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SQUARE-INTEGRABILITY MODULO A SUBGROUP

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ABSTRACT. For locally compact groups we prove a weak form of Frobenius's reciprocity theorem. As a consequence, we propose a definition of square-integrable representation modulo a subgroup that clarifies the relation between coherent states, wavelet transforms and covariant localisation observables. In particular a self-contained proof of Imprimitivity Theorem for covariant positive operator valued measures is given.

1. INTRODUCTION

Given a locally compact group G, a closed subgroup H and a unitary representation σ of H, we prove the existence of an isomorphism between the commuting ring of the representation of G unitarily induced by σ and the space of *admissible* linear maps commuting with respect to the action of H (see Definition 1 below). This result can be viewed as a weak form of Frobenius's reciprocity theorem for locally compact groups, compare with [22]. In particular, it allows to characterise sub-representations of the induced representation in terms of a condition of square-integrability on the quotient space G/H. This equivalence gives a natural definition of square-integrable representation modulo a subgroup that unifies many notions used in literature in the context both of coherent states and of wavelet analysis, for a review see [2]. Finally, we show that, given a unitary representation U of G, there exists a positive operator valued (POV) measures based on G/H and covariant with respect to U if and only if U is square-integrable modulo H, compare with [17] and [32].

This last result is a consequence of Imprimitivity Theorem for covariant POV measures based on G/H. We give a self-contained proof of this theorem for non-normalised POV measures using the notion of Gärding domain, [25]. For projective valued measures, the theorem was proven by Mackey, [21], for separable groups, and by Blattner, [4], in full generality, and it is known as Mackey's imprimitivity theorem. For normalised POV measures, there are many independent proofs. Up to our knowledge, Poulsen, [27], first proves it for Lie groups using elliptic regularity theory, Davies, [9], and Scutaru, [32],

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for topological groups, but with some unnecessary assumption, Neumann, [24], and Cattaneo, [7], for locally compact groups. These last proofs are based on Neumark's dilation theorem, [29], in order to reduce the problem to the projective case, and, hence, on Mackey's imprimitivity theorem. Finally, Castrigiano and Henrichs, [8], show the above result using the theory of positive functions on a C^*-algebra.

Our proof is independent both on Neumark's theorem and on Mackey's theorem, which are corollaries of our result. It is based on the proof of Mackey's imprimitivity theorem given by Orsted, [25], as suggested by a remark in [8] (compare also with [11, Ch. XXII, Sec. 3, Ex. 10]). In particular, we use a realisation of the induced representation inspired by an exercise of [11, Ch. XXII, Sec. 3, Ex. 10]. Our construction is a variation of the one given by Blattner, [4], and, in our opinion, is very elementary and *intrinsic*, it does not use the notion of quasi-invariant measure and the Hilbert space where the representation acts is a space of square-integrable functions, compare with Folland, [13, Ch. 6].

The paper is organised in the following way. In Sec. 2 we introduce the notation and we give our construction of the induced representation. In Sec. 3, we prove the weak Frobenius's reciprocity theorem and we propose a definition of square-integrable representation modulo a subgroup. In Sec. 4 we give the proof of Imprimitivity Theorem for non-normalised transitive POV measures and we characterise them in terms of square-integrable representations modulo a subgroup.

To avoid technical problems with integration theory, we assume that groups and Hilbert spaces are separable, but the results hold without this hypothesis.

2. Notations

In this paper, G is a locally compact second countable topological group and H a closed subgroup of G. We denote by μ_G and μ_H left Haar measures on G and H, respectively. Let Δ_G and Δ_H be the corresponding modular functions.

Let $X = G/H$ be the quotient space of the left cosets with the natural topology and $p : G \to X$ the canonical projection, which is an open map. For all $g \in G$, we denote by $x \mapsto g[x]$ the action of G on X. If f is a function on G and $g \in G$, we let f^g be the map given by $(f^g)(g') = f(g^{-1}g')$, for all $g' \in G$.

Given a locally compact second countable topological space Y, by *Radon measure* on Y, we mean a positive measure defined on the σ-algebra $\mathcal{B}(Y)$ of Borel subsets of Y such that it is finite on compact sets. Since the space is second countable, Radon measures are both outer and inner regular. In particular Haar measures are Radon. We denote by $C_c(Y)$ the space of continuous complex functions on Y with compact support and by $\operatorname{supp} f$ the support of a continuous function f.

We recall the following relation between $\mathrm{C_c}(G)$ and $\mathrm{C_c}(X)$, due to Weil (for the proof see, for example, Prop. 2.48 of [13]).

Lemma 1. *Let $f \in \mathrm{C_c}(G)$ and K its support. There is a unique $\tilde{f} \in \mathrm{C_c}(X)$ such that, for all $g \in G$,*

$$\tilde{f}(p(g)) = \int_H f(gh)d\mu_H(h).$$

Moreover

$$\sup_{x\in X} |\tilde{f}(x)| \leq C_K \sup_{g\in G} |f(g)|, \tag{1}$$

where C_K is a constant depending only on the support of f. Finally, if $f' \in \mathrm{C_c}(X)$ [positive], there is $f \in \mathrm{C_c}(G)$ [positive] such that $\tilde{f} = f'$ and $p(\operatorname{supp} f) = \operatorname{supp} f'$.

By *Hilbert space*, we mean a complex separable Hilbert space, being $\langle\cdot,\cdot\rangle$ the scalar product, linear in the first variable, and $\|\cdot\|$ the corresponding norm. If A is a (bounded) operator, we denote by $\|A\|$ also the norm of A. By a representation of G, we mean a continuous (with respect to the strong operator topology) unitary representation of G acting in a Hilbert space. Given a representation π acting in $\mathcal{H}$, for all $f \in \mathrm{C_c}(G)$, we let

$$\pi(f) = \int_G f(g)\pi(g)d\mu_G(g),$$

where the integral is in the strong operator topology. In particular, one has that, for all $g \in G$ and $f \in \mathrm{C_c}(G)$,

$$\pi(g)\pi(f) = \pi(f^g). \tag{2}$$

We denote by $\mathcal{D}_\pi$ the Gärding domain of π, *i.e.*,

$$\mathcal{D}_\pi = \operatorname{span}\{\pi(f)u \mid f \in \mathrm{C_c}(G),\ u \in \mathcal{H}\},$$

One has the following properties.

Lemma 2. *With the above notations, the Gärding domain of π is a G-invariant dense subspace of $\mathcal{H}$. If π' is another representation of G acting in $\mathcal{H}'$ and W is an operator from $\mathcal{H}$ to $\mathcal{H}'$ intertwining π and π', then*

$$W\mathcal{D}_\pi \subset \mathcal{D}_{\pi'}.$$

Proof. Let $g \in G$, $f \in \mathrm{C_c}(G)$ and $v \in \mathcal{H}$. By Eq. (2), $\pi(g)\pi(f)v = \pi(f^g)v$ and, since $f^g \in \mathrm{C_c}(G)$, it follows that $\mathcal{D}_\pi$ is G-invariant. To show the density, given $v \in \mathcal{H}$ and $\epsilon > 0$, since π is continuous with respect to the strong operator topology, there is a compact neighbourhood K of the identity such that, for all $g \in K$, $\|\pi(g)u - u\| \leq \epsilon$. Since K contains a non-void open set, $\mu_G(K) > 0$ and, by outer regularity of μ_G, there is an open set $V \supset K$ with $\mu_G(V\backslash K) \leq \epsilon\mu_G(K)$. Let $f \in \mathrm{C_c}(G)$ such that $f(g) = 1$ for all $g \in K$,

$0 \le f(g) \le 1$ for all $g \in G$ and $\mathrm{supp} f \subset V$. Then, defined $a = \frac{1}{\mu_G(K)}$,

$$\begin{aligned}
\|\pi(af)v - v\| &= \left\| a\int_K (\pi(g)v - v)d\mu_G(g) + a\int_{V\setminus K} f(g)\pi(g)v d\mu_G(g) \right\| \\
&\le a\int_K \|\pi(g)v - v\| d\mu_G(g) + a\int_{V\setminus K} f(g)\|\pi(g)v\| d\mu_G(g) \\
&\le \epsilon(1 + \|v\|).
\end{aligned}$$

To show the second point, let $f \in \mathrm{C_c}(G)$ and $v \in \mathcal{H}$, then

$$\begin{aligned}
W\pi(f)v &= W\int_G f(g)\pi(g)v d\mu_G(g) = \\
\int_G f(g)W\pi(g)v d\mu_G(g) &= \int_G f(g)\pi'(g)Wv d\mu_G(g) = \pi'(f)Wv.
\end{aligned}$$

□

If G is a Lie group,usually Gärding domain is defined replacing $\mathrm{C_c}(G)$ with $C_c^\infty(G)$, see [33]. We adopted the definition of [25].

We now give a realisation of the induced representation, based on the following lemma.

Lemma 3. *There is a continuous function $\theta : G \to [0, +\infty[$ such that, for all $g \in G$,*

$$\int_H \theta(gh)d\mu_H(h) = 1,$$

and, for any compact subset K of G, $KH \cap \mathrm{supp}\,\theta$ is compact.

Moreover, let $Y \in \mathcal{B}(G)$ such that, for all $h \in H$, $\mu_G(Yh\setminus Y) = 0$. Then Y is negligible with respect to $\theta\mu_G$ if and only if negligible with respect to μ_G, where $\theta\mu_G$ is the measure having density θ with respect to μ_G.

Proof. The existence of θ is proven, for example, in Prop. 2 of [16]. With respect to second part, if $\mu_G(Y) = 0$, then $\int_Y \theta(g)d\mu_G(g) = 0$. Conversely,

$$\begin{aligned}
\int_Y \theta(g)d\mu_G(g) &= \int_Y \theta(g)\int_H \theta(gh)d\mu_H(h)d\mu_G(g) \\
&= \int_H \int_Y \theta(gh)d\mu_G(g)d\mu_H(h) \\
(g \mapsto gh^{-1}) &= \int_H \Delta_G(h^{-1})\int_{Yh} \theta(g)d\mu_G(g)d\mu_H(h) \\
&= \int_H \Delta_G(h^{-1})\int_Y \theta(g)d\mu_G(g)d\mu_H(h) \\
&= 0,
\end{aligned}$$

where we used that $\mu_G(Yh\setminus Y) = 0$. □

Let σ be a representation of H acting in $\mathcal{K}$. Given θ as in Lemma 3, let $\mathcal{F}^\sigma$ be the subspace of functions F from G to $\mathcal{K}$ such that

- F is μ_G-measurable;
- given $h \in H$, for μ_G-almost all $g \in G$,

$$\sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\sigma(h^{-1})F(g) = F(gh); \tag{3}$$

- $\int_G \|F(g)\|^2 \theta(g) d\mu_G(g) < +\infty$.

We notice that, due to Lemma 3, a function F satisfying Eq. (3) is μ_G-measurable if and only if it is $\theta\mu_G$-measurable, so $F \in \mathcal{F}^\sigma$ if and only if $F \in L^2(G, \theta\mu_G, K)$ and Eq. (3) holds.

Given $v \in \mathcal{K}$ and $f \in \mathrm{C_c}(G)$, let $F_{f,v}$ be the function from G to $\mathcal{K}$ defined, for all $g \in G$, as

$$(F_{f,v})(g) = \int_H \sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}} f(gh)\sigma(h)v \; d\mu_H(h).$$

Lemma 4. *With the above notations, the space $\mathcal{F}^\sigma$ is a closed subspace of $L^2(G, \theta\mu_G, K)$, which does not depend on the choice of θ, and each $F \in \mathcal{F}^\sigma$ is locally μ_G-integrable. For each $f \in \mathrm{C_c}(G)$ and $v \in \mathcal{K}$, $F_{f,v}$ is in $\mathcal{F}^\sigma$, it is continuous and* $\operatorname{supp} F_{f,v} \subset (\operatorname{supp} f)H$. *Finally, the space generated by the elements of the form $F_{f,v}$ is dense in $\mathcal{F}^\sigma$.*

Proof. We claim that $\mathcal{F}^\sigma$ is closed. Indeed, let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}^\sigma$. Since $L^2(G, \theta\mu_G, K)$ is a Hilbert space, $(F_n)_{n\in\mathbb{N}}$ converges in L^2 and, possibly passing to a subsequence, $\theta\mu_G$-almost everywhere. Let Y be the complement of the set of elements $g \in G$ such that $(F_n(g))_{n\in\mathbb{N}}$ converges pointwise and denote by $F(g)$ the limit. By hypothesis, Y is $\theta\mu_G$-negligible and, by unicity of the limit, $(F_n)_{n\in\mathbb{N}}$ converges to F in L^2. Let now $h \in H$, by definition of $\mathcal{F}^\sigma$ and the fact that $(F_n)_n$ is denumerable, it exists $Y_h \in \mathcal{B}(G)$ such that $\mu_G(Y_h) = 0$ and, for all $g \in G \backslash Y_h$ and $n \in \mathbb{N}$

$$\sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\sigma(h^{-1})F_n(g) = F_n(gh).$$

If $g \notin Y \cup Y_h$, passing to the limit, one has that $(F_n(gh))_{n\in\mathbb{N}}$ converges and

$$\sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\sigma(h^{-1})F(g) = F(gh). \tag{4}$$

In particular, $gh \notin Y$, that is $Yh^{-1} \subset Y \cup Y_h$. Since $\mu_G(Y_h) = 0$, it follows that $\mu_G(Y_h \backslash Y) = 0$. By Lemma 3, it follows that $\mu_G(Y) = 0$ and, hence, $\mu_G(Y \cup Y_h) = 0$. So Eq. (4) holds μ_G-almost everywhere, that is $F \in \mathcal{F}^\sigma$.

We now prove that $\mathcal{F}^\sigma$ is independent on θ. Let θ' a non-negative continuous function such that $\int_H \theta'(gh) d\mu_H(h) = 1$ for all $g \in G$. Let F from

G to K μ_G-measurable and such that Eq. (3) holds. Then

$$\begin{aligned}
\int_G \|F(g)\|^2 \theta(g) d\mu_G(g) &= \int_G \|F(g)\|^2 \theta(g) \int_H \theta'(gh) d\mu_H(h) d\mu_G(g) \\
&= \int_H \int_G \|F(g)\|^2 \theta(g)\theta'(gh) d\mu_G(g) d\mu_H(h) \\
(g \mapsto gh^{-1}) \quad &= \int_H \int_G \left\|F(gh^{-1})\right\|^2 \theta(gh^{-1})\theta'(g)\Delta_G(h^{-1}) d\mu_G(g) d\mu_H(h) \\
(F \in \mathcal{F}^\sigma, h \mapsto h^{-1}) \quad &= \int_H \int_G \|F(g)\|^2 \theta(gh)\theta'(g) d\mu_G(g) d\mu_H(h) \\
&= \int_G \|F(g)\|^2 \theta'(g) \int_H \theta(gh) d\mu_H(h) d\mu_G(g) \\
&= \int_G \|F(g)\|^2 \theta'(g) d\mu_G(g).
\end{aligned}$$

This shows the claim.

Let now $F \in \mathcal{F}^\sigma$, we prove that F is locally μ_G-integrable. Let $f \in \mathrm{C_c}(G)$ non-negative, then, as before,

$$\begin{aligned}
\int_G \|F(g)\| f(g) d\mu_G(g) &= \int_G \|F(g)\| f(g) \int_H \theta(gh) d\mu_H(h) d\mu_G(g) \\
&= \int_G \|F(g)\| \theta(g) \int_H \sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}} f(gh) d\mu_H(h) d\mu_G(g) \\
&= \int_G \|F(g)\| f'(g)\theta(g) d\mu_G(g)
\end{aligned}$$

where $f'(g) := \int_H \sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}} f(gh) d\mu_H(h)$ is continuous and $\operatorname{supp} f' \subset (\operatorname{supp} f)H$. By Lemma 3, $f'\theta \in L^2(G, \mu_G)$, so that $\|F(g)\| f'(g)\theta(g)$ is μ_G-integrable, hence F is locally μ_G-integrable.

The properties of $F_{f,v} \in \mathcal{F}^\sigma$ are clear (use the proof of Lemma 1). We show the density. Let $F \in \mathcal{F}^\sigma$ such that, for all $f \in \mathrm{C_c}(G)$ and $v \in \mathcal{K}$, $\langle F, F_{f,v}\rangle = 0$. Then, using the same argument as before and Tonelli theorem, one can check that the map

$$(g,h) \mapsto \langle F(g), \sigma(h)v\rangle \, \overline{f(gh)}\theta(g)\sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}}$$

is $\mu_G \otimes \mu_H$-integrable and

$$\begin{aligned}
0 &= \langle F, F_{f,v}\rangle \\
&= \int_G \int_H \langle F(g), \sigma(h)v\rangle\, \overline{f(gh)}\theta(g)\sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}}d\mu_H(h)d\mu_G(g) \\
(g\mapsto gh^{-1}, h\mapsto h^{-1})) &= \int_G \langle F(g), v\rangle\, \overline{f(g)} \int_H \theta(gh)d\mu_H(h)d\mu_G(g) \\
&= \int_G \langle F(g), v\rangle\, \overline{f(g)}d\mu_G(g).
\end{aligned}$$

By standard arguments, one has that $F(g) = 0$ μ_G-almost all $g \in G$, that is $F = 0$. □

Define, for all $g \in G$ and $F \in \mathcal{F}^\sigma$,

$$(L^\sigma_g F)(g') = F(g^{-1}g') \qquad g' \in G,\ \mu_G\text{-a.e. .}$$

One has the following result.

Proposition 1. *Let σ be a representation of H, then L^σ is a representation of G acting in $\mathcal{F}^\sigma$ and is a realisation of the representation induced by σ from H to G. In particular, the Gärding domain of L^σ is a subspace of continuous functions.*

Proof. Given $g \in G$, we prove that L^σ_g is a well-defined isometric operator in $\mathcal{F}^\sigma$. Let $F \in \mathcal{F}^\sigma$, then, for all $h \in H$ and for μ_G-almost all $g' \in G$

$$(L^\sigma_g F)(g'h) = F(g^{-1}g'h) = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\sigma(h^{-1})F(g^{-1}g') = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\sigma(h^{-1})(L^\sigma_g F)(g').$$

Moreover,

$$\begin{aligned}
\int_G \left\|F(g^{-1}g')\right\|^2 \theta(g')d\mu_G(g') &= \int_G \left\|F(g')\right\|^2 \theta(gg')d\mu_G(g') \\
&= \int_G \left\|F(g')\right\|^2 \theta(gg') \int_H \theta(g'h)d\mu_H(h)d\mu_G(g') \\
&= \int_H \int_G \left\|F(g')\right\|^2 \theta(gg')\theta(g'h)d\mu_G(g')d\mu_H(h) \\
(g'\mapsto g'h^{-1}) &= \int_H \int_G \Delta_G(h^{-1}) \left\|F(g')\right\|^2 \theta(gg'h^{-1})\theta(g')d\mu_G(g')d\mu_H(h) \\
(h\mapsto h^{-1}) &= \int_G \left\|F(g')\right\|^2 \theta(g') \int_H \theta(gg'h)d\mu_H(h)d\mu_G(g') \\
&= \int_G \left\|F(g')\right\|^2 \theta(g')d\mu_G(g')
\end{aligned}$$

This proves that L^σ_g is a well defined isometric operator in $\mathcal{F}^\sigma$.

In order to show that $g \mapsto L^\sigma_g$ is continuous, since L^σ_g is isometric and by a density argument, one can reduced to prove that, given $f \in \mathrm{C_c}(G)$, $v \in \mathcal{K}$

and $F' \in \mathcal{F}^\sigma$, the map

$$g \mapsto \left\langle L^\sigma_g F_{f,v}, F' \right\rangle = \int_G \left\langle F_{f,v}(g^{-1}g'), F'(g) \right\rangle \theta(g) d\mu_G(g)$$

is continuous. However, due to Lemma 4, $F_{f,v}$ is continuous and, due to Lemma 3, $\operatorname{supp} F_{f,v} \cap \operatorname{supp} \theta$ is compact, so that the thesis follows by dominated convergence theorem.

We prove that $\mathcal{D}^\sigma_L$ is a subspace of continuous functions. Indeed, let $f \in \mathrm{C_c}(G)$ and $F \in \mathcal{F}^\sigma$. Given $F' \in \mathcal{F}^\sigma$, observe that the function on $G \times G$

$$\Psi(g, g') = f(g) \left\langle F(g^{-1}g'), F'(g') \right\rangle \theta(g')$$

is $\mu_G \otimes \mu_G$-measurable and

$$\begin{aligned}
\int_G \int_G |f(g) \left\langle F(g^{-1}g'), F'(g') \right\rangle \theta(g')| d\mu_G(g') d\mu_G(g) &\leq \\
\int_G \int_G |f(g)| \left\| F(g^{-1}g') \right\| \left\| F'(g') \right\| \theta(g') d\mu_G(g') d\mu_G(g) &\leq \\
\int_G |f(g)| \left(\int_G \left\| F(g^{-1}g') \right\|^2 \theta(g') d\mu_G(g') \right)^{\frac{1}{2}} \left(\int_G \left\| F'(g') \right\|^2 \theta(g') d\mu_G(g') \right)^{\frac{1}{2}} d\mu_G(g) &= \\
\int_G |f(g)| \left\| L^\sigma_g F \right\|_{\mathcal{F}^\sigma} \left\| F' \right\|_{\mathcal{F}^\sigma} d\mu_G(g) &= \\
\int_G |f(g)| \left\| F \right\|_{\mathcal{F}^\sigma} \left\| F' \right\|_{\mathcal{F}^\sigma} d\mu_G(g) &.
\end{aligned}$$

Since f has compact support, the above integral is finite and, by Tonelli theorem, Ψ is integrable with respect to $\mu_G \otimes \mu_G$. Then

$$\begin{aligned}
\left\langle L^\sigma(f) F, F' \right\rangle &= \int_G f(g) \int_G \left\langle F(g^{-1}g'), F'(g') \right\rangle \theta(g') d\mu_G(g') d\mu_G(g) \\
&= \int_G \int_G \langle f(g) F(g^{-1}g'), F'(g') \rangle d\mu_G(g) \theta(g') d\mu_G(g') \\
(g \mapsto g'g, g \mapsto g^{-1}) \quad &= \int_G \int_G \langle \Delta_G(g^{-1}) f(g'g^{-1}) F(g), F'(g') \rangle d\mu_G(g) \theta(g') d\mu_G(g') \\
&= \int_G \left\langle (f \star F)(g'), F'(g') \right\rangle \theta(g') d\mu_G(g'),
\end{aligned}$$

where $(f \star F)(g') = \int_G \Delta_G(g^{-1}) f(g'g^{-1}) F(g) d\mu_G(g)$, which is well defined since F is locally μ_G-integrable. Then, one has that $L^\sigma(f)F = f \star F$. The continuity of $f \star F$ is now consequence of the dominated convergence theorem. □

3. A weak form of Frobenius's reciprocity theorem

The following definition is a possible extension of the notion of admissible vector for square-integrable representations. We fix a function θ as given by Lemma 3.

Definition 1. Let σ be a representation of H acting in $\mathcal{K}$ and π a representation of G acting in $\mathcal{H}$. A linear map $A : \mathcal{D}_\pi \to \mathcal{K}$ such that

- for all $h \in H$ and $v \in \mathcal{D}_\pi$,

$$\sigma(h)Av = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} A\pi(h)v; \tag{5}$$

- for all $v \in \mathcal{D}_\pi$, the map from G to $\mathcal{K}$

$$g \mapsto A\pi(g^{-1})v := (W_A v)(g)$$

  is μ_G-measurable and

$$\int_G \left\| A\pi(g^{-1})v \right\|^2 \theta(g) d\mu_G(g) \leq \beta \left\| v \right\|^2, \tag{6}$$

  where β is a positive constant independent on v,

is called *admissible map for π modulo (H, σ).*

The admissible maps modulo (H, σ) give a characterisation of the commuting ring of the representation induced by σ, compare with the results obtained by Moore, [22].

Theorem 1. *Let σ be a representation of H acting in $\mathcal{K}$ and π a representation of G acting in $\mathcal{H}$. Let $A : \mathcal{D}_\pi \to \mathcal{K}$ be an admissible map for π modulo (H, σ), then*

- *for all $v \in \mathcal{D}_\pi$, $W_A v \in \mathcal{D}_{L^\sigma} \subset \mathcal{F}^\sigma$ (in particular $W_A v$ is a continuous function);*
- *the linear map $v \mapsto W_A v$ extends to a unique bounded operator W_A, called* wavelet transform, *from $\mathcal{H}$ to $\mathcal{F}^\sigma$ that intertwines π and $\mathcal{F}^\sigma$.*

Conversely, given a bounded operator $W : \mathcal{H} \to \mathcal{F}^\sigma$ intertwining π with $\mathcal{F}^\sigma$, there is a unique admissible map A (for π modulo (H, σ)) such that, for all $v \in \mathcal{D}_\pi$, $Wv = W_A v$.

Proof. Let A be an admissible map. Given $v \in \mathcal{D}_\pi$ and $h \in H$,

$$\begin{aligned} (W_A v)(gh) &= A\pi(h^{-1}g^{-1})v = \\ \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} \sigma(h^{-1}) A\pi(g^{-1})v &= \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} \sigma(h^{-1})(W_A v)(g), \end{aligned}$$

for all $g \in G$. Due to Eq. (6), one has that $W_A v \in \mathcal{F}^\sigma$ and $\|W_A v\| \leq \sqrt{\beta} \|v\|$. Since $\mathcal{D}_\pi$ is dense in $\mathcal{H}$, $v \mapsto W_A v$ extends to a unique bounded operator W_A. Moreover, if now $g' \in G$

$$(W_A v)(g'^{-1}g) = A\pi(g^{-1}g')v = (W_A \pi(g')v)(g)$$

for all $g \in G$, so that W_A intertwines π and L^σ. In particular, due to Lemma 2, $W_A \mathcal{D}_\pi \subset \mathcal{D}_{L^\sigma}$, and the elements of $\mathcal{D}_{L^\sigma}$ are continuous functions by Prop. 1.

Conversely, let W be bounded operator from $\mathcal{H}$ to $\mathcal{F}^\sigma$ intertwining π and $\mathcal{F}^\sigma$. By Lemma 2 and Prop. 1, for all $v \in \mathcal{D}_\pi$, Wv is a continuous function and we can define A from $\mathcal{D}_\pi$ to $\mathcal{K}$ as $Av = (Wv)(e)$, where e is the identity of G. Given $h \in H$ and $v \in \mathcal{D}_\pi$,

$$A\pi_h v = (W\pi_h v)(e) = (L^\sigma_h Wv)(e) = (Wv)(h^{-1}) = \sqrt{\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}}\sigma(h)(Wv)(e),$$

so that Eq. (5) holds. Moreover, if $g \in G$

$$A\pi_{g^{-1}}v = (W\pi_{g^{-1}}v)(e) = (L^\sigma_{g^{-1}}Wv)(e) = (Wv)(g),$$

so that $W_A = W$ on $\mathcal{D}_\pi$ and Eq. (6) is satisfied with $\beta = \|W\|^2$. Let B an other admissible map such that, for all $v \in \mathcal{D}_\pi$

$$B\pi_{g^{-1}}v = (Wv)(g) \qquad g \in G \ \theta\mu_G\text{-a.e..}$$

Since both side satisfy Eq. (3), by Lemma 3, the equality holds μ_G-almost everywhere and, by continuity, everywhere. With the choice $g = e$, one has $Bv = (Wv)(e) = A$. □

We add some comments. If a linear map A satisfies Eq. (5) and is closable, its closure is semi-invariant with weight $h \mapsto \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}$ in the sense of [12] and the measurability of $W_A v$ follows from the continuity of π. However, as shown by Example 1 below, there are admissible maps that are not closable. If A is closable and π is irreducible, the condition (6) is equivalent to the existence of $v \in \mathcal{D}_\pi$ such that

$$0 < \int_G \left\|A\pi(g^{-1})v\right\|^2 \theta(g)d\mu_G(g) < +\infty,$$

(use the first part of the proof of Th. 3 of [12]).

If X admits an invariant measure (in particular if both G and H are unimodular), Eq. (5) is the requirement that A is an (algebraic) intertwining map between $\pi_{|H}$ and σ.

Assume now that G is compact and π irreducible. Since $\mathcal{H}$ is finite dimensional, $\mathcal{D}_\pi = \mathcal{H}$ and, taking into account that both G and H are unimodular, Eq. (5) is the condition that A is an intertwining operator between $\pi_{|H}$ and σ. Finally, since the measure μ_G is bounded, Eq. (6) is trivially satisfied. Then, the space of admissible maps is precisely the set of intertwining operators between $\pi_{|H}$ and σ and Theorem 1 reduces to Frobenius's reciprocity theorem for compact groups, due to Weil, see, for example, [13].

In case that G is not compact, the following example shows that it is restrictive to assume that admissible maps are closable.

Example 1. In Theorem 1, let $\pi = L^\sigma$ and choose $W = I$. A simple computation shows that the admissible map A such that $W_A = I$ is, for all $F \in \mathcal{D}_{L^\sigma}$,

$$AF = F(e),$$

which is clearly not closable (if G is not discrete).

In particular, let G be the Poincaré group $\mathbb{R}^4 \times' SO(3,1)$, $H = \mathbb{R}^4 \times SO(3)$ and σ the trivial representation of H. It is well known that G/H has an invariant measure and L^σ is irreducible, so that the multiples of identity are the only intertwining operators. Due to the previous observation, in this example there are neither bounded nor closable admissible maps.

Due to the fact that, in general, Gärding domains do not have a natural topology such that Eq. (6) is equivalent to the continuity of the admissible maps, our result is not good enough to give a *useful* characterisation of the set of intertwining operators between π and L^σ, compare with so-called "intertwining number theorems", see, for an exposition, [33], and the results contained in [28] for Lie groups. However, Theorem 1 allows to characterise completely the representations that are equivalent to a sub-representation of the induced one.

Corollary 1. *With the notations of Theorem* 1*, the following conditions are equivalent:*

- *the representation π is equivalent to a sub-representation of L^σ;*
- *there is an admissible map A_0 such that, for all $v \in \mathcal{D}_\pi$,*

$$\int_G \left\| A\pi(g^{-1})v \right\|^2 \theta(g) d\mu_G(g) = \|v\|^2 . \tag{7}$$

- *there is an admissible map A such that, for all $v \in \mathcal{D}_\pi$,*

$$\alpha \|v\|^2 \leq \int_G \left\| A\pi(g^{-1})v \right\|^2 \theta(g) d\mu_G(g) \leq \beta \|v\|^2 , \tag{8}$$

  where $0 < \alpha \leq \beta$;

If any of the above three conditions is satisfied, we said to π is square-integrable modulo *(H, σ).*

Proof. Assume first condition, then there is an isometry W intertwining π and L^σ. Applying Theorem 1 to W, there exists an admissible map A such that $W = W_A$ and, since W is isometric, Eq. (7) holds. Clearly Eq. (7) implies Eq. (8). Assume now the third condition, the corresponding wavelet operator W_A satisfies, for all $v \in \mathcal{H}$

$$\sqrt{\alpha} \|v\| \leq \|W_A v\| \leq \sqrt{\beta} \|v\| .$$

In particular, W_A is injective, so that, by polar decomposition, there is an isometry W_0 such that $W_A = W_0|W_A|$. Since W_A commutes with the action of G, W_0 intertwines π and L^σ. □

In the framework of wavelet analysis, Eq. (8) says that $\{A\pi(g^{-1})\}_{g\in G}$ is a (vector valued) frame in $\mathcal{H}$ and Eq. (7) that this frame is tight. So one can restate Corollary 1 in the following way. A representation π is square-integrable modulo H if and only if the set $\{A\pi(g^{-1})\}_{g\in G}$ is a frame for some admissible map A, and A can always be chosen in such a way that the corresponding frame is tight.

Example 2. Assume that π is irreducible and let $H = \{e\}$ being σ the trivial representation. Then, π is square-integrable modulo (H, σ) if and only if π is square-integrable in the sense of Godement, see, for example, [16], if G

is unimodular, and [12], if G is non-unimodular. In particular, there exist always bounded admissible maps $A = \langle \cdot, v\rangle$ where v is in the domain of the formal degree of π, [12], such that Eq. (7) holds (compare with Example 1 above and Example 3 below).

The following result gives some informations when the admissible map is bounded.

Corollary 2. *With the notations of Theorem* 1*, let* $A : \mathcal{H} \to \mathcal{K}$ *be a bounded operator satisfying Eq.* (5)*. Then,*

(1) *the space* X *has an invariant measure,* i.e. $\frac{\Delta_H(h)}{\Delta_G(h)} = 1$ *for all* $h \in H$*;*
(2) *if* A *satisfies Eq.* (6)*, the corresponding wavelet operator* W_A *is given by*
$$(W_A v)(g) = A\pi(g^{-1})v,$$
for all $g \in G$ *and* $v \in \mathcal{H}$*;*
(3) *if* A *satisfies Eq.* (8)*, then* π *is square-integrable modulo both* (H, σ) *and* $(H, \pi_{|H})$*.*

Proof. With respect to the first claim, it is clear that, if A satisfies Eq. (5) for all $v \in \mathcal{D}_\pi$, then Eq. (5) holds for all $v \in \mathcal{H}$, *i.e.* A is is semi-invariant with weight $h \mapsto \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}$ in the sense of [12]. However, A is bounded and this is possible only if $\frac{\Delta_H(h)}{\Delta_G(h)} = 1$ for all $h \in H$, compare with [12, Eq. 2].

In order to show the second statement, let v in $\mathcal{H}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_\pi$ such that $v = \lim v_n$. Since $((W_A v_n)(g))_n$ converges pointwisely to the continuous function $\psi_v(g) = A\pi(g^{-1})v$, by Eq. (6) and Fatou lemma, it follows that $\psi_v \in \mathcal{F}^\sigma$. On the other hand, $W_A v = \lim W_A v_n$ and, by unicity of the limit, $\psi_v = W_A v$.

Finally assume that Eq. (8) holds and let $A = U|A|$ be the polar decomposition of A. Clearly $|A|$ commutes with $\pi_{|H}$ and, taking into account that U restricted to the range of $|A|$ is an isometry, Eq. (8) becomes
$$\alpha \left\| v \right\|^2 \leq \int_G \left\| |A|\pi(g^{-1})v \right\|^2 \theta(g) d\mu_G(g) \leq \beta \left\| v \right\|^2,$$
so, by Corollary 1 π is square-integrable modulo $(H, \pi_{|H})$. □

From Corollary 2 it follows that if π is square-integrable modulo (H, σ) with respect to some bounded admissible map, then it is square-integrable modulo $(H, \pi_{|H})$. However, also with this assumption, in general Eq. (7) can not be satisfied by any bounded admissible map, as showed by the following example, adapted from [15] (compare with the notion of weak and strong square-integrability in [15]).

Example 3. Let $G = \mathbb{R}$ and π be the left regular representation, $H = \{e\}$ and σ the trivial representation. Clearly, any bounded admissible map is of the form $\langle \cdot, v\rangle$ for some vector $v \in L^2(\mathbb{R})$. Losert and Rindler, [20], prove that there is a vector $\eta \in L^2(\mathbb{R})$ with compact support and cyclic. Let $A = \langle \cdot, \eta\rangle$, then the corresponding operator W_A is injective so that π is in fact square-integrable (modulo (H,σ)). However, since $\mathbb{R}$ is unimodular and not discrete, Führ and Mayer, [15], show that there are not vectors $v \in \mathcal{H}$ such that $A_0 = \langle \cdot, v\rangle$ satisfies Eq. (7).

Our definition of square-integrability modulo a subgroup unifies many notions used in literature in the fields of wavelet analysis and of generalised coherent states. For example.

(1) Square-integrability modulo the centre: π is irreducible, H is a central subgroup of G, σ is the character of H defined by the restriction of π to H and $A = \langle \cdot, v\rangle$, for some non-zero vector $v \in \mathcal{H}$, [5];
(2) Gilmore-Perelemov coherent states and α-admissible vectors: π is cyclic, H is the stability subgroup, up to a phase factor, of some non-zero vector $v \in \mathcal{H}$ with respect to the action of $\pi_{|H}$, σ is the corresponding character of H and $A = \langle \cdot, v\rangle$, [2], [26], [31] and reference therein.
(3) Systems of coherent states: π is arbitrary, $\sigma = \pi_{|H}$ and A^*A is of trace class, [19], [32].
(4) Vector coherent states and V-admissible vectors: π is arbitrary, σ is a finite dimensional representation contained in the restriction of π to H and A is the projection on the closed subspace left invariant by σ, [1], [2], [30] and reference therein.
(5) Weak and strong integrability: π is arbitrary, $H = \{e\}$ with the trivial representation, and $A = \langle \cdot, v\rangle$ for some $v \in \mathcal{H}$, [3], [14] and [15].

4. Imprimitivity Theorem for systems of covariance

We start with the definition of covariant localisation observable.

Definition 2. Given a representation π of G acting in $\mathcal{H}$, a map E from the Borel subsets $\mathcal{B}(X)$ of X into the set of positive operators in $\mathcal{H}$ such that

(1) $E(\emptyset) = 0$;
(2) $E(X)$ is injective;
(3) for any disjoint sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X)$,

$$E(\cup_n Y_n) = \sum_n E(Y_n),$$

where the series converges in the strong operator topology;
(4) for all $g \in G$ and $Y \in \mathcal{B}(X)$,

$$\pi_g E(Y) \pi_g^{-1} = E(g[Y]), \tag{9}$$

is called a *localisation observable* based on X, covariant with respect to π and acting in $\mathcal{H}$. Moreover,

- if $E(X) = I$, E is said to be *normalised*,
- if, for all $Y \in \mathcal{B}(X)$, $E(Y)$ is a projection operator, E is said to be *projective*.

The first and third requirement is the fact that E is a POV measure on X and the forth that (π, E) is a *system of* G*-covariance*, [7], or a *generalised imprimitivity*, [17] (see, also, [9], [27]). The second requirement is not a constraint, since the kernel of $E(X)$ is invariant with respect to the action of π and is contained by the kernel of $E(Y)$ for any $Y \in \mathcal{B}(X)$. Finally, if E is projective, then it is necessarily normalised and commutative and (π, E) is a system of imprimitivity, [21] (see, also, [13]). The reason to introduce the name *covariant localisation observable*, instead of *system of* G*-covariance*, is to stress the different role between the representation π and the POV measure E. In doing so, we adopt the terminology from Quantum Mechanics, see, for example, [6], [18].

The notion of equivalence is the natural one. Indeed, if E_1 and E_2 are localisation observables covariant with respect to π_1 and π_2, respectively, they are *equivalent* if there is a unitary operator T intertwining π_1 and π_2 such that

$$E_2(Y)T = TE_1(Y) \qquad \forall Y \in \mathcal{B}(X).$$

Example 4. Let σ be a representation of H and L^σ the corresponding induced representation acting in $\mathcal{F}^\sigma$. For all $Y \in \mathcal{B}(X)$, let $E^\sigma(Y)$ be the operator in $\mathcal{F}^\sigma$ defined by

$$(E^\sigma(Y)F)(g) = \chi_Y(p(g))F(g) \qquad g \in G,\ \mu_G\text{-a.e.},$$

where $F \in \mathcal{F}^\sigma$ and χ_Y is the characteristic function of the subset Y. It is well known, see, for example, [13], that $Y \mapsto E(Y)$ is a projective localisation observable based on X and covariant with respect to L^σ.

Let now T be a positive operator in $\mathcal{F}^\sigma$ commuting with L^σ. Define $\mathcal{F}^\sigma_T$ as the closure of the range of T, L^σ_T be the restriction of L^σ to $\mathcal{F}^\sigma_T$ and, for any $Y \in \mathcal{B}(X)$, $E^\sigma(Y)_T = TE^\sigma(Y)T$, regarded as operator in $\mathcal{F}^\sigma_T$. Clearly, the map $Y \mapsto E^\sigma(Y)_T$ is localisation observable based on X and covariant with respect to L^σ_T. The next theorem will show that, up to an equivalence, all the localisation observables are of this form.

Theorem 2. *Let* π *be a representation of* G *acting in* $\mathcal{H}$ *and* E *a localisation observable based on* X *covariant with respect to* π*. There is a unique (up to an equivalence class) representation* σ_E *of* H *and an isometry* W *from* $\mathcal{H}$ *to* $\mathcal{F}^{\sigma_E}$ *such that*

$$W\pi(g) = L^{\sigma_E}(g)W \qquad g \in G \tag{10}$$

$$E(Y) = E(X)^{\frac{1}{2}}W^*E^{\sigma_E}(Y)WE(X)^{\frac{1}{2}} \qquad Y \in \mathcal{B}(X) \tag{11}$$

$$\mathcal{F}^\sigma = \overline{\operatorname{span}}\{E^{\sigma_E}(Y)Wv \mid Y \in \mathcal{B}(X), v \in \mathcal{H}\}. \tag{12}$$

Moreover, E *is projective if and only if* $WE(X)^{\frac{1}{2}}$ *unitary. Finally, if* E' *is another localisation observable equivalent to* E*, then* $\sigma_{E'}$ *is equivalent to* σ_E*.*

Proof. We split the proof in seven steps.
Step 1). We define an operator valued linear form M on $\mathrm{C_c}(G)$ associated with the POV measure E.

Given $u \in \mathcal{H}$, let $d\langle E(x)u,u\rangle$ be the bounded (Radon) measure on X

$$Y \mapsto \langle E(Y)u,u\rangle\,,$$

having total mass $\langle E(X)u,u\rangle \leq \|u\|^2\,\|E(X)\|$ and satisfying, due to Eq. (9),

$$\int_X f(g[x])d\langle E(x)u,u\rangle = \int_X f(x)d\langle E(x)\pi_g u,\pi_g u\rangle\,, \tag{13}$$

for all $g \in G$.

Given $f \in \mathrm{C_c}(G)$ and $u \in \mathcal{H}$, by Lemma 1, the integral

$$\int_X\left(\int_H f(gh)d\mu_H(h)\right)d\langle E(p(g))u,u\rangle\,,$$

is well defined, linear in f, quadratic in u and it is bounded by

$$C_f \sup_{g\in G}|f(g)|\,\|u\|^2\,\|E(X)\|\,,$$

where C_f is a constant depending only on the support on f. Fixed $f \in \mathrm{C_c}(G)$, by polarisation identity, there is a unique operator $M(f)$ on $\mathcal{H}$ such that, for all $u \in \mathcal{H}$,

$$\begin{aligned}\langle M(f)u,u\rangle &= \int_X\left(\int_H f(gh)d\mu_H(h)\right)d\langle E(p(g))u,u\rangle\\ \|M(f)\| &\leq 4C_f\sup_{g\in G}|f(g)|\,\|E(X)\|\,,\end{aligned} \tag{14}$$

(the factor 4 is due to polarisation identity). By Eq. (13), it follows that, for all $g \in G$ and $f \in \mathrm{C_c}(G)$,

$$\pi(g)M(f)\pi(g) = M(f^g). \tag{15}$$

We claim that, for all $h \in H$ and $f \in \mathrm{C_c}(G)$,

$$M(f(\cdot\, h)) = \Delta_H(h^{-1})M(f), \tag{16}$$

where $f(\cdot\, h)$ is the function $g \mapsto f(gh)$. Indeed, if $u \in \mathcal{H}$,

$$\begin{aligned}\langle M(f(\cdot\, h))u,u\rangle &= \int_X\left(\int_H f(gsh)d\mu_H(s)\right)d\langle E(p(g))u,u\rangle\,,\\ &= \int_X\left(\int_H f(gsh)d\mu_H(s)\right)d\langle E(p(g))u,u\rangle\\ (s\mapsto sh^{-1}) \quad &= \Delta_H(h^{-1})\int_X\left(\int_H f(gs)d\mu_H(s)\right)d\langle E(p(g))u,u\rangle\\ &= \Delta_H(h^{-1})\,\langle M(f)u,u\rangle\,.\end{aligned}$$

Step 2). We show that, if $u,v \in \mathcal{D}_\pi$, there is a unique continuous function $\phi_{u,v}$ defined on G such that

$$\langle M(f)u,v\rangle = \int_G f(g)\phi_{u,v}(g)d\mu_G(g) \qquad f \in \mathrm{C_c}(G).$$

The unicity is clear, since μ_G is a Radon measure and $\phi_{u,v}$ is continuous. To prove the existence, given $u, v \in \mathcal{H}$, we define a linear form on $\mathrm{C_c}(G \times G)$ in the following way. Let $\beta \in \mathrm{C_c}(G \times G)$, $K \subset G \times G$ be its support, K_1 and K_2 the projection of K on the first and second space, respectively. Fixed $g \in G$, the map $g' \mapsto \beta(g', g) =: \beta_g$ is in $\mathrm{C_c}(G)$, so the function

$$G \ni g \mapsto \psi(g) := \langle M(\beta_g)\pi(g)u, v\rangle \in \mathbb{C}$$

is well defined. We claim that $\psi \in \mathrm{C_c}(G)$ Indeed, given $g_1, g_2 \in G$

$$\begin{aligned}
|\psi(g_1) - \psi(g_2)| &= |\langle M(\beta_{g_1})\pi(g_1)u, v\rangle - \langle M(\beta_{g_2})\pi(g_2)u, v\rangle| \\
&\leq |\langle M(\beta_{g_1})(\pi(g_1) - \pi(g_2)u), v\rangle| + |\langle M(\beta_{g_2} - \beta_{g_1})\pi(g_2)u, v\rangle| \\
\text{(Eq. (14))} \quad &\leq 4C_{K_1}\|v\|\,\|E(X)\| \left(\sup_{G\times G} |\beta(g', g)|\,\|\pi(g_1) - \pi(g_2)u\| \right. \\
&\qquad \left. + \sup_{g'\in G} |\beta(g', g_1) - \beta(g', g_2)|\,\|u\| \right),
\end{aligned}$$

since π and β are continuous, also ψ is continuous. By Eq. (14), one has that, for all $g \in G$

$$|\psi(g)| \leq 4C_{K_1} \sup_{g'\in G} |\beta(g', g)|\,\|u\|\,\|v\|\,\|E(X)\|,$$

so $\operatorname{supp}\psi \subset K_2$ and $\psi \in \mathrm{C_c}(G)$.

It follows that there is an operator $\Lambda(\beta)$ in $\mathcal{H}$ such that

$$\begin{aligned}
\langle \Lambda(\beta)u, v\rangle &= \int_G \psi(g)d\mu_G(g) = \int_G \langle M(\beta_g)\pi(g)u, v\rangle d\mu_G(g) \\
\|\Lambda(\beta)\| &\leq C_K \sup_{G\times G} |\beta(g', g)|\,\|E(X)\|, \qquad (17)
\end{aligned}$$

where $C_K = 4C_{K_1}\ \mu_G(K_2)$ depends only on the support of β. In particular one has that, if $f_1, f_2 \in \mathrm{C_c}(G)$,

$$\begin{aligned}
\langle \Lambda(f_1 \otimes f_2)u, v\rangle &= \int_G f_2(g)\,\langle M(f_1)\pi(g)u, v\rangle d\mu_G(g) \\
&= \langle M(f_1)\pi(f_2)u, v\rangle, \qquad (18)
\end{aligned}$$

and, for all $h \in H$ and $\beta \in \mathrm{C_c}(G \times G)$,

$$\begin{aligned}
\langle \Lambda(\beta(\cdot\, h, \cdot))u, v\rangle &= \int_G \langle M(\beta_g(\cdot\, h))\pi(g)u, v\rangle d\mu_G(g) \\
\text{(Eq. (16))} \quad &= \Delta_H(h^{-1}) \int_G \langle M(\beta_g(\cdot))\pi(g)u, v\rangle d\mu_G(g) \\
&= \Delta_H(h^{-1})\,\langle \Lambda(\beta(\cdot, \cdot))u, v\rangle. \qquad (19)
\end{aligned}$$

Fixed $u, v \in \mathcal{H}$, by Eq. (17), it follows that the linear form $\beta \mapsto \langle \Lambda(\beta)u, v\rangle$ is continuous with respect to the natural topology of $\mathrm{C_c}(G \times G)$, there is a measurable complex function $\eta_{u,v}$ of modulo 1 and a Radon measure $\lambda_{u,v}$ on $G \times G$ such that

$$\langle \Lambda(\beta)u, v\rangle = \int_{G\times G} \beta(g, g')\eta_{u,v}(g, g')\, d\lambda_{u,v}(g, g'),$$

see, for example, [10, Ch. XIII, Sec. 16],

Given $f_1, f_2 \in \mathrm{C_c}(G)$ and $u, v \in \mathcal{H}$, then, for all $f \in \mathrm{C_c}(G)$

$$\begin{aligned}
\langle M(f)\pi(f_1)u, \pi(f_2)v\rangle &= \int_G \overline{f_2(g)}\, \langle M(f)\pi(f_1)u, \pi(g)v\rangle d\mu_G(g) \\
\text{(Eq. (15))} &= \int_G \overline{f_2(g)} \left\langle M(f^{g^{-1}})\pi(g^{-1})\pi(f_1)u, v\right\rangle d\mu_G(g) \\
\text{(Eqs. (2), (18))} &= \int_G \overline{f_2(g)} \left\langle \Lambda(f^{g^{-1}} \otimes f_1^{g^{-1}})u, v\right\rangle d\mu_G(g) \\
&= \int_G \overline{f_2(g)} \int_{G\times G} f(gg_1)f_1(gg_2)\eta_{u,v}(g_1, g_2)d\lambda_{u,v}(g_1, g_2)d\mu_G(g) \\
& \int_{G\times G\times G} f(gg_1)f_1(gg_2)\overline{f_2(g)}\eta_{u,v}(g_1, g_2)d\mu_G(g)d\lambda_{u,v}(g_1, g_2) \\
(g \mapsto gg_1^{-1}) &= \int_{G\times G\times G} f(g)f_1(gg_1^{-1}g_2)\overline{f_2(gg_1^{-1})}\ \times \\
&\qquad \Delta_G(g_1^{-1})\eta_{u,v}(g_1, g_2)d\mu_G(g)d\lambda_{u,v}(g_1, g_2) \\
&= \int_G f(g)\phi_{\pi(f_1)u,\pi(f_2)v}d\mu_G(g),
\end{aligned}$$

where, for all $g \in G$,

$$\phi_{\pi(f_1)u,\pi(f_2)v}(g) = \int_{G\times G} f_1(gg_1^{-1}g_2)\overline{f_2(gg_1^{-1})}\Delta_G(g_1^{-1})\eta_{u,v}(g_1, g_2)d\lambda_{u,v}(g_1, g_2),$$

which is a continuous function being $f_1, f_2 \in \mathrm{C_c}(G)$. By linearity, it follows that, for all $u, v \in \mathcal{D}_\pi$ and $f \in \mathrm{C_c}(G)$, there is a continuous function $\phi_{u,v}$ such that

$$\langle M(f)u, v\rangle = \int_G f(g)\phi_{u,v}d\mu_G(g).$$

Step 3). We construct a Hilbert space $\mathcal{K}$, which will carry the representation σ_E.

Let ϕ be the sequilinear form defined on $\mathcal{D}_\pi \times \mathcal{D}_\pi$ as $\phi(u, v) = \phi_{u,v}(e)$. Clearly, if $f_1, f_2 \in \mathrm{C_c}(G)$ and $u, v \in \mathcal{H}$

$$\begin{aligned}
\phi(\pi(f_1)u, \pi(f_2)v) &= \int_{G\times G} f_1(g_1^{-1}g_2)\overline{f_2(g_1^{-1})}\Delta_G(g_1^{-1})\eta_{u,v}(g_1, g_2)d\lambda_{u,v}(g_1, g_2) \\
\text{(20)} \qquad &= \langle \Lambda(f_1 \bullet f_2)u, v\rangle
\end{aligned}$$

with $(f_1 \bullet f_2)(g_1,g_2) = f_1(g_1^{-1}g_2)\overline{f_2(g_1^{-1})}\Delta_G(g_1^{-1})$. By Eqs. (2), (20) and dominated convergence theorem, the map

$$(g,g') \mapsto \phi(\pi(g)\pi(f_1)u, \pi(g')\pi(f_2)v)$$

is continuous on $G \times G$ and

$$\phi_{\pi(f_1)u,\pi(f_2)v}(g) = \phi(\pi(g^{-1})\pi(f_1)u, \pi(g^{-1})\pi(f_2)v).$$

By linearity, it follows that, for all $u, v \in \mathcal{D}_\pi$

$$(g,g') \mapsto \phi(\pi(g)u, \pi(g')v) \text{ is continuous} \tag{21}$$

and, for all $f \in \mathrm{C_c}(G)$,

$$\int_G f(g)\phi(\pi(g^{-1})u, \pi(g^{-1})v)d\mu_G(g) = \int_G f(g)\phi_{u,v}(g)d\mu_G(g) = \langle M(f)u,v\rangle . \tag{22}$$

The form ϕ is non-negative, since, by construction,

$$\begin{aligned}\langle M(f)u,u\rangle &= \int_G f(g)\phi_{u,u}(g)d\mu_G(g)\\ &= \int_X \left(\int_H f(gh)d\mu_H(h)\right) d\langle E(p(g))u,u\rangle\end{aligned}$$

which is clearly non-negative, then $\phi_{u,u}(g) \geq 0$ μ_G-almost everywhere and, since $\phi_{u,u}$ is continuous, $\phi_{u,u}(e) \geq 0$.

Let $\mathcal{K}$ be the closure of the quotient space of $\mathcal{D}_\pi$ over the kernel of ϕ with respect to scalar product induced by ϕ and A be the map from $\mathcal{D}_\pi$ to $\mathcal{K}$ mapping $v \in \mathcal{D}_\pi$ into its equivalence class Av, viewed in a natural way as an element of $\mathcal{K}$.

We claim that $\mathcal{K}$ is separable (so that $\mathcal{K}$ is in fact a Hilbert space). Since $\mathcal{N} := A\mathcal{D}_\pi$ is dense in $\mathcal{K}$, it is sufficient to show that $\mathcal{N}$ is separable. Since G is second countable, there is a denumerable family $\{f_n\}_{n\in\mathbb{N}}$ in $\mathrm{C_c}(G)$ such that for any $f \in \mathrm{C_c}(G)$ and $\epsilon > 0$ there is a compact set K and f_n satisfying

$$\begin{aligned}\operatorname{supp}(f - f_n) &\subset K\\ \sup_{g\in G}|f(g) - f_n(g)| &< \epsilon.\end{aligned} \tag{23}$$

Moreover, since $\mathcal{H}$ is separable, there is a denumerable family $\{u_m\}_{m\in\mathbb{N}}$ dense in $\mathcal{H}$. We claim that $\{A\pi(f_n)u_m\}_{n,m\in\mathbb{N}}$ is dense in $\mathcal{N}$.

Indeed, given $f \in \mathrm{C_c}(G)$ and $u \in \mathcal{H}$, let f_n and u_m such that Eq. (23) holds, $\|u - u_m\| < \epsilon$ and $\|u_m\| \leq 2\|u\|$. Then

$$\begin{aligned}\|A\pi(f)u - A\pi(f_n)u_m\|_{\mathcal{K}} &\leq \|A\pi(f - f_n)u_m\|_{\mathcal{K}} + \|A\pi(f)(u - u_m)\|_{\mathcal{K}}\\ &= \sqrt{\phi_{\pi(l)u_m,\pi(l)u_m}} + \sqrt{\phi_{\pi(f)v,\pi(f)v}},\end{aligned}$$

where $l = f - f_n$ and $v = u - u_m$. Then, using Eq. (17) and (20),

$$\begin{aligned}
\phi_{\pi(l)u_m,\pi(l)u_m} &= \langle \Lambda(l \bullet l)u_m, u_m\rangle \\
&\leq C_K \sup_{G\times G} |l(g_1^{-1}g_2)\overline{l(g_1^{-1})}\Delta_G(g_1^{-1})| \, \|E(X)\| \, \|u_m\|^2 \\
&\leq C_K \sup_{g\in K}(\Delta_G(g^{-1}))(\sup_{g\in G}|f(g) - f_n(g)|)^2 \, \|E(X)\| \, \|u_m\|^2 \\
&\leq 4C_K \sup_{g\in K}(\Delta_G(g^{-1})) \, \|E(X)\| \, \|u\|^2 \, \epsilon^2,
\end{aligned}$$

where C_K depends only on K. In the same way,

$$\begin{aligned}
\phi_{\pi(f)v,\pi(f)v} &= \langle \Lambda(f \bullet f)v, v\rangle \\
&\leq C_K \sup_{G\times G} |f(g_1^{-1}g_2)\overline{f(g_1^{-1})}\Delta_G(g_1^{-1})| \, \|E(X)\| \, \|u - u_m\|^2 \\
&\leq C_K \sup_{g\in K}(\Delta_G(g^{-1})) \, \|E(X)\| \, (\sup_{g\in G}|f(g)|)^2\epsilon^2.
\end{aligned}$$

From the above inequalities, one has that

$$\|A\pi(f)u - A\pi(f_n)u_m\|_{\mathcal{K}} \leq C'\epsilon,$$

where C' is a suitable constant depending only on f and u. Since the set $\{A\pi(f)u\}$ spans $A\mathcal{D}_\pi$, the claim follows.

Step 4). We define a representation σ_E, denoted in the following simply by σ, and an isometry W satisfying Eqs. (10), (11) and (12).

To this aim, we first prove that, for all $h \in H$ and $u, v \in \mathcal{D}_\pi$

$$\phi(\pi(h)u, \phi(h)v) = \frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}\phi(u, v). \tag{24}$$

We can always assume that $u = v = \pi(f)w$ for some $f \in \mathrm{C_c}(G)$ and $w \in \mathcal{H}$. Then, by Eq. (2),

$$\begin{aligned}
\phi(\pi(h)\pi(f)w, \pi(h)\pi(f)w) &= \phi(\pi(f^h)w, \pi(f^h)w) \\
&= \left\langle \Lambda(f^h \bullet f^h)w, w \right\rangle \\
&= \Delta_G(h) \, \langle \Lambda((f \bullet f)(\cdot\, h, \cdot))w, w\rangle \\
\text{(Eq. (19))} \quad &= \frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})} \, \langle \Lambda((f \bullet f)(\cdot, \cdot))w, w\rangle \\
&= \frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})}\phi(\pi(f)w, \pi(f)w).
\end{aligned}$$

From Eq. (24), it follows that there is an isometric operator σ_h in $\mathcal{K}$ such that, for all $h \in H$,

$$\sigma_h A u = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} A\pi_h u \qquad u \in \mathcal{D}_\pi. \tag{25}$$

We claim that $h \mapsto \sigma$ is a representation of H. The algebraic properties are clear and, since σ_h is isometric for all $h \in H$, it is sufficient to show that, given $u, v \in \mathcal{D}_\pi$, the map

$$h \mapsto \langle \sigma_h A u, A v \rangle_{\mathcal{K}} = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} \phi(\pi_h u, v)$$

is continuous and this fact follows from Eq. (21). We denote by σ the representation defined by Eq. (25).

Moreover, we claim that A is an admissible map with respect to (σ, H). By construction, Eq. (5) is satisfied. Moreover, for all $u, v \in \mathcal{D}_\pi$, the map $g \mapsto \left\langle A\pi(g^{-1})u, Av \right\rangle_{\mathcal{K}}$ is continuous by Eq. (21) and, hence, $g \mapsto A\pi(g^{-1})u$ is μ_G-measurable. Let now $f \in \mathrm{C_c}(G)$ non-negative and $u \in \mathcal{D}_\pi$, by Eq. (22)

$$\begin{aligned}
\langle M(f)u, u \rangle &= \int_G f(g) \phi(\pi(g^{-1})u, \pi(g^{-1})u) d\mu_G(g) \\
&= \int_G f(g) \left\| A\pi(g^{-1})u \right\|^2 d\mu_G(g) \\
&= \int_G f(g) \left\| A\pi(g^{-1})u \right\|^2 \int_H \theta(gh) d\mu_H(h) d\mu_G(g) \\
(g \mapsto gh^{-1}) &= \int_H \int_G \Delta_G(h^{-1}) f(gh^{-1}) \left\| A\pi(hg^{-1})u \right\|^2 \theta(g) d\mu_G(g) d\mu_H(h) \\
(\text{Eq. } (5), h \mapsto h^{-1}) &= \int_G \left(\int_H f(gh) d\mu_H(h) \right) \left\| A\pi(g^{-1})u \right\|^2 \theta(g) d\mu_G(g).
\end{aligned}$$

By definition of $M(f)$ and with notation of Lemma 1, one has that

$$\int_X \tilde{f}(x) d \langle E(x)u, u \rangle = \int_G \tilde{f}(p(g)) \left\| A\pi(g^{-1})u \right\|^2 \theta(g) d\mu_G(g). \tag{26}$$

By Lemma 1, there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathrm{C_c}(G)$ positive such that $(\tilde{f}_n)_n$ is a partition of the unit of X. Then

$$\begin{aligned}
\langle E(X)u, u \rangle &= \sum_n \int_X \tilde{f}_n(x) d \langle E(x)u, u \rangle \\
(\text{Eq. } (26)) &= \sum_n \int_G \tilde{f}_n(p(g)) \left\| A\pi(g^{-1})u \right\|^2 \theta(g) d\mu_G(g).
\end{aligned}$$

By monotone convergence theorem, the map $g \mapsto \left\| A\pi(g^{-1})u \right\|^2 \theta(g)$ is μ_G-integrable and

$$\langle E(X)u, u \rangle = \int_G \left\| A\pi(g^{-1})u \right\|^2 \theta(g) d\mu_G(g). \tag{27}$$

This shows that A is an admissible map.

Let W_A be the corresponding wavelet operator. From (27), one has that, for all $u \in \mathcal{D}_\pi$, $\langle W_A u, W_A u \rangle = \langle E(X)u, u \rangle$, that is, by density, $W_A^* W_A = E(X)$. By Theorem 1, W_A intertwines π with L^σ and, by definition of localisation observable, $E(X)$ is injective. Then, by polar decomposition, there is an isometry W such that $W_A = WE(X)^{\frac{1}{2}}$. Since W intertwines π with L^σ, W satisfies Eq. (10).

To prove Eq. (11), let $f \in \mathrm{C_c}(G)$ and $u \in \mathcal{D}_\pi$, using the definition of E^σ,

$$\begin{aligned}
\int_X \tilde{f}(x) d\left\langle W_A^* E^\sigma(x) W_A u, u\right\rangle &= \int_G \tilde{f}(p(g)) \left\langle (W_A u)(g), (W_A u)(g)\right\rangle_{\mathcal{K}} \theta(g) d\mu_G(g) \\
&= \int_G \tilde{f}(p(g)) \left\|A\pi(g^{-1})u\right\|^2 \theta(g) d\mu_G(g) \\
\text{(Eq. (26))} \quad &= \int_X \tilde{f}(x) d\left\langle E(x)u, u\right\rangle .
\end{aligned}$$

By Riesz-Markov theorem and the surjectivity of the map $f \mapsto \tilde{f}$, see Lemma 1, it follows that, for all $Y \in \mathcal{B}(X)$ and $u \in \mathcal{D}_\pi$,

$$\left\langle E(Y)u, u\right\rangle = \left\langle W_A^* E^\sigma(Y) W_A u, u\right\rangle .$$

By density and the definition of W it follows the claim.

For Eq. (12), it is sufficient to prove that the closed subspace

$$\mathcal{M} := \{F \in \mathcal{F}^\sigma \mid \left\langle F, E^\sigma(Y) W_A v\right\rangle = 0, \forall Y \in \mathcal{B}(X), v \in \mathcal{H}\}$$

is the null space. Using Eq. (9) and the fact that W_A commutes with the action of G, it follows that $\mathcal{M}$ is a G-invariant closed subspace of $\mathcal{F}^\sigma$. In particular, if $f \in \mathrm{C_c}(G)$ and $F' \in \mathcal{M}$, $F = L^\sigma(f)F'$ is in $\mathcal{M}$ and, due to Prop. 1, F is a continuous function. Let now $Y \in \mathcal{B}(X)$ and $v \in \mathcal{D}_\pi$, then

$$\begin{aligned}
0 &= \left\langle F, E^\sigma(Y) W_A v\right\rangle \\
&= \int_{p^{-1}(Y)} \left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} \theta(g)\ d\mu_G(g). \qquad (28)
\end{aligned}$$

Since $v \in \mathcal{D}_\pi$, by Theorem 1, $W_A v$ is continuous and, hence, also the map $\left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}}$ is continuous. Due to this and the fact that $F, W_A u \in \mathcal{F}^\sigma$, one has that for all $h \in H$ and *for all* $g \in G$

$$\left\langle F(gh), (W_A v)(gh)\right\rangle_{\mathcal{K}} = \frac{\Delta_H(h)}{\Delta_G(h)} \left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} . \qquad (29)$$

Let

$$Y' = \{g \in G \mid \left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} \le 0\},$$

which is closed. Since $\frac{\Delta_H(h)}{\Delta_G(h)}$ is strictly positive, due to Eq. (29), for all $h \in H$, $Y'h = Y'$. Defined $Y = p(Y')$, which is closed since p is an open map, one has that $Y' = p^{-1}(Y)$ and, using Eq. (28), it follows that $\left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} = 0$ $\theta\mu_G$-almost all $g \in Y'$. By Lemma 3 ($Y'h = Y'$) and the continuity, $\left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} = 0$ for all $g \in Y'$. Repeating the above argument, one concludes that $\left\langle F(g), (W_A v)(g)\right\rangle_{\mathcal{K}} = 0$ for all $g \in G$. By definition of W_A, $\left\langle F(g), A\pi(g^{-1})v\right\rangle_{\mathcal{K}} = 0$ and, since $A\mathcal{D}_\pi$ is dense in $\mathcal{K}$ and $\pi(g^{-1})$ is unitary, it follows that $F(g) = 0$ for all $g \in G$, that is $F = 0$. Since $\{\pi(f)F' \mid f \in \mathrm{C_c}(G), F' \in \mathcal{M}\}$ is dense in $\mathcal{M}$, one has that $\mathcal{M} = 0$. The claim is now clear.

Step 5). We show that σ is unique up to an equivalence.

Let τ be another representation of H acting in $\mathcal{K}'$ and W' the isometry from $\mathcal{H}$ to L^τ such that Eqs. (10), (11) and (12) hold. By Theorem 1, there is an admissible map B from π with respect to (H,τ) such that $W_B = WE(X)^{\frac{1}{2}}$. Given $f \in \mathrm{C_c}(G)$ and $u \in \mathcal{D}_\pi$, applying Eq. (11) with W' instead of W, one has that

$$\begin{aligned}
\langle M(f)u,u\rangle &= \int_X \tilde{f}(x)\ d\langle E^\tau(x)W_Bu, W_Bu\rangle \\
&= \int_G \tilde{f}(p(g))\left\|B\pi(g^{-1})u\right\|^2\theta(g)d\mu_G(g) \\
&= \int_H\int_G f(gh)\left\|B\pi(g^{-1})u\right\|^2\theta(g)d\mu_G(g)d\mu_H(h) \\
(g\mapsto gh^{-1}, h\mapsto h^{-1})\quad &= \int_H\int_G f(g)\left\|B\pi(g^{-1})u\right\|^2\theta(gh)d\mu_G(g)d\mu_H(h) \\
&= \int_G f(g)\left\|B\pi(g^{-1})u\right\|^2 d\mu_G(g).
\end{aligned}$$

On the other hand, due to Eq. (22),

$$\int_G f(g)\left\|A\pi(g^{-1})u\right\|^2 d\mu_G(g) = \int_G f(g)\left\|B\pi(g^{-1})u\right\|^2 d\mu_G(g).$$

By usual arguments, it follows that $\|Au\|_{\mathcal{K}} = \|Bu\|_{\mathcal{K}'}$. Hence, there is a unique isometric operator t from $\mathcal{K}$ to $\mathcal{K}'$ such that

$$tAu = Bu \qquad u\in\mathcal{D}_\pi.$$

By definition of σ and the fact that B satisfies Eq. (5), t intertwines σ and τ. Since Eq. (12) holds both for W and W', one has that t is in fact unitary. This shows that τ is equivalent to σ.

Step 6). We characterise the condition that E is projective.

Assume now that $WE(X)^{\frac{1}{2}}$ is unitary, then since E^σ is projective, see Example 4, and Eq. (11), it follows that also E is projective. Conversely, assume that E is projective. Since E is normalised, $E(X)^{\frac{1}{2}} = I$ and $W = W_A$. For all $Y, Z \in \mathcal{B}(X)$ and $u,v\in\mathcal{H}$,

$$\begin{aligned}
\langle WW^*E^\sigma(Y)Wu, E^\sigma(Z)Wv\rangle &= \langle WE(Y)u, E^\sigma(Z)Wv\rangle \\
&= \langle W^*E^\sigma(Z)WE(Y)u, v\rangle \\
&= \langle E(Z\cap Y)u, v\rangle \\
&= \langle W^*E^\sigma(Z\cap Y)WE(Y)u, v\rangle \\
&= \langle E^\sigma(Y)Wu, E^\sigma(Z)Wv\rangle,
\end{aligned}$$

since both E and E^σ are projective. By Eq. (12), $\{E^\sigma(Y)Wu\}$ is total in $\mathcal{F}^\sigma$ so that $WW^* = I$ and, since W is an isometry, W is unitary.

Step 7). We show that the equivalence class of E defines the equivalence class of σ_E.

Let E' as in the statement of the theorem and T a unitary operator such that

$$E(Y)'T = TE(Y) \quad Y \in \mathcal{B}(X) \tag{30}$$

$$\pi(g)'T = T\pi(g) \quad g \in G, \tag{31}$$

where, here and in the following, we denote by a prime the the objects that refer to E'. Given $f \in \mathrm{C_c}(G)$, from Eq. (31), if follows that, $M'(f)T = TM(f)$. Let now $u \in \mathcal{D}_\pi$ and $v \in \mathcal{D}_{\pi'}$, by Lemma 2 $Tu \in \mathcal{D}_{\pi'}$ and, by Eq. (22) applied first to E' and, then, to E,

$$\begin{aligned}\int_G f(g)\phi'(\pi'(g^{-1})Tu, \pi'(g^{-1})v)d\mu_G(g) &= \langle M(f)'Tu, v\rangle \\ &= \langle M(f)u, T^*v\rangle \\ &= \int_G f(g)\phi(\pi(g^{-1})u, \pi(g^{-1})T^*v)d\mu_G(g).\end{aligned}$$

By standard arguments, one has that $\phi'(Tu, v) = \phi(u, T^*v)$. Since T is unitary, for all $u, v \in \mathcal{D}_\pi$,

$$\phi'(Tu, Tv) = \phi(u, v).$$

It follows that there is an unitary operator t from $\mathcal{K}$ to $\mathcal{K}'$ such that, for all $u \in \mathcal{D}_\pi$,

$$tAu = A'tu.$$

By definition of σ and σ' and Eq. (31), one has that, for all $h \in H$ and $u \in \mathcal{D}_\pi$,

$$\sigma'(h)tu = t\sigma(h)u.$$

Hence, by density, σ' is equivalent to σ. □

We add some comments. With the notation of Example 4 and $\sigma = \sigma_E$, Eq. (11) implies that E is equivalent to a localisation observable of the form E^σ_T where $T = WE(X)^{\frac{1}{2}}W^*$ is a positive operator on $\mathcal{F}^\sigma$. In particular, there exists always a *normalised* localisation observable E_0 covariant with respect to π and acting in $\mathcal{H}$ such that

$$E(Y) = E(X)^{\frac{1}{2}}E_0(Y)E(X)^{\frac{1}{2}} \quad Y \in \mathcal{B}(X). \tag{32}$$

If E is normalised, Eq. (12) implies that $(\mathcal{F}^\sigma, E^\sigma)$ is the dilation of $(\mathcal{H}, E)$ in the sense of Neumark's dilatation theorem. Moreover, since our proof is independent on Mackey's imprimitivity theorem, it contains the Mackey's result as a particular case and one can easily show that two *projective* localisation observables E and E' are equivalent, if and only if σ_E and $\sigma_{E'}$ are equivalent. However, if E and E' are not projective one has only the *only if* part, as showed by the following example.

Example 5. Let $G = \mathcal{T}$ be the one dimensional torus and $H = \{e\}$. Denoted by (f_1, f_2) the canonical basis of $\mathcal{H} := \mathbb{C}^2$, define, for all $z \in \mathcal{T}$ and $Y \in \mathcal{B}(\mathcal{T})$,

$$\begin{aligned}
\pi(z) &= z \langle \cdot, f_1 \rangle f_1 + z^2 \langle \cdot, f_2 \rangle f_2 \\
E(Y) &= \mu(Y) \langle \cdot, f_1 \rangle f_1 + \mu(Y) \langle \cdot, f_2 \rangle f_2 \\
E'(Y) &= E(Y) + \int_Y \overline{z}\, d\mu(z) \langle \cdot, f_1 \rangle f_2 + \int_Y z\, d\mu(z) \langle \cdot, f_2 \rangle f_1,
\end{aligned}$$

where μ is the normalised Haar measure on $\mathcal{T}$. By direct computation, one can check that E is a projective localisation observable and E' is a normalised non-projective one, both covariant with respect to π. Moreover, one has that $\sigma_E = \sigma_{E'}$ is the two dimensional trivial representation of the identity, however E and E' are not equivalent.

By Eq. (32), one can always assume that localisation observables are normalised. The following corollary settles a correspondence between normalised localisation observables E and admissible maps A satisfying Eq. (7), compare with [17], [23], [32].

Corollary 3. *Let π be a representation of G and θ as in Lemma 3. Given a normalised localisation observable E covariant with respect to π, there is an admissible map A for π with respect to (H, σ_E) that satisfies Eq. (7) and, for all $u, v \in \mathcal{D}_\pi$ and $Y \in \mathcal{B}(X)$,*

$$\langle E(Y)u, v \rangle = \int_{p^{-1}(Y)} \langle A\pi(g^{-1})u, A\pi(g^{-1})v \rangle \, \theta(g) d\mu_G(g). \tag{33}$$

In particular π is square integrable modulo (H, σ_E).

Conversely, if π is square integrable modulo (H, σ) for some representation σ of H and A is an admissible map for π with respect to (H, σ) such that Eq. (7) holds, then Eq. (33) defines a normalised localisation observable covariant with respect to π.

Proof. For the first part, due to Theorem 2, there is an isometry W such that Eqs. (10) and (11) hold. In particular, π is square-integrable. Since W is an intertwining isometric operator, by Theorem 1, there is an admissible map A such that Eq. (7) holds. By definition of E^{σ_E}, Eq. (33) follows.

Conversely, assume that π is square integrable modulo (H, σ) for some σ. By Corollary 1, there are admissible maps A satisfying Eq. (7). In particular the corresponding wavelet operator W_A is isometric and intertwines π and L^σ. Then, the map

$$Y \mapsto W_A^* E^\sigma(Y) W_A =: E(Y)$$

is a normalised localisation observable covariant with respect to π, explicitely given by Eq. (33). □

The cited result of Scutaru requires that A is bounded and A^*A is of trace class, whereas the one of Holevo is explicitely proven for G compact with H arbitrary and G type I group with $H = \{e\}$.

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