

POSITIVE OPERATOR VALUED MEASURES COVARIANT WITH RESPECT TO AN ABELIAN GROUP

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ABSTRACT. Given a unitary representation U of an Abelian group G and a subgroup H , we characterise the positive operator valued measures based on the quotient group G/H and covariant with respect to U .

1. INTRODUCTION

Let G be a topological group and H a closed subgroup. Given a unitary representation U of G , it is of interest in Quantum Mechanics and in wavelet analysis to describe the positive operator valued measures Q defined on the quotient space G/H and covariant with respect to U . The generalised imprimitivity theorem assures that Q exists if and only if there exists a unitary representation σ of H such that U is a subrepresentation of the representation unitarily induced by σ .

In his seminal papers [7] and [8], Holevo classifies the covariant positive operator valued measures if G is of type I and $H = \{e\}$, and if G is compact and H is arbitrary.

In this paper we extend the above result to the case G Abelian and H arbitrary. We describe the covariant positive operator valued measures in terms of a family $W_x : E_x \rightarrow E$ of isometries, where the index x runs over the dual group of G , $\dim E_x$ equals the multiplicity of the character x in U and E is a fixed (infinite dimensional) Hilbert space. As a byproduct, we define a unitary operator Σ that diagonalises the representation of G unitarily induced by a representation of H with uniform multiplicity.

As an application of our characterisation, in the final section we give three examples.

- (1) The *regular* representation of the real line, where the positive operator valued measure describes the *position observable* in one dimension.
- (2) The *number*-representation of the torus, where the positive operator valued measure describes the *phase observable*.
- (3) The tensor product of two *number*-representations of the torus, where the positive operator valued measure describes the *phase difference observable*.

2. NOTATIONS

In this paper, by *Hilbert space* we mean a separable complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ linear in the first argument, by *group* we mean a locally compact second countable Abelian group and by *representation* a continuous unitary

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representation of a group acting on a Hilbert space. If X is a locally compact second countable topological space, we denote by $\mathcal{B}(X)$ the Borel σ -algebra of X and by $C_c(X)$ the space of continuous complex functions on X with compact support. By *measure* we mean a positive measure defined on $\mathcal{B}(X)$ and finite on compact sets.

In the sequel we shall use rather freely basic results of Harmonic Analysis on Abelian groups, as exposed, for example, in Refs. [4] and [5].

We fix a group G and a closed subgroup H . We denote by \widehat{G} and \widehat{H} the corresponding dual groups and by $\langle x, g \rangle$ the canonical pairing.

We denote by

$$q : G \longrightarrow G/H, \quad q(g) = \dot{g}$$

the canonical projection onto the quotient group G/H . If $a \in G$ and $\dot{g} \in G/H$, we let $a[\dot{g}] = q(ag) = \dot{a}\dot{g}$ be the natural action of a on the point \dot{g} .

Let H^\perp be the annihilator of H in \widehat{G} , that is

$$H^\perp = \left\{ y \in \widehat{G} \mid \langle y, h \rangle = 1 \ \forall h \in H \right\}.$$

The group H^\perp is a closed subgroup of \widehat{G} and $\widehat{G/H}$ can be identified (and we will do) with H^\perp by means of

$$\langle y, \dot{g} \rangle := \langle y, g \rangle \quad \forall y \in H^\perp, \forall \dot{g} \in G/H.$$

Since H^\perp is closed, we can consider the quotient group $\widehat{G/H^\perp}$. We denote by

$$\pi : \widehat{G} \longrightarrow \widehat{G/H^\perp}, \quad \pi(x) = \dot{x}$$

the canonical projection. The group \widehat{H} can be identified (and we will do) with the quotient group $\widehat{G/H^\perp}$ by means of

$$\langle \dot{x}, h \rangle := \langle x, h \rangle \quad \forall \dot{x} \in \widehat{G/H^\perp}, \forall h \in H.$$

Let μ_G , μ_H and $\mu_{G/H}$ be fixed Haar measures on G , H and G/H , respectively.

We denote by μ_{H^\perp} the Haar measure on H^\perp such that the Fourier-Plancherel co-transform $\overline{\mathcal{F}}_{G/H}$ is unitary from $L^2(G/H, \mu_{G/H})$ onto $L^2(H^\perp, \mu_{H^\perp})$, where $\overline{\mathcal{F}}_{G/H}$ is given by

$$(\overline{\mathcal{F}}_{G/H} f)(y) = \int_{G/H} \langle y, \dot{g} \rangle f(\dot{g}) \, d\mu_{G/H}(\dot{g}) \quad y \in H^\perp,$$

for all $f \in (L^1 \cap L^2)(G/H, \mu_{G/H})$.

Given $\varphi \in C_c(\widehat{G})$, let

$$\tilde{\varphi}(\dot{x}) := \int_{H^\perp} \varphi(xy) \, d\mu_{H^\perp}(y) \quad \forall \dot{x} \in \widehat{G/H^\perp}.$$

It is well known that $\tilde{\varphi}$ is in $C_c(\widehat{G/H^\perp})$ and that $\tilde{\varphi} \geq 0$ if $\varphi \geq 0$. Given a measure ν on $\widehat{G/H^\perp}$, the map

$$(2.1) \quad C_c(\widehat{G}) \ni \varphi \longmapsto \int_{\widehat{G/H^\perp}} \tilde{\varphi}(\dot{x}) \, d\nu(\dot{x}) \in \mathbb{R}$$

is linear and positive. Hence, by Riesz-Markov theorem, there is a unique measure $\tilde{\nu}$ on \widehat{G} such that

$$\int_{\widehat{G}} \phi(x) d\tilde{\nu}(x) = \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \phi(xy) d\mu_{H^\perp}(y)$$

for all $\phi \in L^1(\widehat{G}, \tilde{\nu})$. One can check that the correspondence $\nu \mapsto \tilde{\nu}$ preserves equivalence and orthogonality of measures.

Given a finite measure μ on \widehat{G} , we denote by μ^π the image measure of μ with respect to π , i.e. the measure on \widehat{G}/H^\perp given by

$$\mu^\pi(A) = \mu(\pi^{-1}(A)) \quad \forall A \in \mathcal{B}(\widehat{G}/H^\perp).$$

We fix a representation U of G acting on a Hilbert space \mathcal{H} . Let Q be a positive operator valued measure (POVM) defined on G/H and acting on \mathcal{H} . If Q satisfies the following properties

- (1) $Q(G/H) = I$,
- (2) for all $X \in \mathcal{B}(G/H)$,

$$U(g)Q(X)U(g^{-1}) = Q(g[X]) \quad \forall g \in G,$$

it is called *covariant* and (U, Q) is said to be a *covariance system*. In particular, if Q is a projective measure, (U, Q) is called an *imprimitivity system*.

For $\omega \in C_c(G/H)$, we define the operator

$$M(\omega) := \int_{G/H} \omega(\dot{g}) dQ(\dot{g}).$$

The map $\omega \mapsto M(\omega)$ defines uniquely the POVM Q . In the following we use M instead of Q .

Finally, given a representation σ of H , we denote by $(\text{ind}_H^G(\sigma), M_0)$ the imprimitivity system induced by σ from H to G .

The aim of this paper is to describe all the positive operator valued measures covariant with respect to U . The generalised imprimitivity theorem (see, for example, Refs. [2] and [3]) states that

Theorem 2.1. *A POVM M based on G/H and acting on \mathcal{H} is covariant with respect to U if and only if there exists a representation σ of H and an isometry W intertwining U with $\text{ind}_H^G(\sigma)$ such that*

$$M(\omega) = W^* M_0(\omega) W$$

for all $\omega \in C_c(G/H)$.

If σ' is another representation of H such that σ is contained (as subrepresentation) in σ' , then $(\text{ind}_H^G(\sigma), M_0)$ is contained (as imprimitivity system) in $(\text{ind}_H^G(\sigma'), M'_0)$. Hence, we can always assume that σ in the previous theorem has infinite multiplicity.

Moreover, there exist a measure ν on \widehat{G}/H^\perp and an infinite dimensional Hilbert space E such that, up to a unitary equivalence, σ acts diagonally on $L^2(\widehat{G}/H^\perp, \nu; E)$. The first step of our construction is to diagonalise the representation $\text{ind}_H^G(\sigma)$.

3. DIAGONALISATION OF $\text{ind}_H^G(\sigma)$

In this section, given a representation of H with uniform multiplicity, we diagonalise the corresponding induced representation.

Let ν be a measure on \widehat{G}/H^\perp and E be a Hilbert space. Let σ^ν be the diagonal representation of H acting on the space $L^2(\widehat{G}/H^\perp, \nu; E)$, that is

$$(\sigma^\nu(h)\xi)(\dot{x}) = \langle \dot{x}, h \rangle \xi(\dot{x}) \quad \dot{x} \in \widehat{G}/H^\perp,$$

where $h \in H$.

We denote by \mathcal{H}^ν the space of functions $f : G \times \widehat{G}/H^\perp \rightarrow E$ such that

- f is weakly $(\mu_G \otimes \nu)$ -measurable;
- for all $h \in H$,

$$(3.1) \quad f(gh, \dot{x}) = \overline{\langle \dot{x}, h \rangle} f(g, \dot{x}) \quad \forall (g, \dot{x}) \in G \times \widehat{G}/H^\perp;$$

•

$$\int_{G/H \times \widehat{G}/H^\perp} \|f(g, \dot{x})\|_E^2 d(\mu_{G/H} \otimes \nu)(\dot{g}, \dot{x}) < +\infty.$$

We identify functions in \mathcal{H}^ν that are equal $(\mu_G \otimes \nu)$ -a.e.. Let G act on \mathcal{H}^ν as

$$(\lambda^\nu(a)f)(g, \dot{x}) := f(a^{-1}g, \dot{x}) \quad (g, \dot{x}) \in G \times \widehat{G}/H^\perp$$

for all $a \in G$. Define

$$(M_0^\nu(\omega)f)(g, \dot{x}) := \omega(\dot{g})f(g, \dot{x}) \quad (g, \dot{x}) \in G \times \widehat{G}/H^\perp$$

for all $f \in \mathcal{H}^\nu$, $\omega \in C_c(G/H)$.

One can easily prove the following fact.

Proposition 3.1. *The space \mathcal{H}^ν is a Hilbert space with respect to the inner product*

$$\langle f_1, f_2 \rangle_{\mathcal{H}^\nu} = \int_{G/H \times \widehat{G}/H^\perp} \langle f_1(g, \dot{x}), f_2(g, \dot{x}) \rangle_E d(\mu_{G/H} \otimes \nu)(\dot{g}, \dot{x}).$$

If $\varphi \in C_c(G \times \widehat{G}/H^\perp; E)$, let

$$f_\varphi(g, \dot{x}) := \int_H \langle \dot{x}, h \rangle \varphi(gh, \dot{x}) d\mu_H(h) \quad \forall (g, \dot{x}) \in G \times \widehat{G}/H^\perp.$$

Then f_φ is a continuous function in \mathcal{H}^ν such that $(q \times \text{id}_{\widehat{G}/H^\perp})(\text{supp} f_\varphi)$ is compact, and the set

$$\mathcal{H}_0^\nu = \left\{ f_\varphi \mid \varphi \in C_c(G \times \widehat{G}/H^\perp; E) \right\}$$

is a dense subspace of \mathcal{H}^ν . The couple (λ^ν, M_0^ν) is the imprimitivity system induced by σ^ν from H to G .

We now diagonalise the representation λ^ν . First of all, we let $\tilde{\nu}$ be the measure defined in \widehat{G} by Eq. (2.1). Let Λ^ν be the diagonal representation of G acting on $L^2(\widehat{G}, \tilde{\nu}; E)$ as

$$(\Lambda^\nu(g)\phi)(x) = \langle x, g \rangle \phi(x) \quad x \in \widehat{G}$$

for all $g \in G$.

Moreover, given $\phi : \widehat{G} \rightarrow E$ and fixed $x \in \widehat{G}$, define ϕ_x from H^\perp to E as

$$\phi_x(y) := \phi(xy) \quad \forall y \in H^\perp.$$

Theorem 3.2. *There is a unique unitary operator Σ from \mathcal{H}^ν onto $L^2(\widehat{G}, \tilde{\nu}; E)$ such that, for all $f \in \mathcal{H}_0^\nu$,*

$$(3.2) \quad (\Sigma f)(x) = \int_{G/H} \langle x, g \rangle f(g, \dot{x}) \, d\mu_{G/H}(\dot{g}) \quad x \in \widehat{G}.$$

The operator Σ intertwines λ^ν with Λ^ν . Moreover,

$$(3.3) \quad (\Sigma^* \varphi)(g, \dot{x}) = \int_{H^\perp} \overline{\langle xy, g \rangle} \varphi(xy) \, d\mu_{H^\perp}(y) \quad (g, \dot{x}) \in G \times \widehat{H}$$

for all $\varphi \in C_c(\widehat{G}; E)$.

Proof. We first define Σ on \mathcal{H}_0^ν . Let $f \in \mathcal{H}_0^\nu$. Fixed $x \in \widehat{G}$, by virtue of Eq. (3.1) the function

$$g \longmapsto \langle x, g \rangle f(g, \dot{x})$$

depends only on the equivalence class \dot{g} of g and we let f^x be the corresponding map on G/H . Due to the properties of f , f^x is continuous and has compact support, so it is $\mu_{G/H}$ -integrable and we define Σf by means of Eq. (3.2).

We claim that Σf is in $L^2(\widehat{G}, \tilde{\nu}; E)$ and $\|\Sigma f\|_{L^2(\widehat{G}, \tilde{\nu}; E)} = \|f\|_{\mathcal{H}^\nu}$. Since the map

$$(x, \dot{g}) \longmapsto f^x(\dot{g})$$

is continuous, by a standard argument Σf is continuous. Moreover, if $x \in \widehat{G}$ and $y \in H^\perp$,

$$\begin{aligned} (\Sigma f)(xy) &= \int_{G/H} \langle xy, g \rangle f(g, \dot{x}) \, d\mu_{G/H}(\dot{g}) \\ &= \int_{G/H} \langle y, \dot{g} \rangle \langle x, g \rangle f(g, \dot{x}) \, d\mu_{G/H}(\dot{g}) \\ &= \overline{\mathcal{F}_{G/H}}(f^x)(y). \end{aligned}$$

Indeed,

$$\begin{aligned} \|\Sigma f\|_{L^2(\widehat{G}, \tilde{\nu}; E)}^2 &= \int_{\widehat{G}} \|\Sigma f(x)\|_E^2 \, d\tilde{\nu}(x) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|(\Sigma f)(xy)\|_E^2 \, d\mu_{H^\perp}(y) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|\overline{\mathcal{F}_{G/H}}(f^x)(y)\|_E^2 \, d\mu_{H^\perp}(y) \\ &\quad \text{(unitarity of } \overline{\mathcal{F}_{G/H}}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \|f^x(\dot{g})\|_E^2 \, d\mu_{G/H}(\dot{g}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \|f(g, \dot{x})\|_E^2 \, d\mu_{G/H}(\dot{g}) \\ &= \int_{G/H \times \widehat{G}/H^\perp} \|f(g, \dot{x})\|_E^2 \, d(\mu_{G/H} \otimes \nu)(\dot{g}, \dot{x}) \\ &= \|f\|_{\mathcal{H}^\nu}^2. \end{aligned}$$

By density, Σ extends to an isometry from \mathcal{H}^ν to $L^2(\widehat{G}, \tilde{\nu}; E)$. Clearly, Eq. (3.2) holds and it defines uniquely Σ .

The second step is computing the adjoint of Σ . Let $\varphi \in C_c(\widehat{G}; E)$, by standard arguments the right hand side of Eq. (3.3) is a continuous function of (g, \dot{x}) . Moreover, it satisfies Eq. (3.1). We have

$$\int_{H^\perp} \overline{\langle xy, g \rangle} \varphi(xy) \, d\mu_{H^\perp}(y) = \overline{\langle x, g \rangle} \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}) \quad (g, \dot{x}) \in G \times \widehat{H}.$$

First of all, we show that the above function is in \mathcal{H}^ν . Indeed,

$$\begin{aligned} & \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \left\| \overline{\langle x, g \rangle} \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}) \right\|_E^2 d\mu_{G/H}(\dot{g}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \left\| \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}) \right\|_E^2 d\mu_{G/H}(\dot{g}) \\ & \quad (\text{unitarity of } \overline{\mathcal{F}_{G/H}}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|\varphi_x(y)\|_E^2 d\mu_{H^\perp}(y) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|\varphi(xy)\|_E^2 d\mu_{H^\perp}(y) \\ (3.4) \quad &= \|\varphi\|_{L^2(\widehat{G}, \tilde{\nu}; E)}^2. \end{aligned}$$

Moreover, for all $f \in \mathcal{H}_0^\nu$, we have

$$\begin{aligned} \langle \Sigma^* \varphi, f \rangle_{\mathcal{H}^\nu} &= \langle \varphi, \Sigma f \rangle_{L^2(\widehat{G}, \tilde{\nu}; E)} \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \langle \varphi(xy), (\Sigma f)(xy) \rangle_E d\mu_{H^\perp}(y) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \langle \varphi_x(y), \overline{\mathcal{F}_{G/H}}(f^x)(y) \rangle_E d\mu_{H^\perp}(y) \\ & \quad (\text{unitarity of } \overline{\mathcal{F}_{G/H}}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \left\langle \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), f^x(\dot{g}) \right\rangle_E d\mu_{G/H}(\dot{g}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \left\langle \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), \langle x, g \rangle f(g, \dot{x}) \right\rangle_E d\mu_{G/H}(\dot{g}) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} \left\langle \overline{\langle x, g \rangle} \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), f(g, \dot{x}) \right\rangle_E d\mu_{G/H}(\dot{g}) \\ &= \int_{G/H \times \widehat{G}/H^\perp} \left\langle \overline{\langle x, g \rangle} \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), f(g, \dot{x}) \right\rangle_E d(\mu_{G/H} \otimes \nu)(\dot{g}, \dot{x}). \end{aligned}$$

Since \mathcal{H}_0^ν is dense, Eq. (3.3) follows. By Eq. (3.4) Σ^* is isometric, hence Σ is unitary.

Finally, we show the intertwining property. Let $a \in G$ and $f \in \mathcal{H}_0^\nu$. Then $\lambda^\nu(a)f \in \mathcal{H}_0^\nu$, and so one has

$$\begin{aligned} (\Sigma\lambda^\nu(a)f)(x) &= \int_{G/H} \langle x, g \rangle f(a^{-1}g, \dot{x}) \, d\mu_{G/H}(\dot{g}) \\ &= \langle x, a \rangle \int_{G/H} f^x(a^{-1}[\dot{g}]) \, d\mu_{G/H}(\dot{g}) \\ &\quad (\dot{g} \rightarrow a[\dot{g}]) \\ &= \langle x, a \rangle \int_{G/H} \langle x, g \rangle f(g, \dot{x}) \, d\mu_{G/H}(\dot{g}) \\ &= (\Lambda^\nu(a)\Sigma f)(x) \quad x \in \widehat{G}. \end{aligned}$$

By density of \mathcal{H}_0^ν , it follows that $\Sigma\lambda^\nu(a) = \Lambda^\nu(a)\Sigma$. □

Given $\omega \in C_c(G/H)$, let $\widetilde{M}_0^\nu(\omega) = \Sigma M_0^\nu(\omega)\Sigma^*$. Then

Proposition 3.3. *For all $\omega \in C_c(G/H)$ and $\phi \in L^2(\widehat{G}, \tilde{\nu}; E)$,*

$$(3.5) \quad \left(\widetilde{M}_0^\nu(\omega)\phi\right)(x) = \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \phi(xy^{-1}) \, d\mu_{H^\perp}(y) \quad x \in \widehat{G}.$$

Proof. Let $\omega \in C_c(G/H)$. We compute the action of $\widetilde{M}_0^\nu(\omega)$ on $C_c(\widehat{G}; E)$. If $\varphi \in C_c(\widehat{G}; E)$, let

$$\xi(x) := \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \varphi(xy^{-1}) \, d\mu_{H^\perp}(y) \quad \forall x \in \widehat{G},$$

which is well defined and continuous. Moreover, for all $x \in \widehat{G}$ and $y \in H^\perp$,

$$\begin{aligned} \xi(xy) &= \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y') \varphi(xy y'^{-1}) \, d\mu_{H^\perp}(y') \\ &= \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y') \varphi_x(y y'^{-1}) \, d\mu_{H^\perp}(y') \\ (3.6) \quad &= (\overline{\mathcal{F}}_{G/H}(\omega) * \varphi_x)(y). \end{aligned}$$

Here and in the following, convolutions are always taken in H^\perp . If $\varphi, \psi \in C_c(\widehat{G}; E)$,

$$\begin{aligned}
\left\langle \widetilde{M}_0^\nu(\omega) \varphi, \psi \right\rangle_{L^2(\widehat{G}, \tilde{\nu}; E)} &= \langle M_0^\nu(\omega) \Sigma^* \varphi, \Sigma^* \psi \rangle_{\mathcal{H}^\nu} \\
&= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} d\mu_{G/H}(\dot{g}) \left\langle \omega(\dot{g}) \overline{\langle x, g \rangle} \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), \overline{\langle x, g \rangle} \times \right. \\
&\quad \left. \times \overline{\mathcal{F}_{G/H}^*}(\psi_x)(\dot{g}) \right\rangle_E \\
&= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{G/H} d\mu_{G/H}(\dot{g}) \left\langle \omega(\dot{g}) \overline{\mathcal{F}_{G/H}^*}(\varphi_x)(\dot{g}), \overline{\mathcal{F}_{G/H}^*}(\psi_x)(\dot{g}) \right\rangle_E \\
&\quad (\text{unitarity of } \overline{\mathcal{F}_{G/H}^*} \text{ and properties of convolution}) \\
&= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} d\mu_{H^\perp}(y) \langle \overline{\mathcal{F}_{G/H}(\omega)} * \varphi_x(y), \psi_x(y) \rangle_E \\
&= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} d\mu_{H^\perp}(y) \langle \xi(xy), \psi(xy) \rangle_E,
\end{aligned}$$

hence Eq. (3.5) holds on $C_c(\widehat{G}; E)$.

Let now $\phi \in L^2(\widehat{G}, \tilde{\nu}; E)$. Since

$$\|\phi\|_{L^2(\widehat{G}, \tilde{\nu}; E)}^2 = \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|\phi(xy)\|_E^2 d\mu_{H^\perp}(y) < +\infty,$$

by virtue of Fubini theorem there is a ν -negligible set $Y_1 \subset \widehat{G}/H^\perp$ such that, for all $x \in \widehat{G}$ with $\dot{x} \notin Y_1$, $\phi_x \in L^2(H^\perp, \mu_{H^\perp}; E)$. Moreover, using the definition of $\tilde{\nu}$, one can check that $\pi^{-1}(Y_1)$ is $\tilde{\nu}$ -negligible. Then, for $\tilde{\nu}$ -almost all $x \in \widehat{G}$, ϕ_x is in $L^2(H^\perp, \mu_{H^\perp}; E)$. We observe that the map

$$\dot{g} \longmapsto \omega(\dot{g}) \left(\overline{\mathcal{F}_{G/H}^*}(\phi_x) \right)(\dot{g})$$

is then in $(L^1 \cap L^2)(G/H, \mu_{G/H}; E)$ for $\tilde{\nu}$ -almost all $x \in \widehat{G}$, hence its Fourier cotransform is continuous, and we have

$$\begin{aligned}
\overline{\mathcal{F}_{G/H}} \left(\omega \overline{\mathcal{F}_{G/H}^*}(\phi_x) \right)(e) &= \overline{\mathcal{F}_{G/H}(\omega)} * \phi_x(e) \\
(3.7) \qquad \qquad \qquad &= \int_{H^\perp} \overline{\mathcal{F}_{G/H}(\omega)}(y) \phi(xy^{-1}) d\mu_{H^\perp}(y).
\end{aligned}$$

Now, we let $(\varphi_k)_{k \geq 1}$ be a sequence in $C_c(\widehat{G}; E)$ converging to ϕ in $L^2(\widehat{G}, \tilde{\nu}; E)$. Then

$$\int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \|(\varphi_k)_x(y) - \phi_x(y)\|_E^2 d\mu_{H^\perp}(y) \longrightarrow 0$$

and so, possibly passing to a subsequence, there is a ν -negligible set $Y_2 \subset \widehat{G}/H^\perp$ such that

$$\int_{H^\perp} \|(\varphi_k)_x(y) - \phi_x(y)\|_E^2 d\mu_{H^\perp}(y) \longrightarrow 0$$

for all $x \in \widehat{G}$ with $\dot{x} \notin Y_2$. This fact means that, for $\tilde{\nu}$ -almost all $x \in \widehat{G}$,

$$(\varphi_k)_x \longrightarrow \phi_x$$

in $L^2(H^\perp, \mu_{H^\perp}; E)$. It follows that

$$\omega \overline{\mathcal{F}}_{G/H}^*((\varphi_k)_x) \longrightarrow \omega \overline{\mathcal{F}}_{G/H}^*(\phi_x)$$

in $L^1(G/H, \mu_{G/H}; E)$. Then, for $\tilde{\nu}$ -almost all $x \in \widehat{G}$,

$$\overline{\mathcal{F}}_{G/H}(\omega \overline{\mathcal{F}}_{G/H}^*((\varphi_k)_x)) \longrightarrow \overline{\mathcal{F}}_{G/H}(\omega \overline{\mathcal{F}}_{G/H}^*(\phi_x))$$

uniformly, and, using Eqs. (3.6), (3.7),

$$\begin{aligned} \left(\widetilde{M}_0^\nu(\omega) \varphi_k\right)(x) &= \overline{\mathcal{F}}_{G/H}(\omega \overline{\mathcal{F}}_{G/H}^*((\varphi_k)_x))(e) \longrightarrow \\ &\longrightarrow \overline{\mathcal{F}}_{G/H}(\omega \overline{\mathcal{F}}_{G/H}^*(\phi_x))(e) = \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \phi(xy^{-1}) d\mu_{H^\perp}(y). \end{aligned}$$

Since $\widetilde{M}_0^\nu(\omega) \varphi_k$ converges to $\widetilde{M}_0^\nu(\omega) \phi$ in $L^2(\widehat{G}, \tilde{\nu}; E)$, Eq. (3.5) follows from unicity of the limit. \square

4. CHARACTERISATION OF COVARIANT POVMs

We fix in the following an *infinite dimensional* Hilbert space E . According to the results of the previous sections, the generalised imprimitivity theorem for Abelian groups can be stated in the following way.

Theorem 4.1. *A POVM M based on G/H and acting on \mathcal{H} is covariant with respect to U if and only if there exist a measure ν on \widehat{G}/H^\perp and an isometry W intertwining U with Λ^ν such that*

$$M(\omega) = W^* \widetilde{M}_0^\nu(\omega) W$$

for all $\omega \in C_c(G/H)$.

To get an explicit form of W , we assume that U acts diagonally on \mathcal{H} . This means that \mathcal{H} is the orthogonal sum of invariant subspaces

$$(4.1) \quad \mathcal{H} = \bigoplus_{k \in I} L^2(\widehat{G}, \rho_k; F_k),$$

where I is a denumerable set, $(\rho_k)_{k \in I}$ is a family of measures on \widehat{G} , $(F_k)_{k \in I}$ is a family of Hilbert spaces, and the action of U is given by

$$(U(g) \phi_k)(x) = \langle x, g \rangle \phi_k(x) \quad x \in \widehat{G},$$

where $\phi_k \in L^2(\widehat{G}, \rho_k; F_k)$ and $g \in G$. We will denote by P_k the orthogonal projector onto the subspace $L^2(\widehat{G}, \rho_k; F_k)$.

The assumption (4.1) is not restrictive. Indeed, it is well known that there are a family of disjoint measures $(\rho_k)_{k \in \mathbb{N} \cup \{\infty\}}$ and a family of Hilbert spaces $(F_k)_{k \in \mathbb{N} \cup \{\infty\}}$ such that $\dim F_k = k$ and, up to a unitary equivalence, Eq. (4.1) holds.

Given the decomposition (4.1), let ρ be a measure on \widehat{G} such that

$$(4.2) \quad \rho(N) = 0 \iff \rho_k(N) = 0 \quad \forall k \in I.$$

We recall that the equivalence class of ρ is uniquely defined by the family $(\rho_k)_{k \in I}$.

Finally, we observe also that the equivalence class of ρ is independent of the choice of decomposition (4.1). Indeed, if G acts diagonally on another decomposition

$$\mathcal{H} = \bigoplus_{k \in I'} L^2(\widehat{G}, \rho'_k; F'_k),$$

then

$$\rho'_k(N) = 0 \quad \forall k \in I' \iff \rho_k(N) = 0 \quad \forall k \in I.$$

It follows that the representation U defines uniquely an equivalence class \mathcal{C}_U of measures ρ such that relation (4.2) holds. Chosen in this equivalence class a *finite* measure ρ , we denote by \mathcal{C}_U^π the equivalence class of the image measure ρ^π . Clearly \mathcal{C}_U^π depends only on \mathcal{C}_U .

We now give the central result of this section.

Theorem 4.2. *Let U be a representation of G acting diagonally on*

$$\mathcal{H} = \bigoplus_{k \in I} L^2(\widehat{G}, \rho_k; F_k).$$

Given $\nu_U \in \mathcal{C}_U^\pi$, let $\widetilde{\nu}_U$ be the measure given by Eq. (2.1). The representation U admits covariant positive operator valued measures based on G/H if and only if, for all $k \in I$, ρ_k has density with respect to $\widetilde{\nu}_U$. In this case, for every $k \in I$, let α_k be the densities of ρ_k with respect to $\widetilde{\nu}_U$.

Let E be a fixed infinite dimensional Hilbert space. For each $k \in I$, let

$$\widehat{G} \ni x \longmapsto W_k(x) \in \mathcal{L}(F_k; E)$$

be a weakly measurable map such that $W_k(x)$ are isometries for ρ_k -almost all $x \in \widehat{G}$. For $\omega \in C_c(G/H)$, let $M(\omega)$ be the operator given by

$$(4.3) \quad \begin{aligned} (P_j M(\omega) P_k \phi)(x) &= \int_{H^\perp} d\mu_{H^\perp}(y) \overline{\mathcal{F}_{G/H}(\omega)(y)} \sqrt{\frac{\alpha_k(xy^{-1})}{\alpha_j(x)}} \times \\ &\times W_j(x)^* W_k(xy^{-1}) (P_k \phi)(xy^{-1}) \quad x \in \widehat{G}, \end{aligned}$$

for all $\phi \in \mathcal{H}$ and $k, j \in I$. Then, M is a POVM covariant with respect to U .

Conversely, any POVM based on G/H and covariant with respect to U is of the form given by Eq. (4.3).

We add some comments before the proof of the theorem.

Remark 4.3. We observe that Eq. (4.3) is invariant with respect to the choice of the measure $\nu_U \in \mathcal{C}_U^\pi$. Indeed, let $\nu'_U \in \mathcal{C}_U^\pi$, and $\beta > 0$ be the density of ν_U with respect to ν'_U . Clearly

$$\widetilde{\nu}_U = (\beta \circ \pi) \widetilde{\nu}'_U,$$

so that the densities α'_k of ρ_k with respect to $\widetilde{\nu}'_U$ are

$$\alpha'_k = (\beta \circ \pi) \alpha_k.$$

It follows that Eq. (4.3) does not depend on the choice of $\nu_U \in \mathcal{C}_U^\pi$.

Corollary 4.4. *Let H be the trivial subgroup $\{e\}$. The representation U admits covariant positive operator valued measures based on G if and only if the measures ρ_k have density with respect to the Haar measure $\mu_{\widehat{G}}$. In this case, the functions α_k in Eq. (4.3) are the densities of ρ_k with respect to $\mu_{\widehat{G}}$.*

Remark 4.5. The content of the previous corollary was first shown by Holevo in Ref. [7] for non-normalised POVM. In order to compare the two results observe that,

if $\phi \in (L^1 \cap L^2) \left(\widehat{G}, \rho_k; F_k \right)$ and $\psi \in (L^1 \cap L^2) \left(\widehat{G}, \rho_j; F_j \right)$, Eq. (4.3) becomes

$$\begin{aligned} \langle M(\omega) \phi, \psi \rangle_{\mathcal{H}} &= \int_G d\mu_G(g) \omega(g) \int_{\widehat{G} \times \widehat{G}} \langle x, g \rangle \overline{\langle y, g \rangle} \sqrt{\alpha_k(y) \alpha_j(x)} \times \\ &\quad \times \langle W_j(x)^* W_k(y) \phi(y), \psi(x) \rangle d(\mu_{\widehat{G}} \otimes \mu_{\widehat{G}})(x, y) \\ &= \int_G d\mu_G(g) \omega(g) \int_{\widehat{G} \times \widehat{G}} K_{U(g^{-1})\psi, U(g^{-1})\phi}(x, y) d(\mu_{\widehat{G}} \otimes \mu_{\widehat{G}})(x, y), \end{aligned}$$

where

$$K_{\psi, \phi}(x, y) = \sqrt{\alpha_k(y) \alpha_j(x)} \langle W_k(y) \phi(y), W_j(x) \psi(x) \rangle$$

is a bounded positive definite measurable field of forms (compare with Eqs. (4.2) and (4.3) in Ref. [7]).

In order to prove Theorem 4.2, we need the following lemma.

Lemma 4.6. *Let ρ be a finite measure on \widehat{G} . Assume that there is a measure ν on \widehat{G}/H^\perp such that ρ has density with respect to $\tilde{\nu}$. Then ρ has density with respect to ρ^π . In this case, ν uniquely decomposes as*

$$\nu = \nu_1 + \nu_2,$$

where ν_1 is equivalent to ρ^π and $\nu_2 \perp \rho^\pi$.

Proof. Suppose that ν is a measure on \widehat{G}/H^\perp such that $\rho = \alpha \tilde{\nu}$, where α is a non-negative $\tilde{\nu}$ -integrable function on \widehat{G} . Then, for all $\varphi \in C_c(\widehat{G}/H^\perp)$,

$$\begin{aligned} \rho^\pi(\varphi) &= \int_{\widehat{G}} \varphi(\pi(x)) d\rho(x) \\ &= \int_{\widehat{G}/H^\perp} d\nu(\dot{x}) \int_{H^\perp} \varphi(\dot{x}) \alpha(xy) d\mu_{H^\perp}(y) \\ &= \int_{\widehat{G}/H^\perp} \varphi(\dot{x}) \alpha'(\dot{x}) d\nu(\dot{x}), \end{aligned}$$

where the function

$$\alpha'(\dot{x}) := \int_{H^\perp} \alpha(xy) d\mu_{H^\perp}(y) \geq 0$$

is ν -integrable by virtue of Fubini theorem. It follows that

$$(4.4) \quad \rho^\pi = \alpha' \nu.$$

Using Lebesgue theorem, we can uniquely decompose

$$\nu = \nu_1 + \nu_2,$$

where ν_1 has base ρ^π and $\nu_2 \perp \rho^\pi$. From Eq. (4.4), it follows that ν_1 and ρ^π are equivalent, and this proves the second statement of the lemma. If $A, B \in \mathcal{B}(\widehat{G}/H^\perp)$ are disjoint sets such that ν_2 is concentrated in A and ν_1 is concentrated in B , then $\tilde{\nu}_2$ and $\tilde{\nu}_1$ are respectively concentrated in the disjoint sets $\tilde{A} = \pi^{-1}(A)$ and $\tilde{B} = \pi^{-1}(B)$. By definition of ρ^π , we also have

$$\rho(\tilde{A}) = \rho^\pi(A) = 0.$$

Since ρ has density with respect to $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$ and $\tilde{\nu}_2$ is concentrated in \tilde{A} , it follows that ρ has density with respect to $\tilde{\nu}_1 \cong \tilde{\rho}^\pi$. The claim is now clear. \square

Proof of Theorem 4.2. Let ρ be a finite measure in \mathcal{C}_U . By virtue of Theorem 4.1, U admits a covariant POVM \iff there exists a measure ν in \widehat{G}/H^\perp such that U is a subrepresentation of $\Lambda^\nu \iff$ each measure ρ_k has density with respect to $\tilde{\nu} \iff \rho$ has density with respect to $\tilde{\nu}$. From Lemma 4.6, U admits a covariant POVM if and only if ρ has density with respect to $\tilde{\rho}^\pi$. Since $\rho^\pi \in \mathcal{C}_U^\pi$, the first claim follows.

Let now M be a covariant POVM. By Theorem 4.1, there is a measure ν on \widehat{G}/H^\perp and an isometry W intertwining U with Λ^ν such that

$$M(\omega) = W^* \widetilde{M}_0^\nu(\omega) W \quad \forall \omega \in C_c(G/H).$$

Using Lemma 4.6, we (uniquely) decompose

$$\nu = \nu_1 + \nu_2,$$

where ν_1 is equivalent to ν_U and $\nu_2 \perp \nu_U$. Then we have

$$\begin{aligned} \sigma^\nu &\cong \sigma^{\nu_U} \oplus \sigma^{\nu_2} \implies \\ &\implies \left(\Lambda^\nu, \widetilde{M}_0^\nu \right) \cong \left(\Lambda^{\nu_U}, \widetilde{M}_0^{\nu_U} \right) \oplus \left(\Lambda^{\nu_2}, \widetilde{M}_0^{\nu_2} \right), \end{aligned}$$

i.e. the imprimitivity system $\left(\Lambda^\nu, \widetilde{M}_0^\nu \right)$ preserves the decomposition

$$L^2(\widehat{G}, \tilde{\nu}; E) \cong L^2(\widehat{G}, \tilde{\nu}_U; E) \oplus L^2(\widehat{G}, \tilde{\nu}_2; E).$$

Moreover, since each ρ_k has density with respect to $\tilde{\nu}_U$ and $\tilde{\nu}_U$ is disjoint from $\tilde{\nu}_2$, it follows that $W(\mathcal{H}) \subset L^2(\widehat{G}, \tilde{\nu}_U; E)$, then we can always assume that the measure ν on \widehat{G}/H^\perp which occurs in Theorem 4.1 is ν_U .

We now characterise the form of W . For $k \in I$, we can always fix an isometry $T_k : F_k \longrightarrow E$ such that $T_k(F_k)$ are mutually orthogonal subspaces of E . Hence, if we define, for $\phi_k \in L^2(\widehat{G}, \rho_k; F_k)$,

$$(T\phi_k)(x) := \sqrt{\alpha_k(x)} T_k \phi_k(x) \quad x \in \widehat{G},$$

T is an isometry intertwining U with Λ^{ν_U} . We define $W_k = WP_k$. The operator $V = WT^*$ is a partial isometry commuting with Λ^{ν_U} , hence there exists a weakly measurable correspondence $\widehat{G} \ni x \longmapsto V(x) \in \mathcal{L}(E)$ such that $V(x)$ are partial isometries for $\tilde{\nu}_U$ -almost all $x \in \widehat{G}$ and

$$(V\phi)(x) = V(x) \phi(x) \quad x \in \widehat{G},$$

where $\phi \in L^2(\widehat{G}, \tilde{\nu}_U; E)$. We have $W = WT^*T = VT$, then

$$\begin{aligned} (W_k \phi_k)(x) &= \sqrt{\alpha_k(x)} V(x) T_k \phi_k(x) \\ (4.5) \quad &= \sqrt{\alpha_k(x)} W_k(x) \phi_k(x) \quad x \in \widehat{G}, \end{aligned}$$

where we set

$$W_k(x) = V(x) T_k \quad \forall x \in \widehat{G}.$$

Since W is isometric, then $W_k^* W_k$ is the identity operator on $L^2(\widehat{G}, \rho_k; F_k)$, hence

$$T_k^* V(x)^* V(x) T_k = I_k \quad x \in \widehat{G}$$

ρ_k -almost everywhere, where I_k is the identity operator on F_k . Since T_k is isometric and $V(x)$ is a partial isometry for $\tilde{\nu}_U$ -almost every $x \in \widehat{G}$ (that is for ρ_k -almost every $x \in \widehat{G}$), it follows that $V(x)^* V(x)$ is the identity on $\text{ran} T_k$ and that $W_k(x)$ is

isometric, for ρ_k -almost every $x \in \widehat{G}$. Weak measurability of the maps $x \mapsto W_k(x)$ is immediate.

The explicit form of M is then given by

$$\begin{aligned} (P_j M(\omega) P_k \phi)(x) &= \left(W_j^* \widetilde{M}_0'(\omega) W_k \phi \right)(x) \\ &= \frac{1}{\sqrt{\alpha_j(x)}} W_j(x)^* \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \times \\ &\quad \times \sqrt{\alpha_k(xy^{-1})} W_k(xy^{-1}) (P_k \phi)(xy^{-1}) d\mu_{H^\perp}(y) \quad x \in \widehat{G}, \end{aligned}$$

where $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$.

Conversely, let $\widehat{G} \ni x \mapsto W_k(x) \in \mathcal{L}(F_k; E)$ be a weakly measurable map such that $W_k(x)$ are isometries for ρ_k -almost every $x \in \widehat{G}$ and for all $k \in I$. We define, for $\phi_k \in L^2(\widehat{G}, \rho_k; F_k)$,

$$(W \phi_k)(x) := \sqrt{\alpha_k(x)} W_k(x) \phi_k(x) \quad \forall x \in \widehat{G},$$

then W is clearly an intertwining isometry between U and Λ^{ν} and Eq. (4.3) defines a covariant POVM. \square

We now study the problem of equivalence of covariant POVMs. To simplify the exposition, we assume that the measures ρ_k in decomposition (4.1) are orthogonal.

Let M and M' be two covariant positive operator valued measures that are equivalent, i.e. there exists a unitary operator $S : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$(4.6) \quad SU(g) = U(g)S \quad \forall g \in G,$$

$$(4.7) \quad SM(\omega) = M'(\omega)S \quad \forall \omega \in C_c(G/H).$$

We have the following result.

Proposition 4.7. *Let $(W_j)_{j \in I}$ and $(W'_j)_{j \in I}$ be families of maps such that Eq. (4.3) holds for M and M' , respectively.*

The POVMs M and M' are equivalent if and only if, for each $k \in I$, there exists a weakly measurable map $x \mapsto S_k(x) \in \mathcal{L}(F_k)$ such that $S_k(x)$ are unitary operators for ρ_k -almost all x and

$$(4.8) \quad \sqrt{\alpha_k(xy)} W_j(x)^* W_k(xy) = \sqrt{\alpha_k(xy)} S_j(x)^* W'_j(x)^* W'_k(xy) S_k(xy)$$

for $(\rho_j \otimes \mu_{H^\perp})$ -almost all (x, y) .

Proof. By virtue of condition (4.6) and orthogonality of the measures ρ_k , S preserves decomposition (4.1). Moreover, for each $k \in I$, there exists a weakly measurable map $x \mapsto S_k(x) \in \mathcal{L}(F_k)$ such that $S_k(x)$ is unitary for ρ_k -almost all x and, if $\phi_k \in L^2(\widehat{G}, \rho_k; F_k)$,

$$(S \phi_k)(x) = S_k(x) \phi_k(x) \quad x \in \widehat{G}.$$

Condition (4.7) is equivalent to

$$P_j M(\omega) P_k \phi = P_j S^* M'(\omega) S P_k \phi$$

for all $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$ and $j, k \in I$. It is not restrictive to assume that the densities α_k are measurable functions. Let

$$\Omega_{j,k}(x, x') = \sqrt{\frac{\alpha_k(x')}{\alpha_j(x)}} (W_j(x)^* W_k(x') - S_j(x)^* W'_j(x)^* W'_k(x') S_k(x')),$$

using Eq. (4.3), the previous condition becomes

$$(4.9) \quad \int_{H^\perp} \mathcal{F}_{G/H}(\omega)(y) \Omega_{j,k}(x, xy^{-1}) (P_k \phi)(xy^{-1}) d\mu_{H^\perp}(y) = 0$$

ρ_j -almost everywhere for all $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$ and $j, k \in I$.

Let K be a compact set of \widehat{G} and $v \in F_k$. In Eq. (4.9) we choose

$$\phi = \chi_K v \in L^2(\widehat{G}, \rho_k; F_k)$$

and $\omega \in C_c(G/H)$ running over a denumerable subset dense in $L^2(G/H, \mu_{H^\perp})$. It follows that there exists a ρ_j -null set $N \subset \widehat{G}$ such that, for all $x \notin N$,

$$\chi_K(xy^{-1}) \Omega_{j,k}(x, xy^{-1}) v = 0$$

for μ_{H^\perp} -almost all $y \in H^\perp$. Since $\Omega_{j,k}$ is weakly measurable, the last equation holds in a measurable subset $Y \subset \widehat{G} \times H^\perp$ whose complement is a $(\rho_j \otimes \mu_{H^\perp})$ -null set. Define

$$m(x, y) = xy^{-1} \quad \forall (x, y) \in \widehat{G} \times H^\perp.$$

For all $(x, y) \in Y \cap m^{-1}(K)$ we then have

$$\Omega_{j,k}(x, xy^{-1}) v = 0.$$

Since F_k is separable and \widehat{G} is σ -compact, we get

$$\Omega_{j,k}(x, xy) = 0$$

for $(\rho_j \otimes \mu_{H^\perp})$ -almost all $(x, y) \in \widehat{G} \times H^\perp$, that is,

$$\sqrt{\alpha_k(xy)} W_j(x)^* W_k(xy) = \sqrt{\alpha_k(xy)} S_j(x)^* W'_j(x)^* W'_k(xy) S_k(xy)$$

for $(\rho_j \otimes \mu_{H^\perp})$ -almost all (x, y) .

Conversely, if condition (4.8) is satisfied for all $j, k \in I$, then clearly M is equivalent to M' . \square

5. EXAMPLES

5.1. Generalised covariant position observables. Let $\mathcal{H} = L^2(\mathbb{R}, dx)$, where dx is the Lebesgue measure on \mathbb{R} . We consider the representation U of the group \mathbb{R} acting on \mathcal{H} as

$$(U(a)\phi)(x) = e^{iax} \phi(x) \quad x \in \mathbb{R}$$

for all $a \in \mathbb{R}$. By means of Fourier transform, U is clearly equivalent to the regular representation of \mathbb{R} . We classify the POVMs based on \mathbb{R} and covariant with respect to U . With the notations of the previous sections, we have

$$G = \mathbb{R}, \quad H = \{0\}, \quad G/H = \mathbb{R}, \quad \widehat{G} = H^\perp = \mathbb{R}, \quad \widehat{G}/H^\perp = \{0\}.$$

We choose $\mu_{G/H} = \frac{1}{2\pi} dx$, so that $\mu_{H^\perp} = dx$, and $E = \mathcal{H}$.

The representation U is already diagonal with multiplicity equal to 1, so that in the decomposition (4.1) we can set $I = \{1\}$, $\rho_1 = dx$, $F_1 = \mathbb{C}$. Hence, by Corollary 4.4, U admits covariant POVMs based on \mathbb{R} and $\alpha_1 = 1$.

According to Theorem 4.2, any covariant POVM M is defined in terms of a weakly measurable map $x \mapsto W_1(x)$ such that $W_1(x) : \mathbb{C} \rightarrow \mathcal{H}$ is an isometry for every $x \in \mathbb{R}$. This is equivalent to selecting a weakly measurable map $x \mapsto h_x \in \mathcal{H}$, with $\|h_x\|_{\mathcal{H}} = 1 \forall x \in \mathbb{R}$, such that $W_1(x) = h_x \forall x \in \mathbb{R}$. Explicitly, if $\phi \in L^2(\mathbb{R}, dx)$,

$$\begin{aligned}
(M(\omega)\phi)(y) &= \int_{\mathbb{R}} \overline{\mathcal{F}}_{\mathbb{R}}(\omega)(x) \langle h_{y-x}, h_y \rangle \phi(y-x) dx \\
&= \int_{\mathbb{R}} \overline{\mathcal{F}}_{\mathbb{R}}(\omega)(y-x) \langle h_x, h_y \rangle \phi(x) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i(y-x)z} \omega(z) \langle h_x, h_y \rangle \phi(x) \frac{dz}{2\pi} \right) dx \quad y \in \mathbb{R}.
\end{aligned}$$

5.2. Generalised covariant phase observables. We give a complete characterisation of the covariance systems based on the one-dimensional torus

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, 2\pi]\}.$$

We have

$$\begin{aligned}
G &= \mathbb{T}, \quad H = \{1\}, \quad G/H = \mathbb{T}, \\
\widehat{G} &= H^\perp = \{(\mathbb{T} \ni z \mapsto z^n \in \mathbb{C}) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}, \\
\widehat{G}/H^\perp &= \{1\}.
\end{aligned}$$

We choose $\mu_{G/H} = \frac{1}{2\pi} d\theta =: \mu_{\mathbb{T}}$, so that μ_{H^\perp} is the counting measure $\mu_{\mathbb{Z}}$ on \mathbb{Z} .

Let U be a representation of \mathbb{T} . Since \mathbb{T} is compact, we can always assume that U acts diagonally on

$$\mathcal{H} = \bigoplus_{k \in I} F_k,$$

where $I \subset \mathbb{Z}$, and F_k are Hilbert spaces such that $\dim F_k$ is the multiplicity of the representation $k \in \mathbb{Z}$ in U . Explicitly,

$$(U(z)\phi_k) = z^k \phi_k$$

for all $z \in \mathbb{T}$ and $\phi_k \in F_k$.

In order to use Eq. (4.1), we notice that $F_k = L^2(\mathbb{Z}, \delta_k; F_k)$ (where δ_k is the Dirac measure at k), so that $\rho_k = \delta_k$. By Corollary 4.4, one has that U admits covariant POVMs based on \mathbb{T} and that $\alpha_k(j) = \delta_{k,j}$ (where $\delta_{k,j}$ is the Kronecker delta).

Choose an infinite dimensional Hilbert space E and, for each $k \in I$, fix an isometry W_k from F_k to E . The corresponding covariance system is given by

$$\begin{aligned}
P_j M(\omega) P_k \phi &= \overline{\mathcal{F}}_{\mathbb{T}}(\omega)(j-k) W_j^* W_k P_k \phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta}) e^{i(j-k)\theta} W_j^* W_k P_k \phi d\theta,
\end{aligned}$$

where $\phi \in \mathcal{H}$ and $\omega \in \mathcal{C}(\mathbb{T})$.

If $I = \mathbb{Z}$ and $\dim F_k = 1 \forall k \in \mathbb{Z}$, U is the number representation and M represents the phase observable (compare with the result obtained in Ref. [1]).

5.3. Covariant phase difference observables. Let $\mu_{\mathbb{T}}$ as in the previous section. We consider the following representation U of the direct product $G = \mathbb{T} \times \mathbb{T}$ acting on the space $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{T}, \mu_{\mathbb{T}} \otimes \mu_{\mathbb{T}})$ as

$$(U(a, b)f)(z_1, z_2) = f(az_1, b^{-1}z_2) \quad (z_1, z_2) \in \mathbb{T} \times \mathbb{T}$$

for all $(a, b) \in \mathbb{T} \times \mathbb{T}$.

Let H be the closed subgroup

$$H = \{(a, b) \in \mathbb{T} \times \mathbb{T} \mid b = a\} \cong \mathbb{T}.$$

We classify all the POVMs based on G/H and covariant with respect to U (for a different approach to the same problem, see Ref. [6]).

We have

$$\begin{aligned} G &= \mathbb{T} \times \mathbb{T}, \quad G/H \cong \mathbb{T}, \quad \widehat{G} = \widehat{\mathbb{T}} \times \widehat{\mathbb{T}} \cong \mathbb{Z} \times \mathbb{Z}, \\ H^\perp &= \{(j, k) \in \mathbb{Z} \times \mathbb{Z} \mid k = -j\} \cong \mathbb{Z}, \\ \widehat{G}/H^\perp &\cong \mathbb{Z}. \end{aligned}$$

We fix $\mu_{G/H} = \mu_{\mathbb{T}}$, so that $\mu_{H^\perp} = \mu_{\mathbb{Z}}$.

We choose the following orthonormal basis $(e_{i,j})_{i,j \in \mathbb{Z}}$ of \mathcal{H}

$$e_{i,j}(z_1, z_2) = z_1^i z_2^{-j} \quad (z_1, z_2) \in \mathbb{T} \times \mathbb{T},$$

so that

$$U(a, b) e_{i,j} = a^i b^j e_{i,j} \quad \forall (a, b) \in \mathbb{T} \times \mathbb{T}.$$

Let $F_{i,j} = \mathbb{C}e_{i,j}$, then U acts diagonally on $F_{i,j}$ as the character $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Then, one can choose as decomposition (4.1)

$$\mathcal{H} = \bigoplus_{i,j \in \mathbb{Z}} F_{i,j} \cong \bigoplus_{i,j \in \mathbb{Z}} L^2(\mathbb{Z} \times \mathbb{Z}, \delta_i \otimes \delta_j; F_{i,j})$$

With the notations of Section 4, we have $I = \mathbb{Z} \times \mathbb{Z}$ and $\rho_{i,j} = \delta_i \otimes \delta_j$. It follows that \mathcal{C}_U^π is the equivalence class of $\mu_{\mathbb{Z}}$. With the choice $\nu_U = \mu_{\mathbb{Z}}$, it follows that $\tilde{\nu} = \mu_{\mathbb{Z}} \otimes \mu_{\mathbb{Z}}$. According to Theorem (4.2), U admits covariant POVMs and $\alpha_{i,j}(n, m) = \delta_{n,i} \delta_{m,j}$.

With the choice $E = \mathcal{H}$, we select a map $(i, j) \mapsto W_{i,j}$, where $W_{i,j}$ is an isometry from $F_{i,j}$ to \mathcal{H} . Since $F_{i,j}$ are one dimensional, there exists a family of vectors $(h_{i,j})_{i,j \in \mathbb{Z}}$ in \mathcal{H} , with $\|h_{i,j}\|_{\mathcal{H}} = 1 \quad \forall (i, j) \in \mathbb{Z} \times \mathbb{Z}$, such that

$$W_{i,j} e_{i,j} = h_{i,j} \quad \forall (i, j) \in \mathbb{Z} \times \mathbb{Z}.$$

The corresponding covariant POVM M is given, for every $\phi \in \mathcal{H}$, by

$$\begin{aligned} P_{l,m} M(\omega) P_{i,j} \phi &= \sum_{h \in \mathbb{Z}} \mathcal{F}_{\mathbb{T}}(\omega)(h) \delta_{l-h,i} \delta_{m+h,j} \langle h_{i,j}, h_{l,m} \rangle \langle \phi, e_{i,j} \rangle e_{l,m} \\ &= \delta_{l+m,i+j} \mathcal{F}_{\mathbb{T}}(\omega)(j-m) \langle h_{i,j}, h_{l,m} \rangle \langle \phi, e_{i,j} \rangle e_{l,m}. \end{aligned}$$

In particular, if $l+m = i+j$, we have

$$\begin{aligned} \langle M(\omega) e_{i,j}, e_{l,m} \rangle &= \overline{\mathcal{F}_{\mathbb{T}}(\omega)}(j-m) \langle h_{i,j}, h_{l,m} \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta}) e^{i(j-m)\theta} \langle h_{i,j}, h_{l,m} \rangle d\theta. \end{aligned}$$

If $l+m \neq i+j$, one has

$$\langle M(\omega) e_{i,j}, e_{l,m} \rangle = 0.$$

REFERENCES

- [1] G. Cassinelli, E. De Vito, P. Lahti and J. P. Pellonpää, *Covariant localisations in the torus and the phase observables*, J. Math. Phys. **43**, 693-704 (2002).
- [2] G. Cassinelli, E. De Vito, *Square-integrability modulo a subgroup*, to appear on Trans. AMS (2003).
- [3] U. Cattaneo, *On Mackey's imprimitivity theorem*, Comment. Math. Helvetici **54**, 629-641 (1979).
- [4] J. Dieudonné, *Elements d'analyse*, Vol. **6**, Gauthiers-Villars, Paris, 1975.
- [5] G. B. Folland, *A Course in Abstract Armonic Analysis*, CRC Press, Boca Raton, 1995.
- [6] T. Heinonen, P. Lahti and J. P. Pellonpää, *Covariant phase difference observables*, to appear on J. Math. Phys. (2003).
- [7] A. S. Holevo, *Generalized imprimitivity systems for Abelian groups*, Izvestiya VUZ. Matematika **27**, No. 2, 53-80 (1983).
- [8] A. S. Holevo, *On a generalization of canonical quantization*, Math. USSR Izvestiya **28**, No. 1, 175-188 (1987).

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