+;$$
v) $\Phi^{-1}(1) = \mathrm{S}^{d-1}$, where S^{d-1} is the unit sphere of $\widehat{\mathbb{R}}^d$ endowed with the Riemannian measure $\mathrm{d}s$.

From (3.2d) it is clear that

$$\widehat{\pi}(b,h)\hat{f}(\xi) = \beta(h)^{-\frac{1}{2}}\, e^{-2\pi i b\Phi(\xi)}\, \hat{f}({}^t h^{-1}.\xi), \tag{3.4a}$$

where $\xi \in \widehat{\mathbb{R}}^d$, $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$ and $(b,R) \in \mathbb{R} \rtimes (\mathbb{R}_+ \times \mathrm{SO}(d))$, which shows that $\widehat{\pi}$ is the mock-metaplectic representation associated with the map Φ. Theorem 9 of [15] then implies that $\widehat{\pi}$ is a reproducing representation. □

We now study the admissible vectors of π. First, we need to recall some elementary facts.

Let ρ be the regular representation of $\mathrm{SO}(d)$ acting on $L^2(\mathrm{S}^{d-1})$, namely

$$\rho(R)\varphi(s) = \varphi(R^{-1}s) \qquad s \in \mathrm{S}^{d-1}, \varphi \in L^2(\mathrm{S}^{d-1}), R \in \mathrm{SO}(d).$$

There holds that

$$L^2(\mathrm{S}^{d-1}) = \bigoplus_{i\in\mathbb{N}} \mathcal{H}_i, \tag{3.5}$$

where each $\mathcal{H}_i$ is the space of spherical harmonics, namely the complex polynomials in d variables, homogeneous of degree i and harmonic. Here each polynomial is regarded as a function on S^{d-1}, so that $\mathcal{H}_i$ can be identified as a subspace of $L^2(\mathrm{S}^{d-1})$. For an account of the role of spherical harmonics in the representation theory of the orthogonal groups see [8]. It is known that

$$\dim \mathcal{H}_0 = 1, \quad \dim \mathcal{H}_1 = d, \quad \dim \mathcal{H}_i = \binom{d+i-1}{d-1} - \binom{d+i-3}{d-1},\ i \geq 2. \tag{3.6}$$

Moreover,

$$\rho = \bigoplus_{i\in\mathbb{N}} \rho_i, \tag{3.7}$$

where ρ_i is the restriction of ρ to $\mathcal{H}_i$. We denote by P_i the projection from $L^2(\mathrm{S}^{d-1})$ onto $\mathcal{H}_i$.

If $d > 2$, each representation ρ_i is irreducible, and two representations ρ_i and ρ_j are inequivalent whenever $i \neq j$ (the multiplicity of each ρ_i is one). For $d = 2$, every $\mathcal{H}_i$ with $i \geq 1$ has dimension 2 and each ρ_i is the sum of two inequivalent irreducible one-dimensional representations, namely $\rho_i^+(\theta) = e^{in\theta}$, $\rho_i^-(\theta) = e^{-in\theta}$, $\theta \in \mathrm{SO}(2) \simeq \mathcal{T}$. Hence, we still obtain a decomposition into inequivalent irreducible representations if we just replace the index set $\mathbb{N}$ with $\mathbb{Z}$. For ease of notation, we shall proceed assuming $d \geq 3$. The case $d = 2$ is described in Section 2.

Recall that the group $\mathbb{R} \rtimes \mathbb{R}_+$ has only two inequivalent infinite dimensional irreducible representations up to unitary equivalence, which we denote by $\widehat{W}^{\pm}$ (see e.g. [24]). Each of them acts on $L^2(\widehat{\mathbb{R}}_\pm)$ as

$$\widehat{W}^{\pm}(b,a)\varphi(\omega) = a^{\frac{1}{2}}\varphi(a\omega)e^{-2\pi i b\omega} \qquad \omega \in \widehat{\mathbb{R}}_\pm,\ (b,a) \in \mathbb{R} \rtimes \mathbb{R}_+, \tag{3.8}$$

where $\varphi \in L^2(\widehat{\mathbb{R}}_\pm)$.

Now, let $J : L^2(\widehat{\mathbb{R}}^d) \to L^2(\widehat{\mathbb{R}}_+ \times \mathrm{S}^{d-1})$ be the operator defined by

$$J\hat{f}(\omega, s) = \frac{\omega^{\frac{d-2}{4}}}{\sqrt{2}}\hat{f}(\sqrt{\omega}\, s) \qquad \omega \in \widehat{\mathbb{R}}_+,\ s \in \mathrm{S}^{d-1},\ \hat{f} \in L^2(\widehat{\mathbb{R}}^d). \tag{3.9}$$

We have the following simple lemma.

Lemma 3.2. *The operator J is unitary.*

Proof. If $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$, then the changes of variable $\omega = r^2$ and $\xi = rs$ yield

$$\int_{\widehat{\mathbb{R}}_+ \times \mathrm{S}^{d-1}} \omega^{\frac{d-2}{2}} |\hat{f}(\sqrt{\omega}s)|^2 \frac{\mathrm{d}\omega \mathrm{d}s}{2} = \int_{\widehat{\mathbb{R}}_+ \times \mathrm{S}^{d-1}} r^{d-1}|\hat{f}(rs)|^2 \mathrm{d}r\mathrm{d}s = \int_{\widehat{\mathbb{R}}^d} |\hat{f}(\xi)|^2 \mathrm{d}\xi. \tag{3.10}$$

The inverse of J is given by

$$(J^{-1}g)(\xi) = \frac{\sqrt{2}}{(\xi \cdot \xi)^{d-2}} g(\xi \cdot \xi, \frac{\xi}{\sqrt{\xi \cdot \xi}}) \qquad \xi \in \widehat{\mathbb{R}}^d,\ \xi \neq 0,\ g \in L^2(\widehat{\mathbb{R}}_+ \times \mathrm{S}^{d-1}),$$

which proves that J is unitary. □

In what follows, we will freely identify

$$\begin{aligned} L^2(\widehat{\mathbb{R}}^d) &\simeq L^2(\widehat{\mathbb{R}}_+ \times \mathrm{S}^{d-1}) \\ &\simeq L^2(\widehat{\mathbb{R}}_+) \otimes L^2(\mathrm{S}^{d-1}) \\ &\simeq \bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+) \otimes \mathcal{H}_i \\ &\simeq \bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i). \end{aligned} \tag{3.11}$$

We define the unitary operator $S : L^2(\mathbb{R}^d) \to \bigoplus_{i\in\mathbb{N}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$ by

$$(Sf)_i = (\mathrm{Id} \otimes P_i)(J\mathcal{F}f) \qquad f \in L^2(\mathbb{R}^d). \tag{3.12}$$

Proposition 3.3. *With the above notation,*

$$S\pi S^{-1} = \bigoplus_{i \in \mathbb{N}} \widehat{W}^+ \otimes \rho_i \tag{3.13}$$

where each component $\widehat{W}^+ \otimes \rho_i$ is irreducible and inequivalent to the others. A vector $\widehat{\eta} \in L^2(\mathbb{R}^d)$ is admissible for π if and only if

$$\int_0^{+\infty} \|(S\widehat{\eta})_i(\omega)\|^2_{\mathcal{H}_i} \frac{\mathrm{d}\omega}{\omega} = d_i \qquad i \in \mathbb{N}. \tag{3.14}$$

Proof. The proof is based on the general theory developed in [15]. We sketch the main steps. For any $\omega \in \widehat{\mathbb{R}}_+$ we denote by ν_ω the measure on $\widehat{\mathbb{R}}^d$ which is the image measure of $\omega^{\frac{d-2}{2}}\mathrm{d}s/2$ under the map

$$\mathrm{S}^{d-1} \ni s \mapsto \sqrt{\omega}s \in \widehat{\mathbb{R}}^d,$$

so that, for all compactly supported continuous functions φ, we have

$$\int_{\widehat{\mathbb{R}}^d} \varphi(\xi)\mathrm{d}\nu_\omega(\xi) = \int_{\mathrm{S}^{d-1}} \varphi(\sqrt{\omega}s)\frac{\omega^{\frac{d-2}{2}}}{2}\mathrm{d}s.$$

The change of variable in spherical coordinates (as in (3.10)) gives

$$\begin{aligned}\int_{\widehat{\mathbb{R}}^d} \varphi(\xi)\mathrm{d}\xi &= \int_0^{+\infty} \left(\int_{\mathrm{S}^{d-1}} \varphi(rs)r^{d-1}\mathrm{d}s\right)\mathrm{d}r \\ &= \int_0^{+\infty} \left(\int_{\mathrm{S}^{d-1}} \varphi(\sqrt{\omega}s)\frac{\omega^{\frac{d-2}{2}}}{2}\mathrm{d}s\right)\mathrm{d}\omega \\ &= \int_0^{+\infty} \left(\int_{\widehat{\mathbb{R}}^d} \varphi(\xi)\mathrm{d}\nu_\omega(\xi)\right)\mathrm{d}\omega,\end{aligned}$$

where $r^2 = \omega$, so that the disintegration formula

$$\mathrm{d}\xi = \int_0^{+\infty} \nu_\omega \,\mathrm{d}\omega \tag{3.15}$$

holds true. Finally, Weil's formula for quasi-invariant measure on quotient spaces [19] reads

$$\int_H \varphi(a,R)\gamma(a)^{-1}\frac{\mathrm{d}a}{a}\mathrm{d}R = C\int_0^{+\infty}\left(\int_{\mathrm{SO}(d)}\varphi(q(\omega)R)\mathrm{d}R\right)\mathrm{d}\omega, \tag{3.16}$$

for some constant C, to be computed. Recalling (3.1) and $q(\omega) = \omega^{-1}$, we obtain $C = 1$ since

$$\int_0^{+\infty}\left(\int_{\mathrm{SO}(d)}\varphi(\omega^{-1}R)\mathrm{d}R\right)\mathrm{d}\omega = \int_0^{+\infty}\left(\int_{\mathrm{SO}(d)}\varphi(\omega,R)\mathrm{d}R\right)\frac{\mathrm{d}\omega}{\omega^2}.$$

Observe that

i) $L^2(\mathbb{R}^d, 2\nu_1) \simeq L^2(\mathrm{S}^{d-1})$;

ii) the "restriction" of the mock-metaplectic representation $\widehat{\pi}$ to the fiber $\Phi^{-1}(1)$ and to the stability subgroup $\mathrm{SO}(d)$ is precisely ρ. Hence, (3.7) provides the decomposition of ρ into its irreducibles, all of them with multiplicity 1;

iii) up to the normalization factor $1/\sqrt{2}$, the operator $S\mathcal{F}^{-1}$ coincides with the operator introduced in [15], whose main feature is that it decomposes $\widehat{\pi}$ into its irreducibles, each of which is the canonical representation obtained by inducing the irreducible representation of $\mathbb{R} \times \mathrm{SO}(d)$ acting on $\mathcal{H}_i$ as

$$(b,R) \mapsto e^{-2\pi i b}\rho_i(R)$$

from $\mathbb{R} \times \mathrm{SO}(d)$ to G.

Theorem 9 of [15] shows that $\widehat{\eta} \in L^2(\widehat{\mathbb{R}}^d)$ is admissible if and only if, for all $i \in \mathbb{N}$,

$$\int_0^{+\infty} \|(S\widehat{\eta})_i(\omega)\|^2_{\mathcal{H}_i}\gamma(q(\omega))\mathrm{d}\omega = \frac{\dim\mathcal{H}_i}{C} = d_i.$$

Explicitly, this amounts to

$$\int_0^{+\infty} \|(S\widehat{\eta})_i(\omega)\|^2_{\mathcal{H}_i} \frac{\mathrm{d}\omega}{\omega} = d_i. \qquad \square$$

We remark that a second proof can be derived using Proposition 2.23 of [19], and it consists of three steps:

i) a direct computation to show (3.13);
ii) the observation that each component $\widehat{W}^+ \otimes \rho_i$ is a square-integrable representation, whose formal degree operator C_i is the unbounded multiplication operator
$$C_i\varphi(\omega) = d_i\, \omega\varphi(\omega),$$
where $\varphi \in L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$ and $\int_0^\infty \omega^2 |\varphi(\omega)|^2 \mathrm{d}\omega < +\infty$;
iii) a final application of Proposition 2.23 of [19], taking into account that $\widehat{W}^+ \otimes \rho_i$ and $\widehat{W}^+ \otimes \rho_j$ are inequivalent representations of G if $i \neq j$.

3.2. **A family of admissible vectors.** We now give an alternative description of the admissible vectors, which provides a direct strategy to construct them.

Let $\widehat{\eta} \in L^2(\mathbb{R}^d)$ be an admissible vector. For any fixed $i \in \mathbb{N}$, we choose an orthonormal basis $\{e^i_k\}_{k=1}^{d_i}$ of $\mathcal{H}_i$, and define $\varphi_{i,k} : \widehat{\mathbb{R}}_+ \to \mathbb{C}$ by

$$\varphi_{i,k}(\omega) = \langle (S\widehat{\eta})_i(\omega), e^i_k \rangle_{\mathcal{H}_i}.$$

By construction, $\varphi_{i,k} \in L^2(\widehat{\mathbb{R}}_+)$ and

$$\int_0^{+\infty} \frac{|\varphi_{i,k}(\omega)|^2}{\omega} \mathrm{d}\omega < +\infty.$$

If $\varphi_{i,k} \neq 0$, up to a normalization we can always assume that

$$\int_0^{+\infty} \frac{|\varphi_{i,k}(\omega)|^2}{\omega} \mathrm{d}\omega = 1, \tag{3.17a}$$

i.e. $\varphi_{i,k}$ is a $1D$-wavelet for $\widehat{W}^+$. Hence

$$(S\widehat{\eta})_i = \sum_{k=1}^{d_i} \varphi_{i,k} \otimes v_{i,k}, \tag{3.17b}$$

where $\{v_{i,k}\}_{k=1}^{d_i}$ is an orthogonal family in $\mathcal{H}_i$ such that

$$\sum_{k=1}^{d_i} \|v_{i,k}\|^2_{\mathcal{H}_i} = d_i, \tag{3.17c}$$

and all $\varphi_{i,k}$ satisfy (3.17a) (if for some k the function $\langle (S\widehat{\eta})_i(\cdot), e^i_k \rangle_{\mathcal{H}_i}$ is zero, we set $v_{i,k} = 0$ and choose an arbitary $\varphi_{i,k}$ satisfying (3.17a)).

The fact that $\widehat{\eta} \in L^2(\mathbb{R}^d)$ implies

$$\sum_{i=1}^{+\infty} \sum_{k=1}^{d_i} \|\varphi_{i,k}\|_2^2 \, \|v_{i,k}\|^2_{\mathcal{H}_i} < +\infty. \tag{3.17d}$$

Conversely, given a family $(\varphi_{i,k}, v_{i,k},)_{i\in\mathbb{N}, k=1,\dots,d_i}$ such that

a) each $\varphi_{i,k}$ is in $L^2(\widehat{\mathbb{R}}_+)$ and satisfies (3.17a),
b) each family $\{v_{i,k}\}_{k=1}^{d_i}$ is orthogonal in $\mathcal{H}_i$ and satisfies (3.17c) and (3.17d),

then $(\varphi_{i,k}, v_{i,k},)_{i\in\mathbb{N},k=1,\dots,d_i}$ defines an admissible vector via (3.17b). A simple solution is given as follows. Choose a 1D wavelet $\varphi \in L^2(\widehat{\mathbb{R}}_+)$. For all $i \in \mathbb{N}$, fix $\alpha_i > 0$ and $v_i \in \mathcal{H}_i$ with

$$\|v_i\|^2_{\mathcal{H}_i} = d_i$$

and

$$\sum_{i\in\mathbb{N}} \alpha_i d_i < +\infty.$$

Define

$$\varphi_i(\omega) = \varphi(\alpha_i^{-1}\omega).$$

Then, the vector $\widehat{\eta} \in L^2(\mathbb{R}^d)$ such that

$$(S\widehat{\eta})_i = \varphi_i \otimes v_i$$

is admissible.

3.3. **Discretization.** The aim of this section is to construct a Parseval frame for $L^2(\mathbb{R}^d)$ based on a discretization of the reproducing representation π.

We fix a finite subgroup of $\mathrm{SO}(d)$ of cardinality L

$$F = \{R_1, \dots, R_L\},$$

and we choose as grid points those in the family

$$x_{j,k,\ell} = (2^j k, 2^j, R_\ell) \qquad j,k \in \mathbb{Z},\ \ell = 1,\dots,L.$$

We denote by $\widehat{F}$ the set of equivalence classes of irreducible (unitary) representations of F, and for each equivalence class in $\widehat{F}$ we fix a representative $\chi : F \to \mathcal{U}(\mathcal{H}_\chi)$, where $\mathcal{H}_\chi$ is the Hilbert space on which χ acts and $\mathcal{U}(\mathcal{H}_\chi)$ is the corresponding set of unitary operators. The dimension of $\mathcal{H}_\chi$, which is always finite, is denoted by d_χ.

For each $i \in \mathbb{N}$, the representation ρ_i restricted to F decomposes into its irreducibles

$$\mathcal{H}_i = \bigoplus_{\chi\in\widehat{F}} \mathcal{H}_\chi \otimes \mathbb{C}^{m_{i,\chi}} \qquad \rho_i = \bigoplus_{\chi\in\widehat{F}} \chi \otimes \mathrm{I}_{m_{i,\chi}}, \tag{3.18}$$

where $m_{i,\chi} \in \mathbb{N}$ is the multiplicity of χ into ρ_i (with the convention that $\mathbb{C}^0 = \{0\}$ if $m_{i,\chi} = 0$, namely when the representation χ does not enter into the decomposition).

We remark that in the two-dimensional case the picture is clearer (see Section 2 and the remarks that follow (3.7)). Taking $F = \{2\pi l/L : l = 0,\dots,L-1\}$, the set $\widehat{F}$ is given by L one-dimensional representations corresponding to the L-roots of unity, namely $\widehat{F} = \{\chi_l = e^{2\pi i l/L} : l = 0,\dots,L-1\}$. Writing $\mathcal{H}_k = \mathrm{span}\{e^{ik\cdot}\}$ for $k \in \mathbb{Z}$ (as already observed, the natural index set in 2D is $\mathbb{Z}$), a simple calculation shows that ρ_k corresponds to $\chi_{\bar{k}}$, where $\bar{k} = k \mod L$. Therefore, in the above decomposition one has

$$m_{k,\chi_l} = \begin{cases} 1 & \text{if } k - l \in L\mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

or, equivalently, $\mathcal{H}_k = \mathcal{H}_{\chi_{\bar{k}}}$.

From (3.5) and (3.18) we finally obtain the decomposition of ρ into its irreducibles

$$L^2(\mathrm{S}^{d-1}) = \bigoplus_{\chi\in\widehat{F}} \mathcal{H}_\chi \otimes \mathbb{C}^{m_\chi} \qquad \rho = \bigoplus_{\chi\in\widehat{F}} \chi \otimes \mathrm{I}_{m_\chi}, \tag{3.19}$$

where $m_\chi = \sum_{i\in I} m_{i,\chi}$, the operator I_{m_χ} is the identity on $\mathbb{C}^{m_\chi}$ and $\mathbb{C}^\infty = \ell_2(\mathbb{N})$ if $\sum_{i\in I} m_{i,\chi} = \infty$. By (3.11) and (3.19), the following identifications hold true:

$$L^2(\widehat{\mathbb{R}}^d) = \bigoplus_{i\in\mathbb{N},\chi\in\widehat{F}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi) \otimes \mathbb{C}^{m_{i,\chi}} = \bigoplus_{i\in\mathbb{N},\chi\in\widehat{F}} \bigoplus_{\mu=1}^{m_{i,\chi}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi \otimes \mathbb{C}\{\epsilon_\mu\}), \tag{3.20}$$

where $(\epsilon_\mu)_{\mu\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$ and each $\mathbb{C}^{m_{i,\chi}}$ is regarded as a closed subspace of $\ell^2(\mathbb{N})$. According to this decomposition, we denote by $P_{i,\chi,\mu}$ the orthogonal projection from $L^2(\widehat{\mathbb{R}}^d)$ onto the closed subspace $L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi \otimes \mathbb{C}\{\epsilon_\mu\})$ of $L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$.

Next, for each χ, we select an orthogonal family $w_1^\chi, \ldots, w_{d_\chi}^\chi$ in $\mathcal{H}_\chi$ such that

$$\|w_\delta^\chi\|^2 = d_\chi \qquad \delta = 1, \ldots, d_\chi. \tag{3.21}$$

For each $i \in I$, we choose $m_{i,\chi}$-vectors in this family and we denote by $\Delta_{i,\chi} = (\delta_1, \ldots, \delta_{m_{i,\chi}})$ the corresponding family of indices, some of which might be repeated. We set

$$v_{i,\chi,\mu} = w_{\delta_\mu}^\chi \otimes \epsilon_\mu \qquad \mu = 1, \ldots, m_{i,\chi}, \tag{3.22}$$

where each $v_{i,\chi,\mu}$ is a vector in $\mathcal{H}_i$ by means of (3.18).

Finally, we select $m_{i,\chi}$-functions $\varphi_{i,\chi,1}, \ldots, \varphi_{i,\chi,m_{i,\chi}} \in L^2(\widehat{\mathbb{R}}_+)$ such that the following conditions hold true:

a) the series

$$\sum_{i\in\mathbb{N}} \sum_{\chi\in\hat{F}} d_\chi \left(\sum_{\mu=1}^{m_{i,\chi}} \|\varphi_{i,\chi,\mu}\|_2^2 \right) < +\infty; \tag{3.23}$$

b) for each $i \in \mathbb{N}$, $\chi \in \widehat{F}$ and $\mu = 1, \ldots, m_{i,\chi}$

$$\sum_{j\in\mathbb{Z}} |\varphi_{i,\chi,\mu}(2^j\omega)|^2 = \frac{1}{L} \qquad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, \tag{3.24a}$$

and for all odd integers m

$$\sum_{j=0}^{+\infty} \varphi_{i,\chi,\mu}(2^j\omega)\overline{\varphi_{i,\chi,\mu}(2^j(\omega + 2\pi m))} = 0 \qquad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+; \tag{3.24b}$$

c) for all $\chi \in \widehat{F}$, if there exists $i, i' \in \mathbb{N}$ and $\mu = 1, \ldots, m_{i,\chi}$, $\mu' = 1, \ldots, m_{i',\chi}$ such that $(i, \mu) \neq (i', \mu')$, but $w_{\delta_\mu}^\chi = w_{\delta_{\mu'}}^\chi$ (where $\delta_\mu \in \Delta_{i,\chi}$ and $\delta_{\mu'} \in \Delta_{i',\chi}$), then

$$\sum_{j\in\mathbb{Z}} \varphi_{i,\chi,\mu}(2^j\omega)\overline{\varphi_{i',\chi,\mu'}(2^j\omega)} = 0 \qquad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, \tag{3.25a}$$

and for all odd integers m

$$\sum_{j=0}^{+\infty} \varphi_{i,\chi,\mu}(2^j\omega)\overline{\varphi_{i',\chi,\mu'}(2^j(\omega + 2\pi m))} = 0 \qquad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+. \tag{3.25b}$$

Let us comment on the relation between these assumptions and the corresponding ones given in the two-dimensional case. Assumption (3.23) is simply a restatement of the fact that $\widehat{\eta}$ should have finite norm. Assumptions (3.24) and (3.25) correspond to assumptions (2.7) and (2.8), respectively. As we have already anticipated when discussing the 2D case, the condition $(i,\mu) \neq (i',\mu')$ corresponds to $m \neq n$ and $w^{\chi}_{\delta_\mu} = w^{\chi}_{\delta_{\mu'}}$ corresponds to $m - n \in L\mathbb{Z}$.

We are now ready to state the main result of this paper.

Theorem 3.4. *Let $\widehat{\eta} \in L^2(\mathbb{R}^d)$ be defined by*

$$(S\widehat{\eta})_i = \sum_{\chi \in \widehat{F}} \sum_{\mu=1}^{m_{i,\chi}} \varphi_{i,\chi,\mu} \otimes v_{i,\chi,\mu}. \tag{3.26}$$

Then the family $\{\pi(2^j k, 2^j, R_\ell)\widehat{\eta})\}_{j,k\in\mathbb{Z},l=1,\ldots,L}$ is a Parseval frame for $L^2(\mathbb{R}^d)$.

The proof is in Section 3.4. We add a few comments. Since $\sum_{\chi\in\widehat{F}} m_{i,\chi} d_\chi = d_i$, we have

$$\sum_{\chi \in \widehat{F}} \sum_{\mu=1}^{m_{i,\chi}} \|v_{i,\chi,\mu}\|^2_{\mathcal{H}_i} = d_i,$$

hence (3.23) ensures that (3.26) is well defined (compare with (3.17b)).

An important result in wavelet theory [22, Theorem 1.6, Chapter 7] shows that (3.24a) and (3.24b) are equivalent to the fact that for each $i \in \mathbb{N}$, $\chi \in \widehat{F}$ and $\mu = 1, \ldots, m_{i,\chi}$ the family $\{\widehat{W}^+(2^j k, 2^j)\sqrt{L}\varphi_{i,\chi,\mu}\}_{j,k\in\mathbb{Z}}$ is a Parseval frame for $L^2(\widehat{\mathbb{R}}_+)$. Furthermore, (3.24a) implies that

$$\int_{\widehat{\mathbb{R}}_+} \frac{|\varphi_{i,\chi,\mu}(\omega)|^2}{\omega} \mathrm{d}\omega = \frac{\ln 2}{L}, \tag{3.27}$$

so that $\sqrt{L/\ln 2}\,\widehat{\eta}$ is an admissible vector for π by Proposition 3.3.

We now show that there exist families of $\{\varphi_{i,\chi,\mu}\}$, satisfying the above conditions. To this end, fix a function $\varphi \in L^2(\widehat{\mathbb{R}}_+)$ supported in $[0,1]$ and such that

$$\sum_{j\in\mathbb{Z}} |\varphi(2^j\omega)|^2 = \frac{1}{L} \qquad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+. \tag{3.28}$$

Choose a sequence $\{\alpha_{i,\chi,\mu}\}$ such that $0 < \alpha_{i,\chi,\mu} < 1$ and

$$\sum_{i\in\mathbb{N}} \sum_{\chi\in\hat{F}} d_\chi \sum_{\mu=1}^{m_{i,\chi}} \alpha_{i,\chi,\mu} < +\infty. \tag{3.29}$$

Suppose further that, for any $\chi \in \widehat{F}$, if there exists $i, i' \in \mathbb{N}$ and $\mu = 1, \ldots, m_{i,\chi}$, $\mu' = 1, \ldots, m_{i',\chi}$ such that $(i,\mu) \neq (i',\mu')$ but $w^{\chi}_{\delta_\mu} = w^{\chi}_{\delta_{\mu'}}$ (where $\delta_\mu \in \Delta_{i,\chi}$ and $\delta_{\mu'} \in \Delta_{i,\chi}$), then

$$|(\mathrm{supp}(\varphi) \cap \alpha^{-1}_{i,\chi,\mu}\alpha_{i',\chi,\mu'} \,\mathrm{supp}(\varphi)| = 0. \tag{3.30}$$

An explicit example is

$$\varphi = \chi_{(1/2,1]},$$
$$\alpha_{i,\chi,\mu} = \frac{1}{2^{n_{i,\chi,\mu}}},$$

where $(i,\chi,\mu) \mapsto n_{i,\chi,\mu}$ is any bijection from the index set

$$\mathcal{N} = \{(i,\chi,\mu) \mid i \in \mathbb{N},\, \chi \in \widehat{F},\, m_{i,\chi} > 0,\, \mu = 1, \ldots, m_{i,\chi}\}$$

onto $\mathbb{N}$.

With the above choices, define

$$\varphi_{i,\chi,\mu}(\omega) = \varphi(\alpha_{i,\chi,\mu}^{-1}\omega) \qquad \omega \in \widehat{\mathbb{R}}_+.$$

Now, the sum in (3.24b) contains products of the form

$$\varphi\left(\frac{2^j\omega}{\alpha_{i,\chi,\mu}}\right)\varphi\left(\frac{2^j\omega}{\alpha_{i,\chi,\mu}} + \frac{2^j 2\pi m}{\alpha_{i,\chi,\mu}}\right).$$

Since $|2^j 2\pi m/\alpha_{i,\chi,\mu}| > |2^j 2\pi m| > 1$ for every odd integer m and every non-negative integer j, one of the two factors must always vanish, so that (3.24b) holds true. Similarly, (3.30) implies (3.25a) and (3.25b).

3.4. **Proof of Theorem 3.4.** We first prove a technical lemma, which is a variant of a well known result (see Lemma 1.10 of [22]).

We recall that a family $(\psi_i)_{i\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$ is a Parseval frame if one of the following two equivalent conditions is satisfied:

a) for all $f \in \mathcal{H}$

$$\sum_{i\in\mathbb{N}} \langle f, \psi_i\rangle \psi_i = f;$$

b) for all $f \in \mathcal{H}$

$$\sum_{i\in\mathbb{N}} |\langle \psi_i, f\rangle|^2 = \|f\|^2,$$

see Theorem 1.7 Chapter 7 of [22]. Both series convergence unconditionally. For a thorough discussion on frames see e.g. [7, 21].

Lemma 3.5. *Let $(\psi_i)_{i\in\mathbb{N}}$ be a family of vectors in $\mathcal{H}$. If there exists a total subset $\mathcal{S}$ of $\mathcal{H}$ such that*

a) for all $f \in \mathcal{S}$ the sequence $(\langle f, \psi_i\rangle)_{i\in\mathbb{N}}$ is in $\ell^2(\mathbb{N})$;

b) for all $f, g \in \mathcal{S}$

$$\sum_{i\in\mathbb{N}} \langle f, \psi_i\rangle\langle \psi_i, g\rangle = \langle f, g\rangle, \tag{3.31}$$

then the family $(\psi_i)_{i\in\mathbb{N}}$ is a Parseval frame.

Proof. Define

$$\mathcal{D} = \{f \in \mathcal{H} \mid \sum_{i\in\mathbb{N}} |\langle f, \psi_i\rangle|^2 < +\infty\}$$

and $V : \mathcal{D} \to \ell^2(\mathbb{N})$

$$Vf = (\langle f, \psi_i\rangle)_{i\in\mathbb{N}}.$$

By construction, $\mathcal{D}$ is a linear subspace containing $\mathcal{S}$, so that $\mathcal{D}$ is dense and V is a linear operator. It is known that V is a closed operator, see Proposition 2.8 of [19]. By (3.31), the restriction of V to $\mathcal{S}$ preserves the scalar product. By linearity, the same property holds on the linear subspace spanned by $\mathcal{S}$, which is contained in $\mathcal{D}$ and dense in $\mathcal{H}$ since $\mathcal{S}$ is total in $\mathcal{H}$. Then V extends to a unique isometry W from $\mathcal{H}$ into $\ell^2(\mathbb{N})$. Since V is closed, then $\mathcal{D} = \mathcal{H}$ and $V = W$. By definition of V, the family $(\psi_i)_{i\in\mathbb{N}}$ is a Parseval frame. □

The following lemma is a variant of a result given in [19] in the context of admissible representations, see Proposition 2.23.

Lemma 3.6. *Take two countable families* $(\mathcal{H}_j)_{j\in\mathbb{N}}$ *and* $(\mathcal{H}'_j)_{j\in\mathbb{N}}$ *of separable Hilbert spaces, set* $\mathcal{H} = \bigoplus_{j\in\mathbb{N}} \mathcal{H}_j \otimes \mathcal{H}'_j$ *and, for all* $j \in \mathbb{N}$*, denote the canonical projection by* $P_j : \mathcal{H} \to \mathcal{H}_j \otimes \mathcal{H}'_j$*. A family* $(\psi_i)_{i\in\mathbb{N}}$ *is a Parseval frame for* $\mathcal{H}$ *if and only if the following two conditions hold true:*

a) for all $j \in \mathbb{N}$ *and all* $f \in \mathcal{H}_j$*,* $f' \in \mathcal{H}'_j$

$$\sum_{i\in\mathbb{N}} |\langle f \otimes f', P_j\psi_i\rangle|^2 = \|f\|^2_{\mathcal{H}_j}\|f'\|^2_{\mathcal{H}'_j};$$

b) for all $j, k \in \mathbb{N}$*,* $j \neq k$ *and for all* $f \in \mathcal{H}_j$*,* $f' \in \mathcal{H}'_j$*,* $g \in \mathcal{H}_k$*,* $g' \in \mathcal{H}'_k$

$$\sum_{i\in\mathbb{N}} \langle f \otimes f', P_j\psi_i\rangle\langle P_k\psi_i, g \otimes g'\rangle = 0.$$

Proof. Assume that $(\psi_i)_{i\in\mathbb{N}}$ is a Parseval frame for $\mathcal{H}$ and fix $j \in \mathbb{N}$. Given $f \in \mathcal{H}_j$ and $f' \in \mathcal{H}'_j$, we have

$$P_j^*(f \otimes f'_j) = \sum_{i\in\mathbb{N}} \langle P_j^*(f \otimes f'_j), \psi_i\rangle\psi_i.$$

For all $k \in \mathbb{N}$, P_k is a bounded linear operator, and $P_kP_j^* = \delta_{jk}P_jP_j^* = \delta_{jk}\,\mathrm{Id}_{\mathcal{H}_j\otimes\mathcal{H}'_j}$. Then

$$\begin{cases} \sum_{i\in\mathbb{N}} \langle f \otimes f'_j, P_j\psi_i\rangle P_j\psi_i = f \otimes f'_j & k = j \\ \\ \sum_{i\in\mathbb{N}} \langle f \otimes f'_j, P_j\psi_i\rangle P_k\psi_i = 0 & k \neq j, \end{cases}$$

whence a) and b) easily follow.

Conversely, set

$$\mathcal{S} = \bigcup_{j\in\mathbb{N}} \{P_j^*(f \otimes f') \mid f \in \mathcal{H}_j, f' \in \mathcal{H}'_j\},$$

which is total in $\mathcal{H}$ by construction. Conditions a) and b) imply that (3.31) of Lemma 3.5 is satisfied, hence, $(\psi_i)_{i\in\mathbb{N}}$ is a Parseval frame. □

The following result is a restatement of the well known characterization of wavelet Parseval frames. For the sake of clarity, we set $\lambda = (j, k) \in \Lambda = \mathbb{Z}^2$ and $x_\lambda = (2^jk, 2^j) \in \mathbb{R} \times \mathbb{R}_+$.

Lemma 3.7. *If the family* $\{\varphi_{i,\chi,\mu}\}$ *in* $L^2(\widehat{\mathbb{R}}_+)$ *satisfies* (3.24a), (3.24b), (3.25a) *and* (3.25b)*, then*

a) for each $i \in \mathbb{N}$*,* $\chi \in \widehat{F}$ *and* $\mu = 1, \ldots, m_{i,\chi}$

$$\sum_{\lambda\in\Lambda} |\langle \varphi, \widehat{W}^+(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2|^2 = \frac{1}{L}\|\varphi\|_2^2 \tag{3.32}$$

for all $\varphi \in L^2(\widehat{\mathbb{R}}_+)$*;*

b) for all $\chi \in \widehat{F}$*, if there exists* $i, i' \in \mathbb{N}$ *and* $\mu = 1, \ldots, m_{i,\chi}$*,* $\mu' = 1, \ldots, m_{i',\chi}$ *such that* $(i, \mu) \neq (i', \mu')$ *but* $w^\chi_{\delta_\mu} = w^\chi_{\delta_{\mu'}}$ *(where* $\delta_\mu \in \Delta_{i,\chi}$ *and* $\delta_{\mu'} \in \Delta_{i,\chi}$*), then*

$$\sum_{\lambda\in\Lambda} \langle \varphi, \widehat{W}^+(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2\langle \widehat{W}^+(x_\lambda)\varphi_{i',\chi,\mu'}, \varphi'\rangle_2 = 0 \tag{3.33}$$

for all $\varphi, \varphi' \in L^2(\widehat{\mathbb{R}}_+)$*.*

Proof. The fact that (3.32) is equivalent to (3.24a) and (3.24b) is one of the fundamental results at the root of wavelet frames, see Theorem 1.6 of [22]. The fact that (3.25a) and (3.25b) imply (3.33) follows by Lemma 1.18 of [22], which, by polarization, can be rewritten as

$$\begin{aligned}&2\pi\sum_{\lambda\in\Lambda}\langle\varphi,\widehat{W}^{+}(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2\langle\widehat{W}^{+}(x_\lambda)\varphi_{i',\chi,\mu'},\varphi'\rangle_2\\ &=\int_{\widehat{\mathbb{R}}}\varphi(\omega)\overline{\varphi'(\omega)}\sum_{j\in\mathbb{Z}}\varphi_{i',\chi,\mu'}(2^j\omega)\overline{\varphi_{i,\chi,\mu}(2^j\omega)}\mathrm{d}\omega\\ &+\int_{\widehat{\mathbb{R}}}\overline{\varphi'(\omega)}\sum_{j\in\mathbb{Z}}\sum_{m\in 2\mathbb{Z}+1}\varphi_{i',\chi,\mu'}(\omega+2^j2\pi m)h_m(2^j\omega)\mathrm{d}\omega,\end{aligned}$$

where

$$h_m(\omega)=\sum_{n=0}^{+\infty}\varphi_{i',\chi,\mu'}(2^n\omega)\overline{\varphi_{i,\chi,\mu}(2^n(\omega+2\pi m))}.$$

Indeed, (3.25a) implies that the first summand vanishes, whereas (3.25b) implies that h_m vanish for all odd integers, hence the second summand is zero. □

Proof of Theorem 3.4. By means of the unitary operator S, we can prove the result for the family of vectors

$$S\pi(x_{\lambda,\ell})\widehat{\eta}\qquad\lambda\in\Lambda,\quad\ell=1,\dots,L$$

in the space $\bigoplus_{i\in\mathbb{N}}L^2(\widehat{\mathbb{R}}_+,\mathcal{H}_i)$, which by (3.11) and (3.20) can be identified with

$$L^2(\widehat{\mathbb{R}}^d)=\bigoplus_{i\in\mathbb{N},\chi\in\widehat{F}}\bigoplus_{\mu=1}^{m_{i,\chi}}L^2(\widehat{\mathbb{R}}_+,\mathcal{H}_\chi\otimes\mathbb{C}\{\epsilon_\mu\}).$$

We mean to apply Lemma 3.6. So, let us fix $i,i'\in\mathbb{N}$, $\chi,\chi'\in\widehat{F}$ and $\mu\in\{1,\dots,m_{i,\chi}\}$, $\mu'\in\{1,\dots,m_{i,\chi'}\}$. Given $\varphi,\varphi'\in L^2(\widehat{\mathbb{R}}_+)$ and $w\in\mathcal{H}_\chi$, $w'\in\mathcal{H}_{\chi'}$, we look at the quantity

$$\begin{aligned}&A(i,\chi,\mu,i',\chi',\mu')\\ &=\sum_{\lambda\in\Lambda}\sum_{\ell=1}^{L}\langle\varphi\otimes w\otimes\epsilon_\mu,P_{i,\chi,\mu}S\pi(x_{\lambda,\ell})\widehat{\eta}\rangle_2\langle P_{i',\chi',\mu'}S\pi(x_{\lambda,\ell})\widehat{\eta},\varphi'\otimes w'\otimes\epsilon_{\mu'}\rangle_2.\end{aligned}$$

Recall that, since $x_{\lambda,\ell}=(x_\lambda,R_\ell)$,

$$P_{i,\chi,\mu}S\pi(x_{\lambda,\ell})\widehat{\eta}=\widehat{W}^{+}(x_\lambda)\varphi_{i,\chi,\mu}\otimes\chi(R_\ell)w^{\chi}_{\delta_\mu}\otimes\epsilon_\mu,$$

hence we have

$$\begin{aligned}A(i,\chi,\mu,i',\chi',\mu')=&\left(\sum_{\lambda\in\Lambda}\langle\varphi,\widehat{W}^{+}(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2\langle\widehat{W}^{+}(x_\lambda)\varphi_{i',\chi',\mu'},\varphi'\rangle_2\right)\\ &\times\left(\sum_{\ell=1}^{L}\langle w,\chi(R_\ell)w^{\chi}_{\delta_\mu}\rangle_{\mathcal{H}_\chi}\langle\chi'(R_\ell)w^{\chi'}_{\delta_{\mu'}},w'\rangle_{\mathcal{H}_{\chi'}}\right),\end{aligned}$$

where the series are absolutely summable because of (3.32) and the Cauchy-Schwarz inequality.

From the Schur orthogonality relations applied to the pair of irreducible representations χ, χ' of F, we know that

$$\frac{1}{L}\sum_{\ell=1}^{L}\langle w, \chi(R_\ell)w^{\chi}_{\delta_\mu}\rangle_{\mathcal{H}_\chi}\langle \chi'(R_\ell)w^{\chi'}_{\delta_{\mu'}}, w'\rangle_{\mathcal{H}_{\chi'}} = \begin{cases} 0 & \chi \neq \chi' \\ \frac{1}{d_\chi}\langle w^{\chi}_{\delta_{\mu'}}, w^{\chi}_{\delta_\mu}\rangle_{\mathcal{H}_\chi}\langle w, w'\rangle_{\mathcal{H}_\chi} & \chi = \chi'. \end{cases}$$

Thus, if $\chi \neq \chi'$, we get $A(i,\chi,\mu,i',\chi',\mu') = 0$. From now on assume $\chi = \chi'$, for which

$$\begin{aligned} A(i,\chi,\mu,i',\chi,\mu') = L\left(\sum_{\lambda\in\Lambda}\langle \varphi, \widehat{W}^+(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2\langle \widehat{W}^+(x_\lambda)\varphi_{i',\chi,\mu'}, \varphi'\rangle_2\right) \\ \times\frac{1}{d_\chi}\langle w^{\chi}_{\delta_{\mu'}}, w^{\chi}_{\delta_\mu}\rangle_{\mathcal{H}_\chi}\langle w, w'\rangle_{\mathcal{H}_\chi}. \end{aligned}$$

If $(i,\mu) \neq (i',\mu')$ and $\delta_\mu \neq \delta_{\mu'}$, then $A(i,\chi,\mu,i',\chi,\mu') = 0$ since the family $w^{\chi}_1, \dots, w^{\chi}_{d_\chi}$ is orthonormal. If $(i,\mu) \neq (i',\mu')$ but $\delta_\mu = \delta_{\mu'}$, then by (3.33) it follows that $A(i,\chi,\mu,i',\chi,\mu') = 0$. Finally, if $(i,\mu) = (i',\mu')$, then (3.21) yields

$$\begin{aligned} A(i,\chi,\mu,i,\chi,\mu) &= L\left(\sum_{\lambda\in\Lambda}\langle \varphi, \widehat{W}^+(x_\lambda)\varphi_{i,\chi,\mu}\rangle_2\langle \widehat{W}^+(x_\lambda)\varphi_{i,\chi,\mu}, \varphi'\rangle_2\right)\langle w, w'\rangle_{\mathcal{H}_\chi} \\ &= \langle \varphi, \varphi'\rangle_2\langle w, w'\rangle_{\mathcal{H}_\chi} \\ &= \langle \varphi\otimes w\otimes \epsilon_\mu, \varphi'\otimes w'\otimes\epsilon_\mu\rangle_2, \end{aligned}$$

where the second equality is a consequence of (3.32).

Summarizing the above results in a single equation, we obtain

$$A(i,\chi,\mu,i',\chi',\mu') = \begin{cases} \langle \varphi\otimes w\otimes \epsilon_\mu, \varphi'\otimes w'\otimes\epsilon_\mu\rangle_2 & \text{if } \chi = \chi' \text{ and } (i,\mu) = (i',\mu'), \\ 0 & \text{if } \chi \neq \chi' \text{ or } (i,\mu) \neq (i',\mu'). \end{cases}$$

The conclusion follows from Lemma 3.6. □

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