QUANTUM HOMODYNE TOMOGRAPHY AS AN INFORMATIONALLY COMPLETE POSITIVE OPERATOR VALUED MEASURE

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Abstract. We define a positive operator valued measure E on $[0, 2\pi] \times \mathbb{R}$ describing the measurement of randomly sampled quadratures in quantum homodyne tomography, and we study its probabilistic properties. Moreover, we give a mathematical analysis of the relation between the description of a state in terms of E and the description provided by its Wigner transform.

Quantum homodyne tomography; positive operator valued measures; Wigner function.

1. Introduction

Quantum homodyne tomography [1, 2, 3, 4] allows to determine the state of a single mode radiation field by repeated measurements of the quadrature observables X_θ, the phases θ being chosen randomly in $\mathbb{T} = [0, 2\pi]$. This can be seen as a consequence of the fact [5] that, for a large class of observables O, there exists an associated function $f_O : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{tr}\,[O\rho] = \int_0^{2\pi} \left[\int_{\mathbb{R}} f_O(\theta, x) \mathrm{d}\nu_\theta^\rho(x) \right] \frac{\mathrm{d}\theta}{2\pi}, \tag{1}$$

where ν_θ^ρ is the probability distribution of outcomes obtained for the quadrature X_θ measured on the state ρ and $\mathrm{d}\theta/2\pi$ is the uniform probability distribution on $\mathbb{T}$. Actual reconstrunction schemes are strictly related to a statistical interpetation of formulas of this kind. Indeed, quantum tomography experiments output a n-uple $\{(\Theta_i, X_i)\}_{i=1}^n$ of pairs in $\mathbb{T} \times \mathbb{R}$, each one of which represents the outcome X_i of a measurement of the quadrature observable corresponding to the randomly picked phase Θ_i. If one assumes such pairs to be samples from a random variable on $\mathbb{T} \times \mathbb{R}$ distributed accordingly to a probability measure μ^ρ such that

$$\mathrm{d}\mu^\rho(\theta, x) = \mathrm{d}\nu_\theta^\rho(x) \frac{\mathrm{d}\theta}{2\pi}, \tag{2}$$

one can use the experimental outcomes to estimate integrals such as (1) (see Ref. [2] and references therein), for example by replacing $\mathrm{d}\mu^\rho$ with its empirical estimate $\frac{1}{n} \sum_{i=1}^n \delta_{(\Theta_i, X_i)}$.

The above reconstruction formula, although very popular, is not the only scheme used for tomographical state estimation: other ones are known which don't rely on it [2]. The hypotesis that experimental results are distributed accordingly to (2) lies however under both every proposed reconstruction algorithm and its statistical analysis [1, 2, 3, 6, 7]. Actually, although given for granted in the cited literature, well-definiteness of a joint probability distribution such as μ^ρ in (2) is *a priori* not trivial. In the first part of our paper we prove well defineteness of μ^ρ by showing there exists a positive operator valued measure (POVM) E on $\mathbb{T} \times \mathbb{R}$ such that

$$\mu^\rho(Z) = tr[E(Z)\rho] \qquad \text{for any Borel subset } Z \text{ of } \mathbb{T} \times \mathbb{R}.$$

According to the physical meaning of μ^ρ, the POVM E is the *generalized observable* associated with the quantum homodyne tomography experimental setup. In particular, we show that μ^ρ has density $p^\rho(\theta, x)$ with respect to the Lebesgue measure on $\mathbb{T} \times \mathbb{R}$, its support is always an unbounded set, and the mapping $\rho \mapsto \mu^\rho$ is injective (i. e. E is *informationally complete*). The intertwining property $X_\theta = e^{i\theta N} X e^{-i\theta N}$, where N is the number operator and X is the position operator, turns out to be crucial for the definition of E (or, equivalently, for the definition of μ^ρ). We remark that the introduction of a POVM for the homodyne tomography measurement process is already present in physical literature (see section 2.3.2 in Ref. [2]), but it is grounded on a formal construction. We provide here an alternative, rigorous formulation.

In their seminal paper on quantum homodyne tomography [4], Vogel and Risken argued that the the Radon transform of $W(\rho)$, where $W(\rho)$ is the Wigner function associated to ρ, is precisely the probability density function p^ρ generated by the homodyne tomography measurement, so that the following commutative diagram holds:

$$\begin{tikzcd}
 & \text{States on } \mathcal{H} \arrow[dl, "W"'] \arrow[dr, "p^{\cdot}"] & \\
\text{Wigner functions on } \mathbb{R}^2 \arrow[rr, "\text{Radon transform}"] & & \text{probability densities on } \mathbb{T}\times\mathbb{R}
\end{tikzcd}$$

The suggested estimation procedure, applied also in the first homodyne tomography experiments, is then based on the inversion of the Radon transform by means of classical techniques in medical tomography. However, the derivation of this fact is once again rather formal, and never given a rigorous basis in the literature on the subject: in the second part of the paper, we will thus address the problems that arise in looking at such formulation of quantum tomography from a rigorous point of view. First of all, we will recall that in order for the Radon transform to be well-defined, we need the Wigner function $W(\rho)$ to be integrable on $\mathbb{R}^2$. Then we will show that the support of $W(\rho)$ can *never* be bounded. This is a potential problem, since the estimation techniques used in classical tomography are explicitly devised for compactly supported objects. One can however by-pass the problem and still give an inverse for the Radon transform if he assumes that the Wigner function under observation is a Schwartz function on $\mathbb{R}^2$. This is precisely what happens in most homodyne tomography experiments, where the states under observation are linear combinations of coherent or number states. In Section 4 we show that this assumption on the Wigner function is equivalent to suppose that ρ has a kernel which is a Schwartz function on $\mathbb{R}^2$ (since ρ is an Hilbert-Schmidt operator, ρ is an integral operator whose kernel is a function on $\mathbb{R}^2$). Under this assumption on ρ we prove that the Radon transform of $W(\rho)$ is p^ρ and the inversion formula holds true.

2. Preliminaries and notations

In this section, we will introduce the notations and give a very brief description of the mathematical structure of quantum homodyne tomography.

2.1. **Notations.** Let $\mathcal{H}$ be a complex, separable Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot\,,\,\cdot\rangle$ linear in the second entry. Denote by $\mathcal{L}(\mathcal{H})$ the Banach space of the bounded operators on $\mathcal{H}$ with uniform norm $\|\cdot\|_{\mathcal{L}}$. Let $\mathcal{I}_1(\mathcal{H})$ be the Banach space of the trace class operators on $\mathcal{H}$ with trace class norm $\|\cdot\|_1$, and let $\mathcal{S}(\mathcal{H})$ be the convex subset of positive trace one elements in $\mathcal{I}_1(\mathcal{H})$. Finally, let $\mathcal{I}_2(\mathcal{H})$ be the Hilbert-Schmidt operators on $\mathcal{H}$, with norm $\|A\|_2 = [\operatorname{tr}[A^*A]]^{1/2}$. We recall that the elements of $\mathcal{S}(\mathcal{H})$ are the *states* of the quantum system whose associated Hilbert space is $\mathcal{H}$.

Suppose Ω is a Hausdorff locally compact second countable topological space. Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω. We recall the following definition of positive operator valued measure.

definition 1. A *positive operator valued measure* (POVM) on Ω with values in $\mathcal{H}$ is a map $E : \mathcal{B}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ such that

(i) $E(A) \geq 0$ for all $A \in \mathcal{B}(\Omega)$;
(ii) $E(\Omega) = I$;
(iii) if $\{A_i\}_{i\in I}$ is a denumerable sequence of pairwise disjoint sets in $\mathcal{B}(\Omega)$, then

$$E(\cup_i A_i) = \sum\nolimits_i E(A_i),$$

where the sum converges in the weak (or, equivalently, ultraweak or strong) topology of $\mathcal{L}(\mathcal{H})$.

E is a *projection valued measure* (PVM) if $E(A)^2 = E(A)$ for all $A \in \mathcal{B}(\Omega)$.

If E is a POVM and $T \in \mathcal{I}_1(\mathcal{H})$, we define

$$\mu_E^T(A) = \operatorname{tr}[E(A)T] \quad \forall A \in \mathcal{B}(\Omega).$$

Then, μ_E^T is a bounded complex measure on Ω. If $\rho \in \mathcal{S}(\mathcal{H})$, μ_E^ρ is actually a probability measure on Ω, and $\mu_E^\rho(A)$ is the probability of obtaining a result in A when performing a measurement of E on the state ρ.

2.2. **The mathematics of quantum homodyne tomography.** The physical system of quantum homodyne tomography is a single radiation mode of the electromagnetic field. The associated Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$. Let

$$\mathcal{A} = \left\{ p(x)\, e^{-\frac{x^2}{2}} \mid p \text{ is a polinomial,} \right\}$$

which is a dense subspace of $\mathcal{H}$. As usual, we denote by X and P the position and momentum operators, respectively. Their action on $\mathcal{A}$ is explicitly given by

$$(Xf)(x) = xf(x) \qquad \text{and} \qquad (Pf)(x) = -i\frac{\mathrm{d}f}{\mathrm{d}x}(x).$$

Letting $\mathbb{T} = [0, 2\pi]$, for any $\theta \in \mathbb{T}$ the corresponding quadrature is the self-adjoint operator X_θ on $L^2(\mathbb{R})$, whose action on $\mathcal{A}$ is

$$X_\theta = \cos\theta X + \sin\theta P.$$

If $x, y \in \mathbb{R}$, and $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\left[e^{irX_\theta} f\right](z) = \left[e^{i(xX+yP)} f\right](z) = e^{i\left(\frac{xy}{2}+xz\right)} f(z+y)$$

for all $f \in L^2(\mathbb{R})$.

We denote by Π_θ the PVM on $\mathbb{R}$ associated to X_θ by spectral theorem. In particular, $\Pi(A) := \Pi_0(A)$ is just multiplication in $L^2(\mathbb{R})$ by the characteristic function 1_A of A, while $\Pi_{\frac{\pi}{2}}(A) = \mathcal{F}^*\Pi(A)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y)\mathrm{d}y \quad f \in L^1 \cap L^2(\mathbb{R}). \tag{3}$$

The number operator is the essentially self-adjoint operator N whose action on $\mathcal{A}$ is

$$N = \frac{1}{2}\left(X^2 + P^2 - 1\right).$$

For all $\theta \in \mathbb{T}$, we let $V(\theta) = e^{i\theta N}$. Since the spectrum of N is $\mathbb{N}$, the map $\theta \to V(\theta)$ is a unitary continuous representation of $\mathbb{T}$ acting on $L^2(\mathbb{R})$, where we regard $\mathbb{T}$ as a topological abelian group with addition modulo 2π. The number representation V intertwines the quadratures X_θ, in the sense that

$$X_\theta = V(\theta)XV(\theta)^*$$

for all $\theta \in \mathbb{T}$, and

$$\Pi_\theta(A) = V(\theta)\Pi(A)V(\theta)^*$$

for all $\theta \in \mathbb{T}$ and $A \in \mathcal{B}(\mathbb{R})$.

Finally, given $\rho \in \mathcal{S}(\mathcal{H})$ and $\theta \in \mathbb{T}$, we denote by ν_θ^ρ the probability distribution on $\mathbb{R}$ of the outcomes of the quadrature X_θ measured on the state ρ, namely

$$\nu_\theta^\rho(A) = \mathrm{tr}\left[\rho\Pi_\theta(A)\right] = \mathrm{tr}\left[\rho V(\theta)\Pi(A)V(\theta)^*\right] \quad \forall A \in \mathcal{B}(\mathbb{R}). \tag{4}$$

3. Main results

In this section, we will describe explicitly the POVM which intervenes in homodyne tomography and the associated probability distributions on states.

The first result studies some properties of the family of probability measures ν_θ^ρ defined by (4). In its proof and in the statement of some of the following results, we will make use of the concept of section through some $\theta \in \mathbb{T}$ of a Borel set $B \in \mathcal{B}(\mathbb{T}\times\mathbb{R})$, defined as follows:

$$B^\theta = \{x \in \mathbb{R} | (\theta, x) \in B\}.$$

Proposition 1. *Given $\rho \in \mathcal{S}(\mathcal{H})$ and $\theta \in \mathbb{T}$*

(i) *the probability measure ν_θ^ρ has density $p_\theta^\rho \in L^1(\mathbb{R})$ with respect to the Lebesgue measure on $\mathbb{R}$;*
(ii) *the map $\theta \mapsto \nu_\theta^\rho(B^\theta)$ is measurable for any $B \in \mathcal{B}(\mathbb{T}\times\mathbb{R})$.*

Proof. If $A \in \mathcal{B}(\mathbb{R})$ has zero Lebesgue measure, then $\Pi(A)f = 1_A f = 0$ for all $f \in L^2(\mathbb{R})$. Therefore, $\nu_\theta^\rho(A) = \mathrm{tr}\left[\rho V(\theta)\Pi(A)V(\theta)^*\right] = 0$. Thus, the first claim follows.

If $\{e_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of $\mathcal{H}$, then

$$\begin{aligned} \mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right] &= \sum_n \left\langle e_n\, ,\, V(\theta)^*\rho V(\theta)\Pi(B^\theta)e_n\right\rangle \\ &= \sum_n\sum_m \left\langle e_n\, ,\, V(\theta)^*\rho V(\theta)e_m\right\rangle \left\langle e_m\, ,\, \Pi(B^\theta)e_n\right\rangle. \end{aligned}$$

Since the map $\theta \mapsto \langle e_n\, ,\, V(\theta)^*\rho V(\theta)e_m\rangle$ is continuous and the map $\theta \mapsto \left\langle e_m\, ,\, \Pi(B^\theta)e_n\right\rangle = \int 1_B(\theta,x)e_n(x)\overline{e_m(x)}\mathrm{d}x$ is measurable by Fubini theorem, measurability of $\theta \mapsto \mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right]$ follows. □

Next theorem shows the existence of a POVM associated to quantum homodyne tomography. This theorem should be compared with the formal derivation of E given in Ref. [2] (see eq. (2.34) therein).

Theorem 1. *There exists a unique positive operator valued measure E on $\mathbb{T}\times\mathbb{R}$ acting in $L^2(\mathbb{R})$ such that*

$$\mathrm{tr}\,[\rho E(B)] = \int_{\mathbb{T}} \nu_\theta^\rho(B^\theta)\frac{\mathrm{d}\theta}{2\pi} \tag{5}$$

for all $\rho\in\mathcal{S}(\mathcal{H})$ and $B\in\mathcal{B}(\mathbb{T}\times\mathbb{R})$.

Proof. Eq. (4) suggests to define the POVM as

$$E(B) = \int_{\mathbb{T}} V(\theta)\Pi(B^\theta)V(\theta)^*\frac{\mathrm{d}\theta}{2\pi}.$$

To prove that the above definition is correct, we first show that the map $\theta\mapsto V(\theta)\Pi(B^\theta)V(\theta)^*$ is $\frac{\mathrm{d}\theta}{2\pi}$-ultraweakly integrable for all Borel subsets B of $\mathbb{T}\times\mathbb{R}$, and then we prove that $B\mapsto E(B)$ is a POVM.

Now, given $B\in\mathcal{B}(\mathbb{T}\times\mathbb{R})$ and $\rho\in\mathcal{S}(\mathcal{H})$, the map $\theta\mapsto \mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right]$ is measurable by the previous proposition, and

$$\left|\mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right]\right| \leq \|\rho\|_1 \left\|\Pi(B^\theta)\right\|_{\mathcal{L}} \leq 1 \quad \forall\theta\in\mathbb{T}.$$

Therefore, it is $\frac{\mathrm{d}\theta}{2\pi}$-integrable. This shows that $\theta\mapsto V(\theta)\Pi(B^\theta)V(\theta)^*$ is $\frac{\mathrm{d}\theta}{2\pi}$-ultraweakly integrable.

Suppose $T\in\mathcal{I}_1(\mathcal{H})$. Then $T=\sum_{k=0}^3 i^k T_k$, with $T_k\geq 0$ and $\|T_0\|_1+\|T_2\|_1=\|T_1\|_1+\|T_3\|_1\leq\|T\|_1$. Setting $\rho_k = T_k/\|T_k\|_1$ (with $0/0=0$), we see that

$$\begin{aligned}\left|\int_{\mathbb{T}}\mathrm{tr}\left[TV(\theta)\Pi(B^\theta)V(\theta)^*\right]\frac{\mathrm{d}\theta}{2\pi}\right| &\leq \sum_{k=0}^3\|T_k\|_1\int_{\mathbb{T}}\left|\mathrm{tr}\left[\rho_k V(\theta)\Pi(B^\theta)V(\theta)^*\right]\right|\frac{\mathrm{d}\theta}{2\pi}\\ &\leq \sum_{k=0}^3\|T_k\|_1 \leq 2\|T\|_1.\end{aligned}$$

This shows the existence of $E(B)\in\mathcal{L}(\mathcal{H})$. Clearly, $E(B)\geq 0$, and $E(\mathbb{T}\times\mathbb{R})=I$.

If $\{B_n\}_{n\in\mathbb{N}}$ is a monotone increasing family of elements in $\mathcal{B}(\mathbb{T}\times\mathbb{R})$, with $B_n\uparrow B$, then, for all θ,

$$\mathrm{tr}\left[\rho V(\theta)\Pi(B_n^\theta)V(\theta)^*\right] = \nu_\theta^\rho(B_n^\theta)\uparrow\nu_\theta^\rho(B^\theta) = \mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right].$$

By dominated convergence theorem

$$\int_{\mathbb{T}}\mathrm{tr}\left[\rho V(\theta)\Pi(B_n^\theta)V(\theta)^*\right]\frac{\mathrm{d}\theta}{2\pi}\uparrow\int_{\mathbb{T}}\mathrm{tr}\left[\rho V(\theta)\Pi(B^\theta)V(\theta)^*\right]\frac{\mathrm{d}\theta}{2\pi},$$

and ultraweak σ-additivity of E follows. □

We let $\mu^\rho = \mathrm{tr}\,[E(\cdot)\rho]$ be the probability distribution on $\mathbb{T}\times\mathbb{R}$ associated to a measurement of E performed on the state ρ. By definition (5) it follows that

$$\mu^\rho(B) = \int_{\mathbb{T}}\nu_\theta^\rho(B^\theta)\frac{\mathrm{d}\theta}{2\pi} \tag{6}$$

as wanted. The following theorem gives some properties of μ^ρ.

Theorem 2. *Let $\rho\in\mathcal{S}(\mathcal{H})$.*

(i) *The measure μ^ρ has density with respect to $\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{d}x$. We denote such density by p^ρ.*
(ii) *For $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ, $p^\rho(\theta,x)=p_\theta^\rho(x)$ for $\mathrm{d}x$-almost all x.*
(iii) *The marginal probability distribution induced by μ^ρ on $\mathbb{T}$ is the Haar measure $\frac{\mathrm{d}\theta}{2\pi}$, and the conditional probability distribution induced by μ^ρ on $\mathbb{R}$ is ν_θ^ρ for $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ.*

Proof. (i) If $B\in\mathcal{B}(\mathbb{T}\times\mathbb{R})$ is a $\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{d}x$-null set, then B^θ is $\mathrm{d}x$-null for $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ by Fubini theorem, so, for such θ's, $\nu_\theta^\rho(B^\theta)=0$. Therefore, $\mu^\rho(B)=0$ by (6), thus showing that μ^ρ has density with respect to $\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{d}x$.

(ii) If $Z\in\mathcal{B}(\mathbb{T})$, $A\in\mathcal{B}(\mathbb{R})$, we have

$$\int_Z\frac{\mathrm{d}\theta}{2\pi}\int_A p^\rho(\theta,x)\mathrm{d}x = \mu^\rho(Z\times A) = \int_Z\nu_\theta^\rho(A)\frac{\mathrm{d}\theta}{2\pi}.$$

This holds for all Z, implying that there exists a $\frac{\mathrm{d}\theta}{2\pi}$-null set $N_A\in\mathcal{B}(\mathbb{T})$ such that $p^\rho(\theta,\cdot)$ is $\mathrm{d}x$-integrable with

$$\int_A p^\rho(\theta,x)\mathrm{d}x = \nu_\theta^\rho(A)$$

for all $\theta\notin N_A$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(\mathbb{R})$ with the following property: if μ_1, μ_2 are positive measures on $\mathbb{R}$ such that $\mu_1(A_n) = \mu_2(A_n)$ for all n, then $\mu_1 = \mu_2$ (such sequence exists since $\mathbb{R}$ is second countable by Theorems C §5 and A §13 in Ref. [8]). Let $N = \cup_n N_{A_n}$. Then N is $\frac{\mathrm{d}\theta}{2\pi}$-null, and, if $\theta \notin N$, $p^\rho(\theta, \cdot)$ is integrable with

$$\int_{A_n} p^\rho(\theta, x)\mathrm{d}x = \int_{A_n} p^\rho_\theta(x)\mathrm{d}x \quad \forall n.$$

This implies that, if $\theta \notin N$, $p^\rho(\theta, x) = p^\rho_\theta(x)$ for $\mathrm{d}x$-almost all x.

(iii) This is just (6).

□

Remark 1. As a consequence of item (iii) in the above proposition, a well known result on conditional probability distribution ensures that, if ϕ is a μ^ρ-integrable function, then $\phi(\theta, \cdot)$ is ν^ρ_θ-integrable for $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ, the map $\theta \mapsto \int_{\mathbb{R}} \phi(\theta, x)\mathrm{d}\nu^\rho_\theta(x)$ is $\frac{\mathrm{d}\theta}{2\pi}$-integrable, and

$$\int_{\mathbb{T}\times\mathbb{R}} \phi(\theta, x)\mathrm{d}\mu^\rho(\theta, x) = \int_{\mathbb{T}} \left[\int_{\mathbb{R}} \phi(\theta, x)\mathrm{d}\nu^\rho_\theta(x)\right] \frac{\mathrm{d}\theta}{2\pi}.$$

By Theorem 2, E is the POVM associated to the measurement of a quadrature X_θ chosen randomly from $\mathbb{T}$ with uniform probability $\frac{\mathrm{d}\theta}{2\pi}$.

The next corollary shows that the probability distribution μ^ρ can not have compact support for any $\rho \in \mathcal{S}(\mathcal{H})$.

Corollary 1. *For all $R > 0$, we have*

$$\int_{\mathbb{T}} \int_{|x|>R} p^\rho(\theta, x)\mathrm{d}x \frac{\mathrm{d}\theta}{2\pi} > 0.$$

Proof. With $A_R = \{x \in \mathbb{R} \mid |x| > R\}$, we have

$$\begin{aligned} \int_{\mathbb{T}} \int_{|x|>R} p^\rho(\theta, x)\mathrm{d}x \frac{\mathrm{d}\theta}{2\pi} &= \mu^\rho(\mathbb{T} \times A_R) = \int_{\mathbb{T}} \mathrm{tr}\left[\rho V(\theta)\Pi(A_R)V(\theta)^*\right] \frac{\mathrm{d}\theta}{2\pi} \\ &= \mathrm{tr}\left[\rho'\Pi(A_R)\right], \end{aligned}$$

with $\rho' = \int_{\mathbb{T}} V(\theta)^*\rho V(\theta)\frac{\mathrm{d}\theta}{2\pi}$. ρ' is a trace one positive operator. Since it commutes with the representation V of $\mathbb{T}$, it is diagonal in the number basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R})$. Since $\langle e_n\, , \Pi(A_R)e_n\rangle > 0$ for all n, the claim follows. □

As a consequence, the map $\rho \mapsto p^\rho$ from $\mathcal{S}(\mathcal{H})$ to the set $P(\mathbb{T}\times\mathbb{R})$ of probability densities in $L^1(\mathbb{T} \times \mathbb{R})$ is not surjective. The next corollary shows that it is actually injective, i. e. the POVM E is *informationally complete* [9].

Corollary 2. *If $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $\rho \neq \sigma$, then $\mu^\rho \neq \mu^\sigma$.*

Proof. If $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, then $\mu^\rho = \mu^\sigma$ if and only if $p^\rho = p^\sigma$ (in $L^1(\mathbb{T} \times \mathbb{R})$), which amounts to say that $p^\rho_\theta = p^\sigma_\theta$ (in $L^1(\mathbb{R})$) for $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ. This is in turn equivalent to $\nu^\rho_\theta = \nu^\sigma_\theta$ for $\frac{\mathrm{d}\theta}{2\pi}$-almost all θ. For $r \in \mathbb{R}$ and $\theta \in \mathbb{T}$, we have by spectral theorem

$$\int_{\mathbb{R}} e^{irx}\mathrm{d}\nu^\rho_\theta(x) = \int_{\mathbb{R}} e^{irx}\mathrm{tr}\left[\rho\Pi_\theta(\mathrm{d}x)\right] = \mathrm{tr}\left[\rho e^{irX_\theta}\right] = \sqrt{2\pi}\,[V(\rho)](r\cos\theta, r\sin\theta),$$

where

$$[V(\rho)](x, y) = \frac{1}{\sqrt{2\pi}}\mathrm{tr}\left[\rho e^{i(xX+yP)}\right].$$

Since the map $V : \mathcal{I}_1(\mathcal{H}) \longrightarrow C(\mathbb{R}^2)$ is injective (see for example Ref. [10]), injectivity of the map $\rho \mapsto \mu^\rho$ follows. □

4. The Radon transform of the Wigner function and Radon reconstruction formula

In the previous section, by means of the POVM E defined in Theorem 1 we estabilished a convex injective correspondence $\rho \mapsto p^\rho$ between states and the set of probability densities on $\mathbb{T} \times \mathbb{R}$. However, no explicit formula relating ρ to the function p^ρ was given, due to the fact that, if ρ does not have a simple expression in terms of the number basis, the expression $\mathrm{tr}\left[V(\theta)^*\rho V(\theta)\Pi(A)\right]$ can not be explicitly computed.

In this section, we will show that, if the state ρ is sufficiently regular, p^ρ can indeed be evaluated, being in fact the Radon transform of the Wigner function $W(\rho)$ of ρ. This is a very well known fact in quantum tomography, going back to the seminal paper of Vogel and Risken [4]. However, no attention has never been paid in the literature to the fact that performing the Radon transform of $W(\rho)$ makes sense only for a restricted class of states, namely for those $\rho \in \mathcal{S}(\mathcal{H})$ such that $W(\rho) \in L^1(\mathbb{R}^2)$. This constraint becomes even more stringent when one considers the inverse formula reconstructing ρ (or, better, $W(\rho)$) from its associated probability density p^ρ. We will see that, in order to derive mathematically consistent formulas both for p^ρ and the reconstruction of $W(\rho)$, one needs to assume that the state belongs to the set of Schwartz functions on $\mathbb{R}^2$. This seems a rather strong limitation, as the very natural attempt to extend the Radon transform and Radon reconstruction to the whole set $\mathcal{S}(\mathcal{H})$ by means of distribution theory fails in the quantum context (Remark 2). Our main reference to the results below is Ref. [11].

If $T \in \mathcal{I}_1(\mathcal{H})$, we introduce the bounded continuous function $V(T)$ on $\mathbb{R}^2$, given by

$$[V(T)](x,y) = \frac{1}{\sqrt{2\pi}} \operatorname{tr}\left[T e^{i(xX+yP)}\right]. \tag{7}$$

It is well known (see for example Ref. [10]) that $V(T) \in L^2(\mathbb{R}^2)$, and V uniquely extends to a unitary operator $V : \mathcal{I}_2(\mathcal{H}) \longrightarrow L^2(\mathbb{R}^2)$. The *Wigner transform* of $A \in \mathcal{I}_2(\mathcal{H})$ is just (up to a constant) the Fourier transform of $V(A)$, i. e.

$$W(A) = \frac{1}{\sqrt{2\pi}} \mathcal{F}_2 V(A), \tag{8}$$

where $\mathcal{F}_2 = \mathcal{F} \otimes \mathcal{F}$ on $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, with $\mathcal{F}$ defined in (3).

If $f \in L^1(\mathbb{R}^2)$, the *Radon transform* of f is the complex function $Rf \in L^1(\mathbb{T} \times \mathbb{R})$ given by

$$Rf(\theta, r) = \int_{-\infty}^{+\infty} f(r\cos\theta - t\sin\theta, r\sin\theta + t\cos\theta)\mathrm{d}t \tag{9}$$

$\frac{\mathrm{d}\theta}{2\pi}\mathrm{d}r$-almost everywhere.

We have the following fact.

Proposition 2. *If $W(\rho) \in L^1(\mathbb{R}^2)$, then*

$$[RW(\rho)](\theta, r) = p^\rho(\theta, r) \tag{10}$$

for $\frac{\mathrm{d}\theta}{2\pi}\mathrm{d}r$-almost all (θ, r).

Proof. Let $\gamma : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}^2$ be the map

$$\gamma(\theta, r) = (r\cos\theta, r\sin\theta).$$

We have

$$[V(\rho)\circ\gamma](\theta, r) = \frac{1}{\sqrt{2\pi}}\operatorname{tr}\left[\rho e^{irX_\theta}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{irt} p^\rho_\theta(t)\mathrm{d}t = \left[\mathcal{F}^{-1}p^\rho_\theta\right](r)$$

by spectral theorem. On the other hand,

$$\begin{aligned}
[\mathcal{F}_2^{-1}W(\rho)\circ\gamma](\theta, r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(xr\cos\theta + yr\sin\theta)}[W(\rho)](x,y)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi}\int_0^\pi \int_{-\infty}^{+\infty} e^{itr(\cos\phi\cos\theta + \sin\phi\sin\theta)}[W(\rho)](t\cos\phi, t\sin\phi)|t|\mathrm{d}t\frac{\mathrm{d}\phi}{2\pi} \\
&= \frac{1}{2\pi}\int_0^\pi \int_{-\infty}^{+\infty} e^{itr\cos(\phi-\theta)}[W(\rho)](t\cos\phi, t\sin\phi)|t|\mathrm{d}t\frac{\mathrm{d}\phi}{2\pi} \\
&= \frac{1}{2\pi}\int_0^\pi \int_{-\infty}^{+\infty} e^{itr\cos\phi}[W(\rho)](t\cos(\phi+\theta), t\sin(\phi+\theta))|t|\mathrm{d}t\frac{\mathrm{d}\phi}{2\pi} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{irx}[W(\rho)](x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{irx}[RW(\rho)](\theta, x)\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\left[\mathcal{F}^{-1}[RW(\rho)](\theta,\cdot)\right](r).
\end{aligned}$$

By injectivity of Fourier transform, the claim then follows by comparison. □

Corollary 3. *The support of $W(\rho)$ is an unbounded subset of $\mathbb{R}^2$ for all $\rho \in \mathcal{S}(\mathcal{H})$.*

Proof. Suppose by contradiction that $W(\rho)=0$ almost everywhere outside the disk D_R of radius R in $\mathbb{R}^2$. Then $W(\rho)\in L^1\left(\mathbb{R}^2\right)$, and so $[RW(\rho)](\theta,r)=p^\rho(\theta,r)$ by the above proposition. We have

$$\begin{aligned}\int_0^{2\pi}\int_{|r|>R}|RW(\rho)(\theta,r)|\mathrm{d}r\frac{\mathrm{d}\theta}{2\pi} &\leq \int_0^{2\pi}\iint_{\mathbb{R}^2\setminus D_R}|W(\rho)(r\cos\theta-t\sin\theta,r\sin\theta+t\cos\theta)|\mathrm{d}r\mathrm{d}t\frac{\mathrm{d}\theta}{2\pi}\\ &= \int_0^{2\pi}\iint_{\mathbb{R}^2\setminus D_R}|W(\rho)(r,t)|\mathrm{d}r\mathrm{d}t\frac{\mathrm{d}\theta}{2\pi}=0,\end{aligned}$$

which contradicts Corollary 1. □

The first formal derivation of (10) is contained in Ref. [4], without the assumption $W(\rho)\in L^1\left(\mathbb{R}^2\right)$. We stress that if $W(\rho)\notin L^1\left(\mathbb{R}^2\right)$, then (10) does not make sense, and the only possible definition of p^ρ is by means of item 1 in Theorem 2.

If we denote by $\mathcal{S}^1(\mathcal{H})$ the subset of states $\rho\in\mathcal{S}(\mathcal{H})$ such that $W(\rho)\in L^1\left(\mathbb{R}^2\right)$, then we have estabilished the following diagram

$$\begin{tikzcd} & \mathcal{S}^1(\mathcal{H}) \arrow[dl, "W"'] \arrow[dr, "p^{\cdot}"] & \\ L^1\left(\mathbb{R}^2\right) \arrow[rr, "R"] & & P(\mathbb{T}\times\mathbb{R}) \end{tikzcd}$$

Now we turn to the problem of reconstructing $W(\rho)$ given p^ρ. If $W(\rho)\in S(\mathbb{R}^2)$, the space of Schwartz functions on $\mathbb{R}^2$, Radon inversion formula is applicable, and we can obtain $W(\rho)$ from p^ρ in a rather explicit way. Before stating Radon inversion theorem, according to Ref. [11] we need to introduce the set $S_H(\mathbb{P}^2)$ of functions $\phi:\mathbb{T}\times\mathbb{R}\longrightarrow\mathbb{C}$ such that

(i) $\phi\in C^\infty(\mathbb{T}\times\mathbb{R})$;
(ii) $\sup_{\theta,\,r}\left|\left(1+|r|^k\right)\frac{\partial^l}{\partial r^l}\frac{\partial^m}{\partial\theta^m}\phi(\theta,r)\right|<\infty$;
(iii) $\phi(\theta,r)=\phi(2\pi-\theta,-r)$ for all θ,r;
(iv) for each $k\in\mathbb{N}$, $\int_{-\infty}^{+\infty}\phi(\theta,r)r^k\mathrm{d}r$ is a homogeneous polynomial in $\sin\theta$, $\cos\theta$ of degree k.

It is shown in Ref. [11] that $Rf\in S_H(\mathbb{P}^2)$ if $f\in S(\mathbb{R}^2)$, and the map $R:S(\mathbb{R}^2)\longrightarrow S_H(\mathbb{P}^2)$ is one-to-one and onto. Thus, in our case $W(\rho)\in S(\mathbb{R}^2)$ is equivalent $p^\rho\in S_H(\mathbb{P}^2)$ by Proposition 2.

The next theorem is a restatement of Theorem 3.6 in Ref. [11] (see also Ref. [4] for a formal derivation of (11)). We stress that the hypothesis $W(\rho)\in S(\mathbb{R}^2)$ (or, equivalently, $p^\rho\in S_H(\mathbb{P}^2)$) is needed in order to give meaning to (12) and to define the integral in (13).

Theorem 3. *Suppose* $W(\rho)\in S(\mathbb{R}^2)$. *Then*

$$W(\rho)=\frac{1}{4\pi^2}R^\#[\Lambda p^\rho] \tag{11}$$

where

$$\Lambda p^\rho(\theta,r)=\sqrt{\frac{\pi}{2}}\left[\mathcal{F}_t[|t|]*p^\rho(\theta,\cdot)\right](r)=\mathrm{PV}\left[\int_{-\infty}^{+\infty}\frac{1}{r-t}\frac{\partial p^\rho(\theta,t)}{\partial t}\mathrm{d}t\right] \tag{12}$$

and

$$R^\# f(x,y)=\int_0^{2\pi}f(\theta,x\cos\theta+y\sin\theta)\frac{\mathrm{d}\theta}{2\pi}\quad\forall f\in C^\infty(\mathbb{T}\times\mathbb{R}) \tag{13}$$

(in (12)*, the Fourier transform of* $|t|$ *and the convolution are interpreted in the sense of tempered distributions, and* PV *is the Cauchy principal value of the integral).*

We devote the rest of this section to find the subset of states $\rho\in\mathcal{S}(\mathcal{H})$ such that $W(\rho)\in S(\mathbb{R}^2)$, i. e. to which both Radon transform (10) and Radon reconstruction formula (11) are applicable.

Each $T\in\mathcal{I}_1(\mathcal{H})$, being a Hilbert-Schmidt operator on $L^2(\mathbb{R})$, is an integral operator, whose kernel K_T is in $L^2\left(\mathbb{R}^2\right)$. We have the following fact.

Proposition 3. *Suppose* $K\in S(\mathbb{R}^2)$. *Then the integral operator* L_K *with kernel* K *is in* $\mathcal{I}_1(\mathcal{H})$*, and its trace is*

$$\mathrm{tr}\,[L_K]=\int_{-\infty}^{+\infty}K(x,x)\mathrm{d}x. \tag{14}$$

Moreover, $L_K \in \mathcal{S}(\mathcal{H})$ if and only if K is positive semidefinite[1] *and $\int_{-\infty}^{+\infty} K(x,x)\mathrm{d}x = 1$.*

Proof. Let $I = (-\pi, \pi)$, and let $\Phi : L^2(\mathbb{R}) \longrightarrow L^2(I)$ be the following unitary operator

$$\Phi f(y) = (1 + \tan^2 y)^{1/2} f(\tan y).$$

Φ intertwines L_K with the integral operator $L_{\tilde{K}}$ on $L^2(I)$ with kernel

$$\tilde{K}(y_1, y_2) = (1 + \tan^2 y_1)^{1/2} K(\tan y_1, \tan y_2)(1 + \tan^2 y_2)^{1/2} \quad y_1, y_2 \in (-\pi, \pi).$$

Since $\tilde{K}$ extends to a C^∞-function on $\overline{I \times I}$ by setting $\tilde{K} = 0$ in the frontier of $\overline{I \times I}$, by Lemma 10.11 in Ref. [12] $L_{\tilde{K}}$ is a trace class operator on $L^2(I)$, whose trace is given by

$$\mathrm{tr}\,[L_{\tilde{K}}] = \int_{-\pi}^{\pi} \tilde{K}(y,y)\mathrm{d}y = \int_{-\infty}^{+\infty} K(x,x)\mathrm{d}x.$$

Since $L_K = \Phi^{-1} L_{\tilde{K}} \Phi$, eq. (14) follows.

It is easy to check that, if K is positive semidefinite, then the integral operator L_K is positive. Conversely, suppose L_K is a positive operator. Fix a Dirac sequence $\{f_n\}_{n\in\mathbb{N}}$, and let $g_n = \sum_{i=1}^N c_i f_n^{x_i}$, where $f_n^{x_i}(x) = f_n(x - x_i)$. We have

$$\begin{aligned} 0 \quad \leq \quad \langle g_n \,,\, L_K g_n \rangle &= \sum_{i,j=1}^{N} c_i \overline{c_j} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{f_n(x - x_j)} K(x,y) f_n(y - x_i) \mathrm{d}x\mathrm{d}y \\ \underset{n\to\infty}{\longrightarrow} \sum_{i,j=1}^{N} c_i \overline{c_j} K(x_j, x_i), \end{aligned}$$

from which positive definiteness of K follows. The last claim in the statement is thus clear. $\square$

We introduce the following linear subspace of $\mathcal{I}_1(\mathcal{H})$

$$\mathcal{I}_1^S(\mathcal{H}) = \left\{ T \in \mathcal{I}_1(\mathcal{H}) \mid K_T \in S(\mathbb{R}^2) \right\},$$

and define

$$\mathcal{S}^S(\mathcal{H}) = \mathcal{S}(\mathcal{H}) \cap \mathcal{I}_1^S(\mathcal{H}).$$

If $T \in \mathcal{I}_1^S(\mathcal{H})$, we can explicitly evaluate the trace in (7) and the Fourier transform in (8) defining $V(T)$ and $W(T)$ respectively. We find

$$\begin{aligned} [V(T)](x,y) &= \mathcal{F}_t^{-1} \left[K_T(t + y/2, t - y/2) \right](x) \\ [W(T)](x,y) &= \frac{1}{\sqrt{2\pi}} \mathcal{F}_t \left[K_T(x + t/2, x - t/2) \right](y), \end{aligned}$$

where we denoted by $\mathcal{F}_t$ the Fourier transform with respect to the variable t. The second formula proves the next proposition.

Proposition 4. *$W : \mathcal{I}_1^S(\mathcal{H}) \longrightarrow S(\mathbb{R}^2)$ is a bijection.*

Restricting to states in $\mathcal{S}^S(\mathcal{H})$, we have thus arrived at the following diagram.

$$\begin{tikzcd} & \mathcal{S}^S(\mathcal{H}) \arrow[dl, "W"'] \arrow[dr, "p^{\cdot}"] & \\ S(\mathbb{R}^2) \arrow[rr, shift left, "R"] & & S_H(\mathbb{P}^2) \arrow[ll, shift left, "\frac{1}{4\pi^2} R^{\#} \Lambda"] \end{tikzcd}$$

Remark 2. Unfortunately, one can not use the definition of Radon transform of distributions to extend (10) to whole $L^2(\mathbb{R}^2)$, or reconstruction formula (11) to a larger set than $\mathcal{S}^S(\mathcal{H})$. In fact, as explained in §5 of Ref. [11], the distributional Radon transform can be defined only as a map $R : \mathcal{E}'(\mathbb{T}\times\mathbb{R}) \longrightarrow \mathcal{E}'(\mathbb{T}\times\mathbb{R})$, $\mathcal{E}'(\mathbb{T}\times\mathbb{R})$ being the set of compactly supported distributions on $\mathbb{T}\times\mathbb{R}$. Corollary 3 then prevents us from giving any distributional sense to (10). Similarly, eq. (11) has no distributional analogue, as the reconstruction formula $T = \frac{1}{4\pi^2} R^{\#}[\Lambda R T]$ (Theorem 5.5 in Ref. [11]) again holds only for compactly supported distributions T.

[1] We recall that a function $K : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is *positive semidefinite* if $\sum_{i,j=1}^N c_i \overline{c_j} K(x_j, x_i) \geq 0$ for all $N \in \mathbb{N}$, $c_1, c_2 \ldots c_N \subset \mathbb{C}$ and $x_1, x_2 \ldots x_N \subset \mathbb{R}$.

Remark 3. Being able to exhibit an explicit inversion formula of the Radon transform only for Wigner functions which are Schwartz class does *not* imply a failure of quantum tomographical methods in reconstructing states with weaker regularity properties, as the associated POVM remains informationally complete on the whole of $\mathcal{S}(\mathcal{H})$, as we have shown in the first part of this paper. In fact, mainly in order to address issues of numerical stability, actual reconstruction methods usually do not involve $\frac{1}{4\pi^2}R^{\#}\Lambda$ directly, but some approximated technique involving regularizations; proofs of consistency are available [1] for some of these regularized estimators which holds on the whole of quantum state space.

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