A THEOREM OF LUDWIG REVISITED

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Abstract. Using a recent result of Busch and Gudder [1] we reconsider a theorem of Ludwig [2] which allows one to identify a class of effect automorphisms as the symmetry transformations in quantum mechanics.

I propose that we embark on a study of the structure of science – its theories and models – in itself. The clue, I shall suggest, is this: at the most basic level of theorizing, sive model construction, lies the pursuit of symmetry.

Bas van Fraassen, *Laws and Symmetry*, Clarendon, 1989.

1. Introduction

The basic structures of quantum mechanics are coded in two sets, the set of states $\mathbf{S}$ and the set of effects $\mathbf{E}$, and in the duality between them, $\mathbf{S} \times \mathbf{E} \ni (T, E) \mapsto \mathrm{tr}\big[TE\big] \in [0,1]$, with an interpretation that the number $\mathrm{tr}\big[TE\big]$ is the probability for the effect E in the state T.[1] Both of these sets are equipped with natural physically relevant structures. For the purposes of this paper it is enough to recall only the *convex structure* of the set of states (if $T_1, T_2 \in \mathbf{S}, 0 \leq w \leq 1$, then $wT_1 + (1-w)T_2 \in \mathbf{S}$), which reflects the possibility of mixing states into new states, and the $\perp$-*order structure* of $\mathbf{E}$, which is directly related to the possibility of comparing effects in terms of their probabilities: for any $E, F \in \mathbf{E}$, $E \leq F$ if and only if $\mathrm{tr}\big[TE\big] \leq \mathrm{tr}\big[TF\big]$ for all $T \in \mathbf{S}$, and for each $E \in \mathbf{E}$ there is a unique $E^\perp \in \mathbf{E}$ such that $\mathrm{tr}\big[TE\big] + \mathrm{tr}\big[TE^\perp\big] = 1$ for all $T \in \mathbf{S}$. (Clearly, $E^\perp = I - E$). In the $\perp$-order structure of $\mathbf{E}$ the set $\mathbf{D}$ of projection operators appears as a distinguished subset of $\mathbf{E}$: for any $E \in \mathbf{E}$, $E^2 = E$ if and only if the greatest lower bound of E and $E^\perp$ exists and equals to the zero effect,

[1]We assume that the reader is familiar with the ordinary Hilbert space formulation of quantum mechanics with states and effects as the primary concepts; states represented as positive trace one operators and effects as the unit bounded positive operators on a complex separable Hilbert space attached to the physical system.

that is, $E \wedge E^{\perp} = O$. The $\perp$-order structure of $\mathbf{E}$ gives $\mathbf{D}$ the stucture of a complete orthocomplemented lattice.

The above quoted structures of $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{D}$, lead to the following notions of automorphisms.

Definition 1. A function $s : \mathbf{S} \to \mathbf{S}$ is a *state automorphism* if
1) s is a bijection,
2) $s(wT_1 + (1-w)T_2) = ws(T_1) + (1-w)s(T_2)$ for all $T_1, T_2 \in \mathbf{S}$, $0 \leq w \leq 1$.

Definition 2. A function $e : \mathbf{E} \to \mathbf{E}$ is an *effect $\perp$-order automorphism* if
1) e is a bijection,
2) for all $E, F \in \mathbf{E}$, $E \leq F \iff e(E) \leq e(F)$,
3) $e(E^{\perp}) = e(E)^{\perp}$ for all $E \in \mathbf{E}$.

Definition 3. *A function $d : \mathbf{D} \to \mathbf{D}$ is a $\mathbf{D}$-automorphism if*
1) d is a bijection,
2) for all $D_1, D_2 \in \mathbf{D}$, $D_1 \leq D_2 \iff d(D_1) \leq d(D_2)$,
3) $d(D^{\perp}) = d(D)^{\perp}$ for all $D \in \mathbf{D}$.

The sets Aut$\,(\mathbf{S})$, Aut$(\mathbf{E})$, and Aut$\,(\mathbf{D})$ of all state automorphisms, effect $\perp$-order automorphisms, and $\mathbf{D}$-automorphisms form groups with respect to the composition of functions. These groups are among the several natural automorphism groups of quantum mechanics on which the theory of symmetry in quantum mechanics can be based. In a recent paper [3] we have investigated some of these groups, leaving aside the group Aut$(\mathbf{E})$ which was already investigated in great detail in Section V.5 of the monograph [2] of Ludwig. Ludwig's theorem V.5.21 is the key result in showing that the groups Aut$\,(\mathbf{S})$, Aut$(\mathbf{E})$, and Aut$\,(\mathbf{D})$ are isomorphic, and therefore provide equivalent formulations of the notion of symmetry in quantum mechanics. Ludwig's proof is, however, rather complicated and it contains an unnecessary dimension restriction. The purpose of this paper is to offer in the form of Lemma 3 a simplified proof of this crucial result with a proper dimension requirement.

2. A theorem of Ludwig revisited

The set $\mathbf{D}$ of projections is a subset of $\mathbf{E}$. Thus it is natural to consider the restriction $e|_{\mathbf{D}}$ on $\mathbf{D}$ of an effect automorphism $e \in \mathrm{Aut}(\mathbf{E})$. One gets:

Lemma 1. *The function* $\mathrm{Aut}(\mathbf{E}) \ni e \mapsto e|_{\mathbf{D}} \in \mathrm{Aut}\,(\mathbf{D})$ *is a group homomorphism.*

Proof. Let $e \in \mathrm{Aut}(\mathbf{E})$. Then for any $E, F, G \in \mathbf{E}$, G is a lower bound of E and F if and only if $e(G)$ is a lower bound of $e(E)$ and $e(F)$. Since $\mathbf{D}$ consists exactly of those effects $E \in \mathbf{E}$ for which O is the only lower bound of E and $E^{\perp}$ one thus has $e(\mathbf{D}) \subseteq \mathbf{D}$. Clearly, $(e_1 \circ e_2)|_{\mathbf{D}} = e_1|_{\mathbf{D}} \circ e_2|_{\mathbf{D}}$ and $e^{-1}|_{\mathbf{D}} = (e|_{\mathbf{D}})^{-1}$. □

The homomorphism of Lemma 1 is, in fact, injective whenever the dimension of the Hilbert space is, at least, two. We shall prove this result, which is due to Ludwig [2, Theorem 5.21, p. 226], using the following characterization of effects [1]. Here $\mathbf{P}$ denotes the set of one dimensional projections.

Lemma 2. *For any $E \in \mathbf{E}$,*

$$E = \vee_{P\in\mathbf{P}}(E \wedge P) = \vee_{P\in\mathbf{P}}\lambda(E,P)P, \tag{1}$$

where

$$\lambda(E,P) := \sup\{\gamma \in [0,1] \,|\, \gamma P \leq E\}.$$

In fact, $\lambda(E,P) = \max\{\gamma \in [0,1] \,|\, \gamma P \leq E\}$, and if $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, is such that $P\varphi = \varphi$, then $\lambda(E,P) = \left\|E^{-1/2}\varphi\right\|^{-2}$, whenever $\varphi \in$ $\mathrm{ran}(E^{1/2})$, whereas $\lambda(E,P) = 0$, otherwise.

Lemma 3. *If $\dim(\mathcal{H}) \geq 2$, then the function $\mathrm{Aut}(\mathbf{E}) \ni e \mapsto e|_{\mathbf{D}} \in \mathrm{Aut}\,(\mathbf{D})$ is injective.*

Proof. It suffices to show that if $e \in \mathrm{Aut}(\mathbf{E})$ is such that $e(D) = D$, for all $D \in \mathbf{D}$, then e is the identity function. Therefore, assume that $e(D) = D$, for all $D \in \mathbf{D}$. Then, in particular, $e(P) = P$, for all $P \in \mathbf{P}$. Thus, for any $\gamma \in [0,1], P \in \mathbf{P}$, $e(\gamma P) \leq e(P) = P$, so that

$$e(\gamma P) = \tau(\gamma, P)P \tag{2}$$

for some $\tau(\gamma,P) \in [0,1]$. The proof now consists of showing that, for any $\gamma \in [0,1]$ and for any $P \in \mathbf{P}$, $\tau(\gamma, P) = \gamma$. If this is the case, then, for any $E \in \mathbf{E}$,

$$\begin{aligned} e(E) &= \vee_{P\in\mathbf{P}} e(\lambda(E,P)P) \\ &= \vee_{P\in\mathbf{P}} \tau(\lambda(E,P),P)P \\ &= \vee_{P\in\mathbf{P}} \lambda(E,P)P \\ &= E \end{aligned}$$

and we are through. We proceed in three steps.

Step 1. Let $E \in \mathbf{E}$. From (1) we obtain that

$$e(E) = \vee_{P\in\mathbf{P}} e(\lambda(E,P)P) = \vee_{P\in\mathbf{P}} \tau(\lambda(E,P),P)P$$

and also that

$$e(E) = \vee_{P\in\mathbf{P}} \lambda(e(E),P)P.$$

Taking the meet of both expression with any 1 dimensional projection we see that

$$\tau(\lambda(E,P),P) = \lambda(e(E),P) \tag{3}$$

for any $E \in \mathbf{E}, P \in \mathbf{P}$.

Step 2. We next show that the function τ does not depend on P, that is,

$$\tau(\gamma,P) = \tau(\gamma) \tag{4}$$

for each $\gamma \in [0,1], P \in \mathbf{P}$. Clearly $\tau(0,P) = 0$ and $\tau(1,P) = 1$ for all $P \in \mathbf{P}$. Thus, consider a fixed $0 < \gamma < 1$ and let $P, Q \in \mathbf{P}$ be such that $QP \neq O$. Define

$$\mu = \frac{1-\gamma}{1-\gamma(1-\mathrm{tr}\big[PQ\big])}. \tag{5}$$

Observe that $1-\gamma \leq \mu < 1$ and define $E := I - \mu Q$. Then $\mathrm{ran}(E^{1/2}) = \mathcal{H}$, so that, by Lemma 2,

$$\lambda(E,P) = \frac{1}{\mathrm{tr}\big[E^{-1}P\big]} = \frac{\mu - 1}{\mu(1-\mathrm{tr}\big[QP\big]) - 1} = \gamma.$$

Hence, due to (3),

$$\tau(\gamma,P) = \lambda(e(E),P). \tag{6}$$

On the other hand $e(E) = I - \tau(\mu,Q)Q$ and again we have $\mathrm{ran}(e(E)^{1/2}) = \mathcal{H}$, so that,

$$\lambda(e(E),P) = \frac{1}{\mathrm{tr}\big[e(E)^{-1}P\big]} = \frac{\tau(\mu,Q)-1}{\tau(\mu,Q)(1-\mathrm{tr}\big[QP\big]) - 1}. \tag{7}$$

Comparing (6) and (7) we have

$$\tau(\gamma,P) = \frac{\tau(\mu,Q)-1}{\tau(\mu,Q)(1-\mathrm{tr}\big[QP\big]) - 1}.$$

¿From (5) we get

$$(1-\mathrm{tr}\big[PQ\big]) = \frac{\mu+\gamma-1}{\mu\gamma},$$

hence

$$\tau(\gamma,P) = \frac{\gamma\mu[\tau(\mu,Q)-1]}{\tau(\mu,Q)(\mu+\gamma-1) - \gamma\mu}. \tag{8}$$

We then see that $\tau(\gamma,P)$ fulfills equation (8), where Q is any 1-dimensional projection such that $\mathrm{tr}\big[PQ\big] \neq 0$ and μ is defined by (5). On the other hand, μ depends only on γ and $\mathrm{tr}\big[PQ\big]$. Given $P_1, P_2 \in \mathbf{P}$, one can find $Q \in \mathbf{P}$ such that $\mathrm{tr}\big[P_1Q\big] = \mathrm{tr}\big[P_2Q\big] \neq 0$ so that (8) implies

$\tau(\gamma, P_1) = \tau(\gamma, P_2)$ and this proves that τ does not depend on P. Equation (4) is thus established.

Step 3. Now suppose that $\dim \mathcal{H} \geq 2$. It is clear from (5) that if $1 - \gamma \leq \alpha < 1$, then we can choose $P, Q \in \mathbf{P}$ such that $\mu = \alpha$. Hence (8) gives

$$\tau(\gamma) = \frac{\gamma\alpha[\tau(\alpha) - 1]}{\tau(\alpha)(\alpha + \gamma - 1) - \gamma\alpha} \tag{9}$$

for all α such that $1 - \gamma \leq \alpha < 1$. Choosing $\alpha = 1 - \gamma$ in (9), since $\gamma \in (0, 1)$ is arbitrary, we obtain

$$\tau(\gamma) = 1 - \tau(1 - \gamma) \tag{10}$$

for any $\gamma \in (0, 1)$. Observe now that (9) can be rewritten as

$$\tau(\gamma) = \frac{a(\alpha)\gamma}{1 + \gamma(a(\alpha) - 1)} \qquad 1 - \gamma \leq \alpha < 1, \tag{11}$$

with $a(\alpha) = \frac{\alpha}{\alpha - 1}\frac{\tau(\alpha) - 1}{\tau(\alpha)}$. We then obtain from (11) that

$$a(\alpha) = \frac{1 - \gamma}{\gamma}\frac{\tau(\gamma)}{1 - \tau(\gamma)} \qquad 1 - \gamma \leq \alpha < 1,$$

from which we conclude that $a(\alpha)$ is a constant. By comparison with (10) we see that in fact $a(\alpha) = 1$ so that $\tau(\gamma) = \gamma$ for all $\gamma \in [0, 1]$. This concludes the proof. $\square$

3. The group isomorphisms

To show that the groups Aut $(\mathbf{S})$, Aut$(\mathbf{E})$, and Aut $(\mathbf{D})$ are isomorphic we need some further results. We recall first the following proposition [3, Proposition 4.9], which is an application of the Gleason theorem.

Proposition 1. *Let* $\dim(\mathcal{H}) \geq 3$*. Given* $d \in \text{Aut}\,(\mathbf{D})$ *there is a unique* $s_d \in \text{Aut}\,(\mathbf{S})$ *such that* $s_d(P) = d(P)$ *for all* $P \in \mathbf{P}$*. Moreover, the map* $\text{Aut}\,(\mathbf{D}) \ni d \mapsto s_d \in \text{Aut}\,(\mathbf{S})$ *is an injective group homomorphism.*

Consider next the set $\mathbf{U} \cup \overline{\mathbf{U}}$ of unitary and antiunitary operators. It is a group and its kernel is set $\mathbf{T} = \{zI \mid z \in \mathbb{C}, |z| = 1\}$. The quotient space $\Sigma = \mathbf{U} \cup \overline{\mathbf{U}}/\mathbf{T}$, with the elements $[U] = \{U' \in \mathbf{U} \cup \overline{\mathbf{U}} \mid U' = zU \text{ for some } z \in \mathbb{C}, |z| = 1\}$, is a group with the multiplication $[U_1][U_2] = [U_1U_2]$. Let $\sigma \in \Sigma$ and $U \in \mathbf{U} \cup \overline{\mathbf{U}}$ be such that $U \in \sigma$. Define the function $s_\sigma : \mathbf{S} \to \mathbf{S}$, by $s_\sigma(T) := UTU^*$. Clearly, s_σ is well defined and it is an element of Aut $(\mathbf{S})$. Moreover, it is an easy exercise to confirm that $s_{\sigma_1} = s_{\sigma_2}$ if only if $\sigma_1 = \sigma_2$. Therefore, the function

$\Sigma \ni \sigma \mapsto s_\sigma \in \mathrm{Aut}\,(\mathbf{S})$ is an injective group homomorphism. The fact that it is also surjective is a formulation of the Wigner theorem [3]:

Proposition 2. *For any $s \in \mathrm{Aut}\,(\mathbf{S})$ there is a $\sigma \in \Sigma$ such that $s = s_\sigma$. Hence the function $\Sigma \ni \sigma \mapsto s_\sigma \in \mathrm{Aut}\,(\mathbf{S})$ is a group isomorhism.*

Finally, define the function $e_\sigma : \mathbf{E} \to \mathbf{E}$, by $e_\sigma(E) := UEU^*$. As above, the function $\Sigma \ni \sigma \mapsto e_\sigma \in \mathrm{Aut}(\mathbf{E})$ is an injective group homomorphism.

Now suppose that the dimension of the Hilbert space is greater than two and consider the following diagram.

$$\begin{tikzcd}
\mathrm{Aut}(\mathbf{E}) \arrow[r] & \mathrm{Aut}\,(\mathbf{D}) \arrow[d] \\
\Sigma \arrow[u] & \mathrm{Aut}\,(\mathbf{S}) \arrow[l]
\end{tikzcd}$$

Each arrow in the diagram is an injective group homomorphism. To show that the groups are isomorphic it suffices to show that the map $\Sigma \to \Sigma$ obtained by composing the arrows is the identity function. Let $\sigma \in \Sigma$ and $U \in \sigma$. Then $e_\sigma(E) = UEU^*$ and $d_{e_\sigma}(D) = UDU^*$. By Propositon 1 there is a unique $s_{d_{e_\sigma}} \in \mathrm{Aut}\,(\mathbf{S})$ such that $s_{d_{e_\sigma}}(P) = d_{e_\sigma}(P) = UPU^*$ for all $P \in \mathbf{P}$. Since any $T \in \mathbf{S}$ can be expressed in the form $T = \sum w_i P_i$ (where the series converges in the trace norm) and the state automorphisms are continuous (in the trace norm) we then have $s_{d_{e_\sigma}}(T) = \sum w_i s_{d_{e_\sigma}}(P_i) = \sum w_i U P_i U^* = UTU^* = s_\sigma(T)$ for all $T \in \mathbf{S}$. Hence $s_{d_{e_\sigma}} = s_\sigma$ and the proof is complete because the map of Proposition 2 is an isomorphism.

References

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