# PHASE SPACE OBSERVABLES AND ISOTYPIC SPACES

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Abstract. We give necessary and sufficient conditions for the set of Neumark projections of a countable set of phase space observable to constitute a resolution of the identity, and we give a criteria for a phase space observable to be informationally complete. The results will be applied to the phase space observables arising from an irreducible representation of the Heisenberg group.

## 1. Introduction

Phase space observables have turned out to be highly useful in various fields of quantum physics, including quantum communication and information theory, quantum tomography, quantum optics, and quantum measurement theory. Also many conceptual problems, like the problem of joint measurability of noncommutative quantities, or the problem of classical limit of quantum mechanics have greatly advanced by this tool. The monographs [1, 2, 3, 4, 5, 6, 7] exhibit various aspects of these developments.

Any positive trace one operator $T$ (a state) defines a phase space observable $Q_T$ according to the rule

$$Q_T(E) = \frac{1}{2\pi} \int_E e^{i(qP+pQ)} T e^{-i(qP+pQ)} dq\, dp,$$

where $E$ is a Borel subset of the (two dimensional) phase space. It is well known that all the phase space observables generated by *pure states* have the same minimal Neumark dilation to a canonical projection measure on $L^2(\mathbb{R}^2)$. On the other hand, the corresponding Neumark projections depend on the pure state in question. If $T$ is a pure state $|u\rangle\langle u|$ defined by a unit vector $u$, we let $P_u$ denote the Neumark projection associated with $Q_{|u\rangle\langle u|}$ . If two unit vectors $u$ and $v$ are orthogonal then also $P_u P_v = 0$. One could then pose the problem of determining a set of orthonormal vectors $\{u_i\}$ such that the associated Neumark projections $\{P_{u_i}\}$ of the phase space observables $Q_{|u_i\rangle\langle u_i|}$ constitute a resolution of the identity. In [8] it was shown that the set of number eigenvectors possesses this property. This was proved by a direct method using the properties of the Laguerre polynomials.

It turns out that this problem has a group theoretical background. This follows from the work of A. Borel [9] on the group representations that are square integrable modulo the centre. Using the results of Borel this problem can be traced back to the study of the isotypic spaces of the representations induced by a central character of the Heisenberg group $H^1$. (We recall that a representation $(\pi, \mathcal{H})$ is called isotypic if it is the direct sum of copies of the same irreducible representation). More precisely, the phase space observables arise from an irreducible representation of $H^1$ that is square integrable modulo the centre. This is actually a general result: any irreducible representation $\pi$ of a group $G$ that is square integrable modulo the centre gives rise to covariant "phase space observables" with the above properties. We prove that a necessary and sufficient condition for the set of Neumark projections $\{P_{u_i}\}$ to be a resolution of the identity is that the representation of $G$ induced by the central character of $\pi$ be isotypic. This phenomenon occurs in particular for the Heisenberg group, which is behind the phase space observables.

Phase space observables $Q_T$ that are generated by states $T$ such that $\mathrm{tr}\,[Te^{i(qP+pQ)}] \neq 0$ for almost all $(q,p) \in \mathbb{R}^2$, are known to have another important property. They are informationally complete, namely, if $W_1$ and $W_2$ are two states for which $\mathrm{tr}\,[W_1 Q_T(E)] = \mathrm{tr}\,[W_2 Q_T(E)]$ for all $E$, then $W_1 = W_2$, see, eg. [10, 11]. We show that, under suitable conditions, this property holds in general for "phase space observables" associated with any irreducible representations $\pi$ of $G$ square integrable modulo centre.

We hope that these results could bring further light on some of the many applications of the phase space observables in quantum mechanics.

## 2. Preliminaries and notations

In this paper we use freely the basic concepts and results of harmonic analysis, referring to [13] as our standard source. Let $G$ be a Hausdorff, locally compact, second countable topological group, and let $Z$ be its centre. $Z$ is a closed, abelian, normal subgroup of $G$. We denote by $X = G/Z$ the quotient space. It is a Hausdorff, locally compact, second countable topological group, and it is also a locally compact $G$-space with respect to the natural action by left multiplication. Let $p : G \to X$ be the canonical projection and $s : X \to G$ a Borel section for $p$, fixed throughout the paper.

Assume further that $G$ is unimodular so that its left Haar measures are also right Haar measures. As an abelian subgroup $Z$ is also unimodular. We denote by $\mu$ and $\mu_0$ two (arbitrarily fixed) Haar measures of $G$ and $Z$, respectively. Then there is a unique $G$-invariant positive Borel measure $\alpha$ on $X$ such that for each compactly supported continuous function $f \in C_c(G)$

$$\int_G f(g)\, d\mu(g) = \int_X \left( \int_Z f(s(x)h)\, d\mu_0(h) \right) d\alpha(x). \tag{1}$$

Moreover, $f \in L^1(\mu)$ if and only if the function $(x,h) \mapsto f(s(x)h)$ is in $L^1(\alpha \otimes \mu_0)$ and in this case (1) holds for $f$. The measure $\alpha$ is also a Haar measure for $X$ (regarded as a group), both right and left.

We denote by $(\pi, \mathcal{H})$ a continuous unitary irreducible representation of $G$ acting on a complex separable Hilbert space $\mathcal{H}$. Let $h \in Z, g \in G$. Then $\pi(h)\pi(g) = \pi(hg) = \pi(gh) = \pi(g)\pi(h)$, so that $\pi(h)$ commutes with all $\pi(g), g \in G$. By Schur's lemma,

$$\pi(h) = \chi(h)\, I,$$

where $I$ is the identity operator on $\mathcal{H}$ and $\chi$ is a $\mathbb{T}$-valued character of $Z$, $\mathbb{T}$ denoting the group of complex numbers of modulus one. We call $\chi$ the central character of $\pi$.

In the following we describe explicitly the imprimitivity system for $G$, based on $X$, induced by the irreducible unitary representation $\chi$ of $Z$. There are several equivalent realizations of this object, and we choose those which are most appropriate for our purposes.

Let $\mathcal{H}^\chi$ denote the space of ($\mu$-equivalence classes of) measurable functions $f : G \to \mathbb{C}$ for which

1. $f(gh) = \chi(h^{-1})f(g)$ for all $h \in Z$,
2. $f \circ s \in L^2(X, \alpha)$.

The definition of the space $\mathcal{H}^\chi$ does not depend on the section $s$. Indeed, if $s'$ is another Borel section for $p$, then for any $x \in X$, $s'(x) = s(x)h$ for some $h \in Z$, so that

$$|f(s'(x))|^2 = |f(s(x)h)|^2 = |\chi(h^{-1})f(s(x))|^2 = |f(s(x))|^2.$$

The space $\mathcal{H}^\chi$ is a complex separable Hilbert space with respect to the scalar product

$$\langle f_1, f_2 \rangle_{\mathcal{H}^\chi} := \int_X \overline{f_1(s(x))} f_2(s(x))\, d\alpha(x),$$

which is independent of $s$.

A description of the structure of $\mathcal{H}^\chi$ is given by the following property. Let $K(G)^\chi$ denote the set of continuous functions $f : G \to \mathbb{C}$ with the properties

1. $f(gh) = \chi(h^{-1})f(g)$ for all $g \in G, h \in Z$,
2. $p(\operatorname{supp} f)$ is compact in $X$.

If $\varphi \in C_c(G)$, then the function $f_\varphi$, defined by

$$f_\varphi(g) := \int_Z \chi(h)\varphi(gh)\, d\mu_0(h),$$

is in $K(G)^\chi$. Moreover, any function $f \in K(G)^\chi$ is of the form $f = f_\varphi$ for some $\varphi \in C_c(G)$ (see, e.g., [13], Proposition 6.1, p. 152). Obviously, $K(G)^\chi \subset \mathcal{H}^\chi$ and $K(G)^\chi$ is dense in $\mathcal{H}^\chi$.

The Hilbert space $\mathcal{H}^\chi$ carries a continuous unitary representation $l$ of $G$ explicitly given by

$$(l(a)f)(g) = f(a^{-1}g), \quad g \in G.$$

It is a realization of the representation of $G$ induced by the representation $\chi$ of $Z$.

Let $\mathcal{B}(X)$ be the $\sigma$-algebra of the Borel subsets of $X$. We define a projection measure on $(X, \mathcal{B}(X))$ by

$$(P(E)f)(g) := \chi_E(p(g))f(g),$$

where $E \in \mathcal{B}(X)$ and $f \in \mathcal{H}^\chi$. Clearly, $\mathcal{B}(X) \ni E \mapsto P(E) \in B(\mathcal{H}^\chi)$ is a projection measure and $(l, P)$ is an imprimitivity system for $G$, based on $X$, and acting on $\mathcal{H}^\chi$. Indeed,

$$l(a)P(E)l(a)^{-1} = P(a.E), \quad a \in G, E \in \mathcal{B}(X).$$

It is a realization of the imprimitivity system canonically induced by $\chi$ and it is irreducible since $\chi$ is irreducible.

## 3. Representations that are square integrable modulo the centre

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of $G$ in a complex separable Hilbert space $\mathcal{H}$. Given $u, v \in \mathcal{H}$, we denote by $c_{u,v}$ the function on $G$ defined through the formula

$$c_{u,v}(g) := \langle \pi(g)u, v\rangle.$$

This function is called a *coefficient* of $\pi$ and it is continuous and bounded,

$$|c_{u,v}(g)| = |\langle \pi(g)u, v\rangle| \leq \|\pi(g)u\| \, \|v\| \leq \|u\| \, \|v\|, \quad g \in G,$$

and it has the property $c_{u,v}(gh) = \chi(h)^{-1}c_{u,v}(g)$ for all $h \in Z$.

**Definition 1.** Let $(\pi, \mathcal{H})$ be a continuous unitary irreducible representation of $G$. We say that $\pi$ is *square integrable modulo the centre* of $G$, when, for all $u, v \in \mathcal{H}$, $c_{u,v} \circ s \in L^2(X, \alpha)$.

This definition is independent of the choice of the function $s$. Indeed, if $s'$ is another section for $p$, then $s'(x) = s(x)h$, $h \in Z$, for all $x \in X$, so that $\pi(s'(x)) = \pi(s(x)h) = \chi(h)\pi(s(x))$, and thus $|\langle \pi(s'(x))u, v\rangle|^2 = |\langle \pi(s(x))u, v\rangle|^2$.

We shall list next the basic properties of the square integrable representations modulo the centre. These results are due to A. Borel [9], and they generalize the classical results of R. Godement [14] for square integrable representations.

1. Let $\pi$ be a unitary irreducible representation of $G$ with central character $\chi$. Then the following three statements are equivalent:
   *a*) $\pi$ is square integrable modulo $Z$;
   *b*) there exist $u, v \in \mathcal{H} \setminus \{0\}$ such that $c_{u,v} \circ s \in L^2(X, \alpha)$;
   *c*) $\pi$ is equivalent to a subrepresentation of $(l, \mathcal{H}^\chi)$.
2. If any (hence all) of the preceding conditions is satisfied, then $c_{u,v} \in \mathcal{H}^\chi$ for all $u, v \in \mathcal{H}$.
3. If $(\pi, \mathcal{H})$ is square integrable modulo $Z$, there exists a number $d_\pi > 0$, called *the formal degree* of $\pi$, such that
$$\langle c_{u,v}, c_{u',v'}\rangle_{\mathcal{H}^\chi} = \frac{1}{d_\pi}\langle u', u\rangle_{\mathcal{H}} \langle v, v'\rangle_{\mathcal{H}}.$$
   The formal degree depends on the normalisation of the Haar measure $\mu$ so that, possibly redefining $\mu$, one can always assume that $d_\pi = 1$ so that
$$\langle c_{u,v}, c_{u',v'}\rangle_{\mathcal{H}^\chi} = \langle u', u\rangle_{\mathcal{H}} \langle v, v'\rangle_{\mathcal{H}}. \tag{2}$$
4. If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are two representations of $G$ which are square integrable modulo $Z$, whose central characters $\chi$ and $\chi'$ coincide, and which are not equivalent, then
$$\langle c_{u,v}, c'_{u',v'}\rangle_{\mathcal{H}^\chi} = 0, \tag{3}$$
   where $c'_{u',v'}$ are coefficients of $(\pi', \mathcal{H}')$.

## 4. Canonical POM associated with a square integrable representation modulo the centre

Let $(\pi, \mathcal{H})$ be a fixed representation with central character $\chi$ and square integrable modulo the centre. Fix $u \in \mathcal{H} \setminus \{0\}$, and define $W_u : \mathcal{H} \to \mathcal{H}^\chi$ by
$$W_u v := c_{u,v}, \quad v \in \mathcal{H}.$$
$W_u$ is a linear map and it is a multiple of an isometry. Indeed, if $v, w \in \mathcal{H}$, then
$$\langle W_u v, W_u w\rangle_{\mathcal{H}^\chi} = \|u\|^2_{\mathcal{H}} \langle v, w\rangle_{\mathcal{H}}. \tag{4}$$

The range of $W_u$ is a closed subspace of $\mathcal{H}^\chi$, and $1/\left\|u\right\|_{\mathcal{H}} W_u$ is a unitary operator from $\mathcal{H}$ to the range of $W_u$. The operator $W_u$ intertwines the action of $\pi$ on $\mathcal{H}$ with the action of $l$ on $\mathcal{H}^\chi$. In fact, for any $a \in G$,

$$\begin{aligned}(W_u(\pi(a)v))(g) &= c_{u,\pi(a)v}(g) = \langle \pi(g)u, \pi(a)v\rangle_{\mathcal{H}} \\ &= \langle \pi(a^{-1}g)u, v\rangle_{\mathcal{H}} = c_{u,v}(a^{-1}g) \\ &= (W_u v)(a^{-1}g) = (l(a)(W_u v))(g),\end{aligned}$$

showing that

$$W_u\, \pi(a) = l(a)\, W_u$$

for all $a \in G$. Hence $\operatorname{ran} W_u$ is invariant with respect to $l$ and the unitary operator $1/\left\|u\right\|_{\mathcal{H}} W_u$ defines an isomorphism of the unitary irreducible representations $(\pi, \mathcal{H})$ and $(l|_{\operatorname{ran} W_u}, \operatorname{ran} W_u)$ of $G$,

$$(\pi, \mathcal{H}) \simeq (l|_{\operatorname{ran} W_u}, \operatorname{ran} W_u).$$

We are in a position to associate to any state $T$ a natural positive operator measure (POM) on $(X, \mathcal{B}(X))$, with values in the positive operators on $\mathcal{H}$. Given a state $T$, for all $E \in \mathcal{B}(X)$ we define

$$Q_T(E) = \int_E \pi(s(x))T\pi(s(x))^{-1}\, d\alpha(x), \tag{5}$$

where the integral is in the weak sense. The definition is well posed. Indeed, let $T = \sum_i \lambda_i |e_i\rangle\langle e_i|$ be the spectral decomposition of $T$ and fix a trace class operator $B$ with the decomposition $B = \sum_k w_k |u_k\rangle\langle v_k|$, where $w_k \geq 0$ and $(u_k), (v_k) \subset \mathcal{H}$ are orthonormal sequences. Since $\pi$ is square integrable modulo $Z$, the function

$$\phi_{ik}(x) = \overline{c_{e_i,v_k}(s(x))}c_{e_i,u_k}(s(x)) = \langle v_k, \pi(s(x))e_i\rangle\langle e_i, \pi(s(x))^{-1}u_k\rangle$$

is $\alpha$-integrable on $X$ and, using the Hölder inequality and the orthogonality relations (2),

$$\begin{aligned}\int_E |\phi_{ik}(x)|\, d\alpha(x) &\leq \left(\int_E |c_{e_i,v_k}(s(x))|^2\, d\alpha(x)\right)^{\frac{1}{2}} \times \\ &\qquad \left(\int_E |c_{e_i,u_k}(x)|^2\, d\alpha(x)\right)^{\frac{1}{2}} \\ &\leq \left\|c_{e_i,v_k}\right\|_{\mathcal{H}^\chi} \left\|c_{e_i,u_k}\right\|_{\mathcal{H}^\chi} \\ &\leq \|e_i\|^2_{\mathcal{H}} \|v_k\|_{\mathcal{H}} \|u_k\|_{\mathcal{H}} = 1.\end{aligned}$$

Since $\sum_{i,k} \lambda_i w_k = \left\|T\right\|_1 \left\|B\right\|_1 = \left\|B\right\|_1$, the series $\sum_{i,k} \lambda_i w_k \phi_{ik}$ converges $\alpha$-almost everywhere to an integrable function $\phi$ and

$$\int_E \phi(x)\, d\alpha(x) = \sum_{i,k} \lambda_i w_k \int_E \phi_{ik}\, d\alpha(x).$$

On the other hand, for $\alpha$-almost all $x \in X$, $\phi(x) = \mathrm{tr}\big[B\pi(s(x))T\pi(s(x))^{-1}\big]$. Hence $\int_E |\mathrm{tr}\big[B\pi(s(x))T\pi(s(x))^{-1}\big]|\, d\alpha(x) \leq \|B\|_1$ and the linear form

$$B \mapsto \int_E \mathrm{tr}\big[B\pi(s(x))T\pi(s(x))^{-1}\big]\, d\alpha(x)$$

is continuous on the Banach space of the trace class operators. Therefore it defines a bounded operator $Q_T(E)$ such that

$$\begin{aligned}
\mathrm{tr}\big[BQ_T(E)\big] &= \int_E \mathrm{tr}\big[B\pi(s(x))T\pi(s(x))^{-1}\big]\, d\alpha(x) \\
&= \sum_{i,k} \lambda_i w_k \int_E \langle v_k, \pi(s(x))e_i\rangle \langle e_i, \pi(s(x))^{-1}u_k\rangle\, d\alpha(x) \\
&= \sum_{i,k} \lambda_i w_k \int_E \overline{c_{e_i,v_k}(s(x))}c_{e_i,u_k}(s(x))\, d\alpha(x).
\end{aligned}$$

By choosing $B = |u\rangle\langle v|$ we see that $Q_T(E)$ has the expression (5).

The mapping $E \mapsto Q_T(E)$ defines a POM on $X$. Indeed, $Q_T(E)$ is a positive operator and, given $u, v \in \mathcal{H}$, the map $E \mapsto \langle u, Q_T(E)v\rangle_{\mathcal{H}}$ is a complex measure on $(X, \mathcal{B}(X))$, due to the $\sigma$-additivity of the integral.

Moreover, $Q_T(X) = I$. Indeed, for all $u, v \in \mathcal{H}$,

$$\begin{aligned}
\langle u, Q_T(X)v\rangle_{\mathcal{H}} &= \sum_i \lambda_i \int_X \overline{c_{e_i,u}(s(x))}c_{e_i,v}(s(x))\, d\alpha(x) \\
&= \sum_i \lambda_i \langle c_{e_i,u}, c_{e_i,v}\rangle \\
&= \sum_i \lambda_i \|e_i\|^2_{\mathcal{H}} \langle u, v\rangle_{\mathcal{H}} = \langle u, v\rangle_{\mathcal{H}}.
\end{aligned}$$

The operator measure $E \mapsto Q_T(E)$ is covariant under the representation $(\pi, \mathcal{H})$, that is, for all $E \in \mathcal{B}(X), a \in G$,

$$\pi(a)Q_T(E)\pi(a)^{-1} = Q_T(a.E).$$

Indeed,

$$
\begin{aligned}
\pi(a)Q_T(E)\pi(a)^{-1} &= \int_E \pi(a)\pi(s(x))T\pi(s(x))^{-1}\pi(a)^{-1}\, d\alpha(x) \\
&= \int_E \pi(as(x))T\pi(as(x))^{-1}\, d\alpha(x) \\
&= \int_E \pi(s(a.x))T\pi(s(a.x))^{-1}\, d\alpha(x) \\
&= \int_{a.E} \pi(s(x))T\pi(s(x))^{-1}\, d\alpha(x) \\
&= Q_T(a.E)
\end{aligned}
$$

where we used the fact that $as(x) = s(a.x)h$, for some $h \in Z$.

## 5. The minimal Neumark dilation of $Q_u$

In this section we consider the operator measure $Q_{|u\rangle\langle u|}$ associated with a pure state $|u\rangle\langle u|$ and we show that the canonical projection measure $P$ defined in Section 2 is the minimal Neumark dilation of $Q_{|u\rangle\langle u|}$ for any $u$.

Given a unit vector $u \in \mathcal{H}$, we denote simply by $Q_u$ the POM $Q_{|u\rangle\langle u|}$. Then for any $E \in \mathcal{B}(X)$ and for all $v, w \in \mathcal{H}$,

$$
\begin{aligned}
\langle W_u v, P(E) W_u w\rangle_{\mathcal{H}^\chi} &= \int_X \overline{(W_u v)(s(x))}\, \chi_E(x)\, (W_u w)(s(x))\, d\alpha(x) \\
&= \int_E \overline{c_{u,v}(s(x))} c_{u,w}(s(x))\, d\alpha(x) \\
&= \langle v, Q_u(E) w\rangle_{\mathcal{H}},
\end{aligned}
$$

which shows that $P$ is a Neumark dilation of $Q_u$.

Furthermore, $P$ is minimal in the sense that $\mathcal{H}^\chi$ is the smallest closed space containing all the vectors of the form $P(E)f$, as $E$ varies in $\mathcal{B}(X)$ and $f$ varies in $\operatorname{ran} W_u$,

$$\mathcal{H}^\chi = \overline{\operatorname{span}}\,\{P(E)f \mid E \in \mathcal{B}(X), f \in \operatorname{ran} W_u\}.$$

We go on to prove this fact. Due to the irreducibility of $\pi$, all the vectors of $\mathcal{H}$ are cyclic for $\pi$ itself. Hence for any $v \in \mathcal{H}$, $v \neq 0$,

$$\mathcal{H} = \overline{\operatorname{span}}\,\{\pi(a)v \mid a \in G\}.$$

Therefore,

$$\operatorname{ran} W_u = \overline{\operatorname{span}}\,\{W_u(\pi(a)v) \mid a \in G\} = \overline{\operatorname{span}}\,\{(l(a)W_u)(v) \mid a \in G\},$$

so that

$$\begin{aligned}
&\overline{\operatorname{span}}\,\{P(E)f \mid E \in \mathcal{B}(X), f \in \operatorname{ran} W_u\} \\
= \;&\overline{\operatorname{span}}\,\{P(E)(l(a)W_u)(v) \mid E \in \mathcal{B}(X), a \in G\} \\
= \;&\overline{\operatorname{span}}\,\{l(a)P(a^{-1}.E)W_u(v) \mid E \in \mathcal{B}(X), a \in G\}.
\end{aligned}$$

Now $W_u v$ is a nonzero element of $\mathcal{H}^\chi$ and $(l,P)$ is an irreducible imprimitivity system for $G$, acting in $\mathcal{H}^\chi$, so that

$$\overline{\operatorname{span}}\,\{l(a)P(a^{-1}.E)W_u(v) \mid E \in \mathcal{B}(X), a \in G\} = \mathcal{H}^\chi,$$

which completes the proof of the statement.

As a final remark we notice that the Neumark projection $P_u : \mathcal{H}^\chi \to \mathcal{H}^\chi$ onto the range of $W_u$ is explicitly given by $P_u = W_u W_u^*$.

## 6. A decomposition of the space $\mathcal{H}^\chi$

In this section we describe a decomposition of the space $\mathcal{H}^\chi$ associated with the representation $(\pi, \mathcal{H})$ of $G$. We denote

$$\begin{aligned}
M(\pi)_0 \;&:=\; \sum_{u\in\mathcal{H}} \operatorname{ran} W_u \\
&=\; \operatorname{span}\{c_{u,v} \mid u, v \in \mathcal{H}\} \\
M(\pi) \;&:=\; \overline{M(\pi)_0}.
\end{aligned}$$

$M(\pi)$ is the smallest closed subspace of $\mathcal{H}^\chi$ that contains all the ranges of the maps $W_u$. If $\pi$ and $\pi'$ are equivalent representations, then

$$M(\pi) = M(\pi').$$

In other words, $M(\pi)$ depends only on the equivalence class of $\pi$. On the other hand, if $\pi$ and $\pi'$ are not equivalent, but they have the same central character $\chi$, then the orthogonality condition (3) imply that

$$M(\pi) \perp M(\pi').$$

We proceed to study the structure of the subspace $M(\pi)$.

1. $M(\pi)$ is invariant under the action of $l$. This is clear since $M(\pi)_0$ is invariant with respect to $l$, hence, for any $a \in G$, $l(a)M(\pi) = \overline{l(a)M(\pi)_0} = \overline{M(\pi)_0} = M(\pi)$.

2. Let $(e_n)_{n\geq 1}$ be a basis of $\mathcal{H}$. Then $(W_{e_p}e_n)_{n,p\geq 1}$ is a basis of $M(\pi)$. To show this, observe that $\langle W_{e_p}e_n, W_{e_q}e_m\rangle_{\mathcal{H}^\chi} = \langle e_q, e_p\rangle_{\mathcal{H}}\langle e_n, e_m\rangle_{\mathcal{H}}$, so that $(W_{e_p}e_n)_{n,p\geq 1}$ is an orthonormal set in $M(\pi)$. Given $u, v \in \mathcal{H}$, one has that $\sum_{n,p} |\langle u, e_n\rangle\langle e_p, v\rangle|^2 = \|u\|^2\|v\|^2$. Hence the series $\sum_{n,p} \langle u, e_n\rangle\langle e_p, v\rangle W_{e_p}e_n$ converges in $M(\pi)$. Since, for all

$g \in G$, $\sum_{n,p} \langle u, e_n \rangle \langle e_p, v \rangle W_{e_p} e_n(g)$ converges to $W_u v(g)$, the set $(W_{e_p} e_n)_{n,p \geq 1}$ generates $M(\pi)_0$, hence $M(\pi)$.

3. The space $M(\pi)$ is isotypic, in fact it can be decomposed as

$$M(\pi) = \oplus_{p \geq 1} \operatorname{ran} W_{e_p}$$

and, for any $p$ the representation $(l|_{\operatorname{ran} W_{e_p}}, \operatorname{ran} W_{e_p})$ is unitarily equivalent to $(\pi, \mathcal{H})$.

The Hilbert sum of the subspaces $M(\pi)$, as $\pi$ runs through the (inequivalent) irreducible representations of $G$ with the same central character $\chi$ that are square integrable modulo the centre, does not exhaust $\mathcal{H}^\chi$, in general. This Hilbert sum is the *discrete part* of $\mathcal{H}^\chi$. In fact, let $V$ be a closed subspace of $\mathcal{H}^\chi$ which is invariant and irreducible under $l$, and denote by $\sigma$ the restriction of $l$ to $V$. Then $\sigma$ is a square integrable representation of $G$ modulo the centre, with the same central character $\chi$, and one has the following result.

**Proposition 1.** *The subspace $V$ is contained in $M(\sigma)$.*

*Proof.* Let $f \in V$ and denote by $S : \mathcal{H}^\chi \to V$ the orthogonal projection onto $V$. For all $g \in \mathcal{H}^\chi$ and $a \in G$ we have

$$\langle \sigma(a) Sg, f \rangle_{\mathcal{H}^\chi} = \langle S l(a) g, f \rangle_{\mathcal{H}^\chi} = \langle l(a) g, f \rangle_{\mathcal{H}^\chi}.$$

Since $Sg$ and $f$ are in $V$ and $(\sigma, V)$ is square integrable modulo $Z$, we have

$$(a \mapsto \langle l(a) g, f \rangle_{\mathcal{H}^\chi}) \in M(\sigma).$$

Explicitly,

$$\langle l(a) g, f \rangle_{\mathcal{H}^\chi} = \int_X \overline{g(a^{-1} s(x))} f(s(x))\, d\alpha(x).$$

For any $\phi \in C_c(G)$ the function $f_\phi$ defined in section 2 is in $K(G)^\chi \subset \mathcal{H}^\chi$ and we get

$$\langle l(a) f_\phi, f \rangle_{\mathcal{H}^\chi} = \int_X d\alpha(x) f(s(x)) \int_Z d\mu_0(h) \overline{\chi(h) \phi(a^{-1} s(x) h)}.$$

We claim that

$$\left(x, h \mapsto f(s(x)) \overline{\chi(h) \phi(a^{-1} s(x) h)}\right) \in L^1(\alpha \otimes \mu_0).$$

Indeed

$$\int_Z |f(s(x)) \chi(h) \phi(a^{-1} s(x) h)|\, d\mu_0(h) = |f(s(x))| \int_Z |\phi(a^{-1} s(x) h)|\, d\mu_0(h)$$

and the function

$$x \mapsto \int_Z |\phi(a^{-1}s(x)h)| \, d\mu_0(h)$$

is in $C_c(X)$ (see, for instance, [13]). Hence its product with $|f(s(x))|$ is in $L^1(\alpha)$ and the claim follows by Tonelli's theorem. Now we can apply Equation (1) to the function

$$f(s(x))\overline{\chi(h)\phi(a^{-1}s(x)h)} = f(s(x)h)\overline{\phi(a^{-1}s(x)h)}$$

to conclude that

$$\begin{aligned} \langle l(a)f_\phi, f\rangle_{\mathcal{H}^\chi} &= \int_G f(g)\overline{\phi(a^{-1}g)}\, d\mu(g) \\ &= (f * \tilde{\phi})(a), \end{aligned}$$

where $\tilde{\phi}(a) := \overline{\phi(a^{-1})}$, and $*$ denotes the convolution. In particular $f * \tilde{\phi} \in M(\sigma)$. If we let $\phi$ run over a sequence of functions on $G$ which is an approximate identity, see for example [13], one can prove that $f * \tilde{\phi} \to f$ in $\mathcal{H}^\chi$ (see the below remark) and, since $M(\sigma)$ is closed, $f \in M(\sigma)$. This shows that $V \subseteq M(\sigma)$. □

**Remark 1.** The proof of the above proposition uses the fact that $f * \tilde{\phi} \to f$ in $\mathcal{H}^\chi$ when $\phi$ runs over a sequence of functions on $G$ which is an approximate identity. To show this technical result one can mimic the standard proof in $L^2(G)$, taking into account that there exists a Borel measure $\nu$ on $G$ having density with respect to $\mu$ such that the induced representation $(l, \mathcal{H}^\chi)$ can be realized on a suitable subspace of $L^2(G, \nu)$ (compare Ex. 6, Sect XXII.3 of [15]).

To summarize,

$$\mathcal{H}^\chi = \oplus_\pi M(\pi) \oplus R,$$

where the first direct sum ranges over the inequivalent irreducible representations of $G$ with central character $\chi$ that are square integrable modulo the centre and the orthogonal complement $R$ is the continuous part of the decomposition.

We can now state the main result of the paper.

**Proposition 2.** *Let $(\pi, \mathcal{H})$ be a square integrable representation of $G$ modulo the centre. Let $\{e_i\}$ be a basis of $\mathcal{H}$. Then the set of orthogonal projections $\{W_{e_i} W_{e_i}^*\}$ is a resolution of the identity in $\mathcal{H}^\chi$ if and only if $(l, \mathcal{H}^\chi)$ is an isotypic representation.*

*Proof.* From items 2 and 3 above it follows that the set $\{W_{e_i} W_{e_i}^*\}$ is a resolution of the identity of $M(\pi)$ and $(l, M(\pi))$ is an isotypic representation. Hence, $\{W_{e_i} W_{e_i}^*\}$ is a resolution of the identity in $\mathcal{H}^\chi$ if and only if $M(\pi) = \mathcal{H}^\chi$ and, in this case, $(l, \mathcal{H}^\chi)$ is isotypic. Conversely, assume that $(l, \mathcal{H}^\chi)$ is an isotypic representation. Let $(\sigma, V)$ be an irreducible subrepresentation of $(l, \mathcal{H}^\chi)$, then $\sigma$ is square integrable modulo the centre and, by Proposition 1, $V \subset M(\sigma)$. Since $(l, \mathcal{H}^\chi)$ is isotypic and $\pi$ is equivalent to a subrepresentation of $(l, \mathcal{H}^\chi)$, $\sigma$ is equivalent to $\pi$, so that $M(\sigma) = M(\pi)$. Since $\mathcal{H}^\chi$ is direct sum of copies of $(\sigma, V)$, it follows that $\mathcal{H}^\chi = M(\pi)$. □

## 7. The informational completeness

An interesting property of the phase space observables is related to the notion of informational completeness. We say that the operator measure $Q_T$, associated with the state $T$, is informationally complete if the set of operators $\{Q_T(E) \mid E \in \mathcal{B}(X)\}$ separates the set of states, [11, 16]. An extensive study of the conditions assuring the informational completeness is given in [12]. In this section, we prove some results suited to our case. First of all,

**Lemma 1.** *Let $T$ be a state in $\mathcal{H}$ and $Q_T$ the corresponding POM generated by the representation $\pi$. Then the following conditions are equivalent:*

1. *$Q_T$ is informationally complete;*
2. *if $B$ is a trace class operator and $\operatorname{tr}\big[B\pi(g)T\pi(g^{-1})\big] = 0$ for all $g \in G$, then $B = 0$.*

*Proof.* It is known, see for example [11], that $Q_T$ is informationally complete if and only if it separates the set of trace class operators. Let $B$ be a trace class operator, then $\operatorname{tr}\big[Q_T(E)B\big] = 0$ for any $E \in \mathcal{B}(X)$ if and only if $\operatorname{tr}\big[B\pi(s(x))T\pi(s(x)^{-1})\big] = 0$ for $\alpha$-almost all $x \in X$. Observing that $\pi(s(x))T\pi(s(x)^{-1}) = \pi(g)T\pi(g^{-1})$ for all $g \in G$ such that $p(g) = x$, this last condition is equivalent to $\operatorname{tr}\big[B\pi(g)T\pi(g^{-1})\big] = 0$ for $\mu$-almost all $g \in G$. Since the map $g \mapsto \operatorname{tr}\big[B\pi(g)T\pi(g^{-1})\big]$ is continuous, the lemma is proved. □

Let $G_1$ be the commutator subgroup of $G$, *i.e.* the subgroup of $G$ generated by the elements of the form $ghg^{-1}h^{-1}$, where $g, h \in G$, and assume that there is subspace $\mathcal{K}$ of $\mathcal{H}$ such that for all $g \in G_1$ and $v \in \mathcal{K}$, $\pi(g)v = c(g)v$ where $c$ is a character of $G_1$. Then the following result is obtained, compare with Th. 15 of [12].

**Proposition 3.** *If $T$ is a state such that $T\mathcal{H} \subset \mathcal{K}$ and $\operatorname{tr}\big[T\pi(g)\big] \neq 0$ for $\mu$-almost all $g \in G$, then $Q_T$ is informationally complete.*

*Proof.* Let $B$ be a trace class operator, and consider the decompositions of $T$ and $B$ as given in Section 4, *i.e.* $T = \sum_i \lambda_i |e_i\rangle\langle e_i|$ and $B =$

$\sum_k w_k |u_k\rangle\langle v_k|$ . Since $T\mathcal{H} \subset \mathcal{K}$, it follows that $\pi(g)e_i = c(g)e_i$ for all $g \in G_1$. Given $g \in G$, using the orthogonality relations (2), one has

$$\begin{aligned}
\operatorname{tr}\big[T\pi(g)\big]\operatorname{tr}\big[B\pi(g^{-1})\big] &= \sum_{i,k} \lambda_i w_k \langle e_i, \pi(g)e_i\rangle_{\mathcal{H}} \langle \pi(g)v_k, u_k\rangle_{\mathcal{H}} \\
&= \sum_{i,k} \lambda_i w_k \langle c_{\pi(g)e_i,\pi(g)v_k}, c_{e_i,u_k}\rangle_{\mathcal{H}^\chi} \\
&= \sum_{i,k} \lambda_i w_k \int_X \overline{c_{\pi(g)e_i,\pi(g)v_k}(s(x))} c_{e_i,u_k}(s(x))\, d\alpha(x) \\
&= \sum_{i,k} \lambda_i w_k \int_X \langle v_k, \pi(s(x))\pi(s(x)^{-1}g^{-1}s(x)g)e_i\rangle_{\mathcal{H}} \langle \pi(s(x))e_i, u_k\rangle_{\mathcal{H}} d\alpha(x) \\
&= \sum_{i,k} \lambda_i w_k \int_X c(s(x)^{-1}g^{-1}s(x)g) \langle v_k, \pi(s(x))e_i\rangle_{\mathcal{H}} \langle \pi(s(x))e_i, u_k\rangle_{\mathcal{H}} d\alpha(x) \\
&= \int_X c(s(x)^{-1}g^{-1}s(x)g)\operatorname{tr}\big[T\pi(s(x)^{-1})B\pi(s(x))\big]\, d\alpha(x),
\end{aligned}$$

since $\sum_{i,k} \lambda_i w_k \langle v_k, \pi(s(x))e_i\rangle \langle \pi(s(x))e_i, u_k\rangle$ converges in $L^1(X, \alpha)$ to $\operatorname{tr}\big[T\pi(s(x)^{-1})B\pi(s(x))\big]$, as shown in Section 4, and $c$ is bounded. Hence

$$\operatorname{tr}\big[T\pi(g)\big]\operatorname{tr}\big[B\pi(g^{-1})\big] = \int_X c(s(x)^{-1}g^{-1}s(x)g)\operatorname{tr}\big[T\pi(s(x)^{-1})B\pi(s(x))\big]\, d\alpha(x)$$

and, if $\operatorname{tr}\big[B\pi(g)T\pi(g^{-1})\big] = 0$ for all $g \in G$, then $\operatorname{tr}\big[B\pi(g^{-1})\big] = 0$ for $\mu$-almost all $g \in G$. On the other hand, if $\{e_n\}$ is a basis of $\mathcal{H}$,

$$\begin{aligned}
\operatorname{tr}\big[B\pi(g^{-1})\big] &= \sum_{n,p} \langle e_n, Be_p\rangle \langle \pi(g)e_p, e_n\rangle \\
&= \sum_{n,p} \langle e_n, Be_p\rangle (W_{e_p}e_n)(g),
\end{aligned}$$

where the double series converges in $\mathcal{H}^\chi$. Since the set $\{W_{e_p}e_n\}_{n,p}$ is orthonormal in $\mathcal{H}^\chi$, the condition $\operatorname{tr}\big[B\pi(g^{-1})\big] = 0$ for $\mu$-almost all $g \in G$ implies $\langle e_n, Be_p\rangle = 0$ for all $n, p$, *i.e.* $B = 0$ and this proves that $Q_T$ is informationally complete. □

**Remark 2.** The condition that $G_1$ is represented by a character is automatically fulfilled (on the whole $\mathcal{H}$) if $G_1$ is contained in the centre of $G$, whence $\pi|_{G_1} = \chi|_{G_1}$.

**Remark 3.** Suppose $G$ is a Lie group and let $\mathcal{H}^\omega$ be the dense subspace of $\mathcal{H}$ of analytic vectors for $\pi$. If $T$ has range in $\mathcal{H}^\omega$, then the function

$G \ni g \mapsto \operatorname{tr}\big[T\pi(g)\big]$ is analytic. This guarantees that $\operatorname{tr}\big[T\pi(g)\big] \neq 0$ for $\mu$-almost all $g \in G$.

## 8. An example

To discuss an example it is convenient to work with another realization of the induced representation $(l, \mathcal{H}^\chi)$.

Let $J$ be the unitary operator from $\mathcal{H}^\chi$ onto $L^2(X, \alpha)$ given by

$$(Jf)(x) := f(s(x)), \quad x \in X.$$

$J$ intertwines the imprimitivity system $(l, P)$ with $(\tilde{l}, \tilde{P})$, where

$$\begin{aligned}(\tilde{l}(a)f)(x) &= \chi(s(x)^{-1}as(a^{-1}.x))\, f(a^{-1}.x), \quad a \in G,\\ (\widetilde{P}(E)f)(x) &= \chi_E(x)f(x), \quad E \in \mathcal{B}(X),\end{aligned}$$

with $f \in L^2(X, \alpha)$.

Given $u \in \mathcal{H}$, if we compose $W_u : \mathcal{H} \to \mathcal{H}^\chi$ of Section 4 with $J$ we obtain an operator $\widetilde{W}_u : \mathcal{H} \to L^2(X, \alpha)$ explicitly given by

$$(\widetilde{W}_u v)(x) = c_{u,v}(s(x)) = \langle \pi(s(x))u, v\rangle_{\mathcal{H}}.$$

If $u$ is a unit vector, $\widetilde{W}_u$ intertwines the operator measure $Q_u$, defined in Section 4, with the projection measure $\widetilde{P}$, which is the minimal Neumark dilation of $Q_u$.

We denote by $\widetilde{M}(\pi)$ the image of $M(\pi)$ under the map $J$. The analysis of $M(\pi)$, made in Section 6, can easily be translated into an analysis of $\widetilde{M}(\pi)$.

1. $\widetilde{M}(\pi)$ is a closed subspace of $L^2(X, \alpha)$, invariant under $\tilde{l}$.
2. Let $(e_n)_{n\geq 1}$ be a basis of $\mathcal{H}$. Then $\widetilde{M}(\pi) = \oplus_{p\geq 1} \operatorname{ran} \widetilde{W}_{e_p}$.
3. For each $n \geq 1$, $(\tilde{l}_{\operatorname{ran} \widetilde{W}_{e_n}}, \operatorname{ran} \widetilde{W}_{e_n})$ is equivalent to the irreducible unitary representation $(\pi, \mathcal{H})$ of $G$.
4. For each $n, p \geq 1$, we define

   $$f_{n,p}(x) := \widetilde{W}_{e_n} e_p.$$

   For each $n \geq 1$, $(f_{n,p})_{p\geq 1}$ is a basis of $\operatorname{ran} \widetilde{W}_{e_n}$, and $(f_{n,p})_{n,p\geq 1}$ is a basis of $\widetilde{M}(\pi)$.

8.1. **The Heisenberg group.** We denote by $H^1$ the Heisenberg group. It is $\mathbb{R}^3$ as a set and we denote its elements by $(t, q, p)$. The product rule is given by

$$(t_1, q_1, p_1)(t_2, q_2, p_2) = (t_1 + t_2 + \frac{p_1q_2 - q_1p_2}{2}, q_1 + q_2, p_1 + p_2).$$

$H^1$ is a connected, simply connected, unimodular Lie group. Its centre is $Z = \{(t, 0, 0) \,|\, t \in \mathbb{R}\}$, and the quotient space $X = H^1/Z$ can be identified with $\mathbb{R}^2$ (with respect to all relevant structures). For the sake of convenience we choose the Haar measures $\mu, \mu_0$, and $\alpha$ on $G$, $Z$, and $X$, respectively, as $\frac{1}{2\pi} dtdqdp$, $dt$, and $\frac{1}{2\pi} dqdp$. The canonical projection $p : G \to X$ is the coordinate projection $p((t, q, p)) = (q, p)$, and we choose the natural, smooth section $s((q, p)) = (0, q, p)$, $q, p \in \mathbb{R}$. With these choices the integral formula of Section 2, which links together the measures $\mu, \mu_0$, and $\alpha$ reads

$$\int_{\mathbb{R}^3} f(t, q, p) \frac{dtdqdp}{2\pi} = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} f((0, q, p)(t, 0, 0))\, dt \right) \frac{dqdp}{2\pi},$$

for all $f \in C_c(\mathbb{R}^3)$, and is simply a consequence of Fubini's theorem.

Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space, and let $(e_n)_{n \geq 1}$ be an orthonormal basis of $\mathcal{H}$. There is a natural action of $H^1$ on $\mathcal{H}$. Let $a, a^*$ denote the ladder operators associated with the basis $(e_n)_{n \geq 1}$, and define

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}}(a + a^*) \\ P &= \frac{1}{\sqrt{2}i}(a - a^*) \end{aligned}$$

on their natural domains. Then

$$(t, p, q) \mapsto e^{i(t + qP + pQ)}$$

is a *unitary irreducible* representation of $H^1$ on $\mathcal{H}$. It is the *only* unitary irreducible representation of $H^1$ whose central character is $t \mapsto e^{it}$, see for instance [17]. It is *unitarily equivalent* to the representation of $H^1$ which acts on $L^2(\mathbb{R})$ as

$$(\pi(t, q, p)\phi)(x) = e^{i(t + px + qp/2)} \phi(x + q), \quad \phi \in L^2(\mathbb{R}).$$

We show that $(\pi, L^2(\mathbb{R}))$ is a representation of $H^1$ that is square integrable modulo the centre $Z$. According to item 1 of section 3, it suffice to show that $c_{\phi,\phi} \circ s \in L^2(\mathbb{R}^2)$ for some $\phi \in L^2(\mathbb{R})$. Explicitly

$$c_{\phi,\phi}(s(q, p)) = \langle \pi(s(q, p))\phi, \phi \rangle = e^{-i\frac{pq}{2}} \int e^{-ipx} \overline{\phi(x + q)} \phi(x)\, dx.$$

Choose $\phi \in C_c(\mathbb{R})$, then, for any $q \in \mathbb{R}$

$$\left( x \mapsto \overline{\phi(x + q)} \phi(x) \right) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Properties of the Fourier transform tell us that

$$\left(p \mapsto \int_{\mathbb{R}} e^{-ipx}\overline{\phi(x+q)}\phi(x)\,dx\right) \in L^2(\mathbb{R}), \quad q \in \mathbb{R}.$$

Thus we have, by the Plancherel theorem,

$$\int_{\mathbb{R}} \left| e^{-i\frac{pq}{2}} \int_{\mathbb{R}} e^{-ipx}\overline{\phi(x+q)}\phi(x)\,dx \right|^2 dp = 2\pi \int_{\mathbb{R}} \left|\overline{\phi(x+q)}\phi(x)\right|^2 dx,$$

and, by the Fubini theorem,

$$\int_{\mathbb{R}} \left(2\pi \int_{\mathbb{R}} |\overline{\phi(x+q)}\phi(x)|^2 dx\right) dq = 2\pi \left\|\phi\right\|^4_{L^2(\mathbb{R})}.$$

Tonelli's theorem tells us that the function $c_{\phi,\phi} \circ s$ is in $L^2(\mathbb{R}^2)$. Moreover, recalling that $d\alpha = \frac{dq\,dp}{2\pi}$,

$$\left\|c_{\phi,\phi} \circ s\right\|_{L^2(\mathbb{R}^2,\alpha)} = \left\|\phi\right\|^2_{L^2(\mathbb{R})}.$$

This shows that $\pi$ is square integrable modulo the centre and that its formal degree is 1. Since $\pi$ is the only irreducible representation of $H^1$ with the central character $e^{it}$ and it is square integrable modulo the centre we conclude that

$$\widetilde{M}(\pi) = L^2(\mathbb{R}^2, \alpha),$$

namely, that $(\tilde{l}, L^2(\mathbb{R}^2, \alpha))$ is an isotypic representation. To exhibit this representation, let us observe that the map $\widetilde{W}_u : \mathcal{H} \to L^2(\mathbb{R}^2, \alpha)$ is given by

$$(\widetilde{W}_u v)(x, y) = \langle e^{i(xQ+yP)}u, v\rangle_{\mathcal{H}}.$$

The functions $f_{n,p}$, $p \geq 1$, which constitute a basis of $\operatorname{ran} \widetilde{W}_{e_n}$, are

$$f_{n,p}(x, y) = \sqrt{2\pi}\langle e^{i(xQ+yP)}e_n, e_p\rangle_{\mathcal{H}}.$$

The operator measure $Q_u$ is given by

$$\langle v, Q_u(E)w\rangle = \frac{1}{2\pi}\int_E \langle v, \pi(0, q, p)u\rangle_{\mathcal{H}}\langle u, \pi(0, q, p)^{-1}w\rangle_{\mathcal{H}} dqdp,$$

which can be written as

$$Q_u(E) = \frac{1}{\pi}\int_E D_z|u\rangle\langle u|D_z^{-1} d\lambda(z),$$

where $z = \frac{-q+ip}{\sqrt{2}}$, $D_z = e^{it+za^*-\overline{z}a}$, and $\lambda$ is the Lebesgue measure on $\mathbb{C}$. The action of $\tilde{l}$ on $L^2(\mathbb{R}^2, \alpha)$ can directly be computed and we get

$$(\tilde{l}(t, q, p)\tilde{f})(x, y) = e^{i(t+\frac{xp-yq}{2})}\tilde{f}(x - q, y - p).$$

As a final remark we note that the commutator group of the Heisenberg group is contained in its center so that if $T$ is a state such that

$\mathrm{tr}\big[T\pi(g)\big] \neq 0$ for almost all $g \in H^1$, then by Proposition 3 the operator measure $Q_T$ is informationally complete. This holds, in particular, if the range of $T$ is contained in the subspace of $\mathcal{H}$ of analytic vectors.

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