INTEGRABILITY OF THE QUANTUM ADIABATIC EVOLUTION AND GEOMETRIC PHASES

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ABSTRACT. We show that the cyclic adiabatic evolution of a quantum system is completely integrable as a classical Hamiltonian system. In this context the Berry phases arise naturally as cohomology of the invariant tori.

1. INTRODUCTION

In order to present the physical context of our work, we briefly recall the framework of the adiabatic quantum evolution at an intuitive, non-rigorous, level. Let us consider a quantum system with a time dependent Hamiltonian

$$H(t) = \sum_{k=1}^{\infty} E_k(t) P_k(t)$$

with a pure point spectrum and $t \in [0, T]$, where $T > 0$ represents the time duration of the evolution. We assume that the eigenprojections $P_k(t)$ and the eigenvalues $E_k(t)$ are T-periodic functions satisfying the no-crossing condition

$$E_k(t) \neq E_{k'}(t) \quad \forall t \in [0, T], \; k \neq k'.$$

The adiabatic theorem in quantum mechanics, [8], states that, in the *adiabatic limit* $T \to \infty$, we can approximate the physical evolution determined by $H(t)$ with the one determined by the *adiabatic Hamiltonian*

$$H_{\mathrm{A}}(t) = H(t) + K(t),$$

where

$$K(t) = \frac{\mathrm{i}}{2} \sum_{k=1}^{\infty} [P'_k(t), P_k(t)]$$

is the selfadjoint operator introduced by Kato. The time evolution $U_{\mathrm{A}}(t)$ determined by $H_{\mathrm{A}}(t)$ is completely characterised by the following two *adiabatic* properties:

$$P_k(t) U_{\mathrm{A}}(t) = U_{\mathrm{A}}(t) P_k(0) \tag{1}$$

$$(\phi, H_{\mathrm{A}}(t)\phi) = E_k(t) \|\phi\|^2 \quad \phi \in \mathrm{range}(P_k(t)). \tag{2}$$

The advantage of the adiabatic Hamiltonian is that in most cases one can calculate explicitly $U_{\mathrm{A}}(t)$ that has a particularly simple expression. In fact,

Date: June 30, 1999.

if the projections $P_k(t)$ are finite dimensional, one can compute the unitary evolutor $V(t)$, determined by the *Hamiltonian* $K(t)$, by solving a finite dimensional linear problem, [9]. Moreover, defining the *Berry phase operator* S by $\mathrm{e}^{\mathrm{i}TS} = V(T)$, one can show that there is a Hilbert basis $\{e_{kj}\}$ such that

$$\begin{aligned} P_k(0)e_{kj} &= e_{kj} \\ Se_{kj} &= \beta_{kj}e_{kj}, \end{aligned}$$

where $k = 1, \ldots, \infty$ and $j = 1, \ldots, \dim P_k(0) = m_k$. Letting, for each $t \in \mathbb{R}$, the basis

$$e_{kj}(t) = V(t)e_{kj},$$

one shows that

$$U_{\mathrm{A}}(t) = \sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\int_0^t E_k(\tau)d\tau} \sum_{j=1}^{m_k} (e_{kj}, \cdot)e_{kj}(t).$$

Hence, for $t = T$,

$$U_{\mathrm{A}}(T) = \sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\int_0^T E_k(\tau)d\tau} \sum_{j=1}^{m_k} \mathrm{e}^{\mathrm{i}T\beta_{kj}}(e_{kj}, \cdot)e_{kj}$$

where $\left(-\int_0^T E_k(\tau)d\tau\right)$ is the usual *dynamical phase* and $(T\beta_{kj})$ is the *geometric phase*, first discovered by Berry in his seminal paper, [4].

In this paper we deal with the problem of clarifying the *geometric* role of the phases β_{kj}, that is of the operator S. In order to do this we can ignore the physical Hamiltonian $H(t)$ (giving rise only to the dynamical phases) in the adiabatic Hamiltonian $H_{\mathrm{A}}(t)$. In other words we can consider a fictitious quantum system with the *geometric* Hamiltonian $K(t)$ and study the corresponding dynamical evolution $V(t)$.

From a mathematical point of view, the geometric evolution $V(t)$ can be characterised in terms of the following two conditions

$$P_k(t)V(t) = V(t)P_k(0) \tag{3}$$

$$(V(t)\phi, \tfrac{d}{dt}V(t)\phi) = 0 \quad \phi \in \mathrm{range}(P_k(0)). \tag{4}$$

The first property refers to the fact that $V(t)$ is *adiabatic*, as well as $U_{\mathrm{A}}(t)$, see Eq. (1), and the second one is equivalent to the assumption that the dynamical phases are zero, see Eq. (2).

The first interpretation of the geometric phases has been found by Simon in his paper, [11], where they are seen as holonomy of a connection and Eq. (4) is exactly the condition of parallel transport for the vector $V(t)\phi$ in the eigensubspace $P_k(t)$. We propose a different point of view and a different interpretation. We associate to the quantum system an equivalent classical Hamiltonian one and we show that the classical evolution is completely integrable (in the spirit of the Arnold-Liouville theorem). In this way we obtain an integral representation of the Berry phase operator S in terms of the cohomology of the invariant tori. The idea of connecting the geometric phases to symplectic geometry can be traced back to [5].

In the finite dimensional case the action variables associated to the classical Hamiltonian system could also be expressed in terms of the "geometric angles" introduced by [1] and [2] in a different context. This fact suggests the possibility of a further investigation of the relation between the approaches of the present paper and of [2].

The paper is organised in the following way. In Sec. 2 we state precisely the mathematical framework of the paper and we define the classical Hamiltonian system associated to the quantum adiabatic evolution. In particular, we suppose that the classical flow, determined by the quantum Hamiltonian $K(t)$, is sufficiently smooth; since $K(t)$ depends only on the family $(P_k(t))_{k\geq 1}$ we express all the hypotheses in terms of the family $(P_k(t))_{k\geq 1}$. Moreover, to simplify the notation we assume $T = 2\pi$. In Sec. 3 we propose a new definition of complete integrability for a classical Hamiltonian system, since the usual notion in terms of action-angle variables does not work in the present case, having our system an infinite number of degrees of freedom. In the final section we prove that the classical system defined by $K(t)$ is completely integrable and we obtain the integral representation of S. This result is the central core of the paper. In the appendix we collect the proofs of the various statements.

2. The classical Hamiltonian system

Let $\mathcal{H}$ be a complex separable Hilbert space with the scalar product denoted by $(\cdot,\cdot)$. Let

$$\mathbb{R} \ni t \mapsto (P_k(t))_{k=1}^{\infty}$$

be a family of 2π-periodic projection valued measures (PVM) on $\mathcal{H}$, that is, for all $t \in \mathbb{R}$,

$$\begin{aligned} P_k(t)P_n(t) &= 0 \qquad k \neq n \quad k,n \geq 1 \\ \text{s-}\textstyle\sum_k P_k(t) &= I \\ P_k(t+2\pi) &= P_k(t) \quad k \geq 1, \end{aligned}$$

where we denote with s$-$ the convergence in the strong operator topology. In the sequel we refer to this topology on $\mathcal{L}(\mathcal{H})$ as SOT. The SOT-derivative of a map $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H})$ will be denoted by $A'(t)$ while the derivative of mappings $\mathbb{R} \ni t \mapsto \phi(t) \in \mathcal{H}$ will be denoted by $\frac{d\phi}{dt}(t)$.

We say that an SOT-differentiable map

$$\mathbb{R} \ni t \mapsto V(t) \in \mathcal{L}(\mathcal{H})$$

is a *geometric evolution* (with respect to the family $(P_k(t))_{k=1}^{\infty}$) if, for all $t \in \mathbb{R}$, $V(t)$ is unitary, $V(0) = 0$ and

$$P_k(t)V(t) = V(t)P_k(0) \tag{5}$$

$$(V(t)\phi, V'(t)\phi) = 0 \quad \phi \in \text{range}(P_k(0)). \tag{6}$$

The following assumptions on the family $t \mapsto (P_k(t))_{k\geq 1}$ are equivalent to the existence of an SOT-C^2-geometric evolution, that is enough to assure that the Hamiltonian function and the classical flow will be C^1.

H1: For all $k \geq 1$ the map $t \mapsto P_k(t)$ is SOT-C^2.

H2: For all $t \in \mathbb{R}$ the series $\frac{\mathrm{i}}{2}\mathrm{s}-\sum_k [P_k'(t), P_k(t)]$ converges to a (selfadjoint, bounded) operator $K(t)$ and the map $t \mapsto K(t)$ is SOT-continuous.

H3: For all $t \in \mathbb{R}$ the series $\frac{\mathrm{i}}{2}\mathrm{s}-\sum_k [P_k''(t), P_k(t)]$ converges to a (selfadjoint, bounded) operator $K_1(t)$ and the map $t \mapsto K_1(t)$ is SOT-continuous.

We have

Proposition 1. *If the conditions* H1*,* H2 *and* H3 *hold, then there is a unique SOT-C^2 geometric evolution*

$$\mathbb{R} \ni t \mapsto V(t) \in \mathcal{L}(\mathcal{H})$$

with respect to the family $(P_k(t))_{k=1}^{\infty}$.

Moreover the following properties hold:

1. $t \mapsto V(t)$ *is the unique solution of the Cauchy problem*

$$\left\{ \begin{array}{rcl} \mathrm{i}V'(t) & = & K(t)V(t) \\ V(0) & = & I; \end{array} \right. \tag{7}$$

2. $V(t+2\pi) = V(t)V(2\pi) \qquad \forall t \in \mathbb{R};$
3. *the map $t \mapsto K(t)$ is SOT-C^1 with derivative $K_1(t)$;*
4. *the Hilbert dimension of the range of $P_k(t)$ does not depend on t and will be denoted by m_k (possibly $m_k = \infty$).*

On the other hand:

Proposition 2. *Suppose that $\mathbb{R} \ni t \mapsto \widehat{V}(t) \in \mathcal{L}(\mathcal{H})$ is an SOT-C^2 geometric evolution, then* H1*,* H2 *and* H3 *hold and $\widehat{V}(t)$ is the geometric evolution $V(t)$ defined by Proposition* 1*.*

From now on we assume that H1, H2 and H3 hold and we consider the quantum evolution given by the Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\psi(t) = K(t)\psi(t).$$

Following Chernoff and Marsden, [6], we can define an Hamiltonian *function* on $\mathcal{H}$ such that the corresponding Hamilton equations are the Schrödinger equation for the quantum evolution. However in this way the Hamiltonian function is time-dependent. With a standard trick we pass to a time-independent classical Hamiltonian system; to this aim we enlarge the phase space introducing two fictitious degrees of freedom corresponding to energy and time. In order to discuss the complete integrability of the system we force the time coordinate to be 2π-periodic; this is possible since $K(t)$ is periodic.

We recall that $\mathcal{H}$ is in a natural way a real symplectic manifold with respect to the differential structure of $\mathcal{H}$ as a real Banach space and to the symplectic 2-form

$$\omega_\phi(\psi_1, \psi_2) = -2\Im\mathrm{m}(\psi_1, \psi_2) \qquad \phi, \psi_1, \psi_2 \in \mathcal{H}.$$

Moreover, ω is exact since $\omega = d\theta$, with

$$\theta_\phi(\psi) = \Im\mathrm{m}(\psi, \phi) \qquad \phi, \psi \in \mathcal{H}.$$

Let $(\mathcal{M}, \Omega, h)$ be the Hamiltonian system given by

$$\begin{aligned} \mathcal{M} &= \mathcal{H} \times \mathbb{R} \times \mathcal{T} \\ \Omega &= \omega - dE \wedge dt \\ h(\phi, E, t) &= (\phi, K(t)\phi) - E \end{aligned}$$

where $\mathcal{T}$ is the one dimensional torus $[0, 2\pi]$ with 0 and 2π identified and we denote by (ϕ, E, t) the points of $\mathcal{M}$. We slightly abuse of the notation since the 1-form dt is closed, but not exact on the torus $\mathcal{T}$. The 2-form Ω is exact since $\Omega = d\Theta$, with $\Theta = \theta - Edt$. We collect the main properties of $(\mathcal{M}, \Omega, h)$ in the following proposition.

Proposition 3. *The function h is C^1 and its Hamilton equations are*

$$\begin{aligned} \mathrm{i}\frac{d\phi}{d\eta}(\eta) &= K(t(\eta))\phi(\eta) \\ \frac{dE}{d\eta}(\eta) &= \frac{\partial h}{\partial t}((\phi(\eta)), E(\eta), t(\eta)) \\ \frac{dt}{d\eta}(\eta) &= 1. \end{aligned}$$

The corresponding flow $F : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ exists globally and is C^1, that is

- *for all $\eta \in \mathbb{R}$, $F_\eta : \mathcal{M} \to \mathcal{M}$ is C^1;*
- *for all $m \in \mathcal{M}$, $F_m : \mathbb{R} \to \mathcal{M}$ is C^1.*

We notice that there is a one to one correspondence between the solutions of the classical Hamiltonian equations with the initial conditions

$$\begin{aligned} \phi(0) &= \phi_0 \\ E(0) &= (\phi_0, K(0)\phi_0) \\ t(0) &= 0 \end{aligned}$$

and the solutions of the Schrödinger equation

$$\mathrm{i}\frac{d\phi}{dt}(t) = K(t)\phi(t)$$

with the initial condition

$$\phi(0) = \phi_0.$$

3. The complete integrability

We are going to prove that the Hamiltonian system defined in the previous section is *completely integrable.*

We recall that a finite dimensional Hamiltonian system is completely integrable on an open set O if there exists a system of *action-angle* variables defined on O such that the Hamiltonian is a function of the action variables only, see for example [3]. It is a well known fact, see for example [7] and references therein, that one has problems in using this notion of complete integrability when the manifold is infinite dimensional.

The following example shows which kind of difficulties arise in the case we are dealing with.

Example 1. Consider the Hilbert space $\ell^2_{\mathbb{C}}$ as a real, symplectic Hilbert manifold with respect to the 2-form

$$\omega_x(y_1, y_2) = -2\Im\mathrm{m}(y_1, y_2) \quad x, y_1, y_2 \in \ell^2_{\mathbb{C}}.$$

Let $(\nu_i)_{k\geq 1}$ be a bounded real sequence and h be the C^∞ real function on $\ell^2_{\mathbb{C}}$

$$h(x) = \sum_{i\geq 1} \nu_i |x_i|^2 \qquad x = (x_i)_{k\geq 1} \in \ell^2_{\mathbb{C}}.$$

It is clear that $(\ell^2_{\mathbb{C}}, \omega, h)$ is a system of an *infinite* number of independent, harmonic oscillators. It is natural to require that this system is completely integrable with respect to a good definition of complete integrability. Nevertheless the corresponding action-angle variables are globally ill defined. In fact, defining the real canonical coordinates

$$\begin{aligned} p_i(x) &= \sqrt{2}\,\Im\mathrm{m}\, x_i \\ q_i(x) &= \sqrt{2}\,\Re\mathrm{e}\, x_i, \end{aligned}$$

one has

$$h = \sum_{i\geq 1} \nu_i \frac{p_i^2 + q_i^2}{2},$$

where the series converges pointwise on $\ell^2_{\mathbb{C}}$. Each harmonic oscillator (p_i, q_i) defines a corresponding couple of action-angle variables

$$\begin{aligned} I_i &= \frac{p_i^2 + q_i^2}{2} \\ \tan\varphi_i &= \frac{q_i}{p_i}. \end{aligned}$$

For each $i \geq 1$ the coordinate φ_i is well defined in the open set

$$\{x \in \ell^2_{\mathbb{C}} : x_i \neq 0\},$$

but the intersection of all these sets *is not open* in $\ell^2_{\mathbb{C}}$, contrary to the finite dimensional case $M = \mathbb{C}^n$.

This example shows that the difficulty in using the classical notion of action-angle variables is not at a fundamental level, but is due to a topological obstruction present in the infinite dimensional case. We propose an alternative definition of integrability that does not suffer from these problems.

First of all we observe that, if μ is a Radon measure on $\mathbb{R}$, $L^2(\mu)$ is a symplectic manifold with respect to the two form

$$\Omega^{\mathrm{c}}_\phi(\psi_1,\psi_2) = -2\Im\mathrm{m}\int \overline{\psi_1}\psi_2 \, d\mu,$$

that is exact since $\Omega^{\mathrm{c}} = d\theta^{\mathrm{c}}$ where

$$\theta^{\mathrm{c}}_\phi(\psi) = \Im\mathrm{m}\int \overline{\psi}\phi \, d\mu.$$

Let G be the group (with respect to pointwise multiplication)

$$G = \{u \in L^\infty(\mu) \ : \ |u(x)| = 1 \ \ \mu\text{-a.e.}\}.$$

Regarding G as a submanifold of the Banach space $L^\infty(\mu)$, it is an abelian Lie group whose Lie algebra is

$$\mathrm{Lie}(G) = \{f \in L^\infty(\mu) \ : \ f(x) \in \mathbb{R} \ \ \mu\text{-a.e.}\} = L^\infty(\mu,\mathbb{R}).$$

G acts on $L^2(\mu)$ by multiplication of functions

$$G \times L^2(\mu) \ni (u,\phi) \mapsto u\phi \in L^2(\mu).$$

The definition we propose is the following.

Definition 1. A classical Hamiltonian system $(\mathcal{M},\omega,h)$ is *completely integrable* on an open set $O \subset \mathcal{M}$ if there exists a Radon measure μ on $\mathbb{R}$ and a C^1 diffeomorphism

$$\Delta : O \to L^2(\mu)$$

from O onto $\Delta(O)$, open in $L^2(\mu)$, such that

1. Δ is symplectic, i.e. $\Delta^*(\Omega^{\mathrm{c}}) = \omega$;
2. for all $u \in G$

$$\begin{array}{rcll} u\Delta(m) & \in & \Delta(O) & \forall m \in O \\ h \circ \Delta^{-1}(u\phi) & = & h \circ \Delta^{-1}(\phi) & \forall \phi \in \Delta(O). \end{array}$$

In this case we say that Δ is a system of *generalised action-angle variables* for the Hamiltonian system $(\mathcal{M},\omega,h)$.

Our definition extends the notion of action-angle variables in the sense that the set $\{|\Delta(m)(x)|^2, \arg(\Delta(m)(x))\}_{x\in\mathbb{R}}$ plays the role of the action-angle variables at the point m. To show this fact we apply the classical method of reduction of phase space with symmetry due to Souriau, [12], and to Marsden and Weinstein, [10].

The map Δ defines a symplectic C^1-action of the abelian Lie group G on the open set O, namely

$$u[m] = \Delta^{-1}(u\Delta(m)) \quad m \in O, \ u \in G. \tag{8}$$

The group G plays the role of the phase group $\mathcal{T}^N$ in the N dimensional case.

The two form ω is exact on O since $\omega = d\Delta^*(\theta^c)$; hence the action (8) admits the *comomentum map*

$$\begin{array}{rcl} \widehat{J}: \mathrm{Lie}(G) & \to & C^1(O) \\ f & \mapsto & \langle \Delta^*(\theta^c), X_f \rangle \end{array}$$

where X_f is the vector field on O associated to f by the action (8). A simple calculation shows that

$$\widehat{J}(f)(m) = \int \overline{\Delta(m)(x)} f(x) \Delta(m)(x) d\,\mu(x). \tag{9}$$

The corresponding *momentum map* is given by

$$\begin{array}{rcl} J: O & \to & \mathrm{Lie}(G)^* \\ m & \mapsto & |\Delta(m)|^2 \end{array} \tag{10}$$

where $|\Delta(m)|^2$ is the $L^1(\mu, \mathbb{R})$ function

$$\mathbb{R} \ni x \mapsto |\Delta(m)(x)|^2 \in \mathbb{R},$$

regarding $L^1(\mu, \mathbb{R})$ as a subspace of $L^\infty(\mu, \mathbb{R})^*$.

For all $f \in L^1(\mu, \mathbb{R})$, let

$$\mathcal{M}_f = J^{-1}(f) = \{m \in O \ : \ |\Delta(m)(x)|^2 = f(x) \ \mu\text{-a.e.}\}.$$

G acts transitively on $\mathcal{M}_f$ and $\left(|\Delta(m)(x)|^2\right)_{x \in \mathbb{R}}$ labels the different orbits $\mathcal{M}_f$ (in the finite dimensional case the orbits are tori).

We are going to show that the characteristic functions of the Borel sets of $\mathbb{R}$ play the role of action variables. In fact, let B be a Borel subset of $\mathbb{R}$, then χ_B is an element of $\mathrm{Lie}(G)$. The flow F_B generated by the corresponding vector field X_B is given by

$$F_B(t,m) = \Delta^{-1}(\mathrm{e}^{-\mathrm{i}t\chi_B}\Delta(m)) \qquad m \in O, \ t \in \mathbb{R}$$

and is 2π periodic

$$F_B(t,m) = F_B(t + 2\pi, m) \quad m \in O, \ t \in \mathbb{R}.$$

It follows that, if $m \in \mathcal{M}_f$, the curve $\mathcal{C}$

$$[0, 2\pi] \ni t \mapsto F_B(t,m) \in O$$

is closed and lies on $\mathcal{M}_f$. Hence $\mathcal{C}$ defines a *cycle* on the torus $\mathcal{M}_f$. Since F_B leaves invariant its *Hamiltonian* function $\widehat{J}(\chi_B)$, we have

$$\begin{aligned} 2\pi \widehat{J}(\chi_B)(m) &= \int_0^{2\pi} \widehat{J}(\chi_B)(F_B(\eta, m)) d\eta \\ &= \int_0^{2\pi} \langle \Delta^*(\theta^c), X_B \rangle_{F_B(\eta,m)} d\eta \\ &= \int_{\mathcal{C}} \Delta^*(\theta^c) \end{aligned}$$

This formula shows that $J(\chi_B)$ can be interpreted as a *generalised action variable*, denoted simply by I_B, explicitly

$$I_B(m) = \int_B |\Delta(m)(x)|^2 d\,\mu(x) = \frac{1}{2\pi}\int_{\mathcal{C}} \Delta^*(\theta^{\mathrm{c}}). \tag{11}$$

Fixed $f \in L^1(\mu, \mathbb{R})$, let S_f be the support of f, then the stabilizer G_f (the same for all points lying on $\mathcal{M}_f$) is

$$G_f = \{u \in G : u(x) = 1 \text{ for almost all } x \in S_f\}.$$

G_f has the property that the quotient group G/G_f is in fact the subgroup of G

$$\{u \in G : u(x) = 1 \text{ for almost all } x \notin S_f\}.$$

Moreover if we define the origin of the orbit $\mathcal{M}_f$ as the point

$$m_0 = \Delta^{-1}(\sqrt{f}) \in \mathcal{M}_f,$$

for all $m \in \mathcal{M}_f$ we can interpret the only element $u_m \in G/G_f \subset G$ such that $u_m[m_0] = m$ as the set of *angle* variables of m, labelled by the index x running on S_f (we notice that, if $x \notin S_f$, the corresponding *angle* variable is ill defined). Explicitly, we obtain

$$u_m(x) = \frac{\Delta(m)(x)}{|\Delta(m)(x)|} \quad \text{for almost all } x \in S_f.$$

Finally, due to the second condition in Definition 1, the dynamical flow F of the Hamiltonian function commutes with the action of G. For each $f \in L^1(\mu, \mathbb{R})$ the flow leaves invariant the orbit $\mathcal{M}_f$; so there is a unique $u_f(t) \in G/G_f \subset G$ such that

$$u_f(t)[m] = F(t, m) \quad \forall m \in \mathcal{M}_f \ t \in \mathbb{R}.$$

Moreover, the one parameter group $t \mapsto u_f(t)$ defines an element $h_f \in \mathrm{Lie}(G)$ for the dynamics on the orbit. The element h_f is an L^∞-function on $\mathbb{R}$ and $h_f(x)$ is the frequency of the *angle* variable corresponding to the index $x \in S_f$ on the orbit.

The usual definition of action-angle variables in the finite dimensional case is a special case of the one given above. In fact let μ be the counting measure on the set $\{1, \ldots, N\}$, then

$$\begin{aligned} L^2(\mu) &= \mathbb{C}^N \\ G &= \mathcal{T}^N. \end{aligned}$$

If Δ defines a generalised system of action-angle variables for $\mathcal{M}$, we denote

$$O' = \{m \in \mathcal{M} \ : \ (\Delta(m))_k \neq 0 \ \ 1 \leq k \leq N\}$$

and we define on O' the C^1 functions

$$\begin{aligned} I_k(m) &= |(\Delta(m))_k|^2 \\ \phi_k(m) &= -\arg(\Delta(m))_k \end{aligned}$$

for all $1 \leq k \leq N$. A standard computation shows that $(I_k, \phi_k)_{k \geq 1}$ is a set of action-angle variables for $\mathcal{M}$ on O'. Conversely let $(I_k, \phi_k)_{k \geq 1}$ be a set of action-angle variables for $\mathcal{M}$. Choosing $m_0 \in \mathcal{M}$, there exists an open set $O \subset \mathcal{M}$ and $C \in \mathbb{R}$ such that

$$I_k(m) > C \quad m \in O,\ 1 \leq k \leq N.$$

Define Δ from O to $\mathbb{C}^N$ as

$$\Delta(m) = \sum_{k=1}^{N} \sqrt{I_k(m) - C}\, \mathrm{e}^{-\mathrm{i}\phi_k(m)} \mathbf{e}_k$$

where $(\mathbf{e}_k)_{k \geq 1}$ is the canonical basis of $\mathbb{C}^N$. It is straightforward to show that Δ is a system of generalised action-angle variables for $\mathcal{M}$ on O.

4. The complete integrability of the quantum adiabatic evolution

The principal object of study of this section is the Berry phase operator S which is defined as the unique bounded selfadjoint operator on $\mathcal{H}$ with spectrum contained in $[0, 1[$ such that

$$V(2\pi) = \mathrm{e}^{\mathrm{i}2\pi S}.$$

Due to Eq. (5), for all $k \geq 1$, $[V(2\pi), P_k(0)] = 0$ and

$$[S, P_k(0)] = 0.$$

Our aim is to obtain an integral representation of S. This goal will be achieved by proving that the classical Hamiltonian system $(\mathcal{M}, \Omega, h)$, equivalent to the quantum adiabatic evolution, has a system of generalised action-angle variables.

Roughly speaking, the idea is the following. Let us suppose for the moment that $\mathcal{H}$ is finite dimensional. The fact that the quantum evolution $V(t)$ satisfies the intertwining property (5) gives a natural set of first integrals for the system. In fact, let $(e_{kj})_{k \geq 1, 1 \leq j \leq m_k}$ be a Hilbert basis of $\mathcal{H}$ such that

$$\begin{aligned} P_k(0) e_{kj} &= e_{kj} \\ S e_{kj} &= \beta_{kj} e_{kj}. \end{aligned}$$

We define the C^1 functions f_{kj} from $\mathcal{M}$ to $\mathbb{R}$ as

$$f_{kj}(\phi, E, t) = |(\phi, V(t) e_{kj})|^2 \tag{12}$$

(note that they are well defined since $V(t + 2\pi) e_{kj} = \mathrm{e}^{\mathrm{i}2\pi\beta_{kj}} V(t) e_{kj}$). Due to the fact that the e_{kj} are orthogonal, the functions f_{kj} are in involution and the properties of $V(t)$ guarantee that they are first integrals. In this way the set

$$\mathcal{S} = \{f_{kj}\} \cup \{h\}$$

is a maximal, involutive set of first integrals and its elements are linearly independent on the open set

$$\{(\phi, E, t) \in \mathcal{M} \ :\ f_{kj}(\phi, E, t) > 0\}.$$

Moreover the common level surfaces of these first integrals are compact in $\mathcal{M}$ (we use here the periodicity of the time coordinate). It follows from this that we can apply the Arnold-Liouville theorem to get the action-angle variables and the equivalence with our definition in order to determine the map Δ. In the general case we need some preliminary steps.

For all $\eta \in \mathbb{R}$, let $\Gamma(\eta)$ be the unitary operator on $\mathcal{H}$

$$\Gamma(\eta) = V(\eta)\mathrm{e}^{-\mathrm{i}\eta S}$$

whose properties are summarised below.

1. $\eta \mapsto \Gamma(\eta)$ is SOT-C^1;
2. $\eta \mapsto \Gamma(\eta)$ is the unique solution of the Cauchy problem
$$\left\{ \begin{array}{rcl} \mathrm{i}\Gamma'(\eta) & = & \gamma(\eta)\Gamma(\eta) \\ \Gamma(0) & = & I \end{array} \right.$$
where, for all $\eta \in \mathbb{R}$
$$\gamma(\eta) = K(\eta) + V(\eta)SV(\eta)^{-1}$$
is a selfadjoint bounded operator;
3. for all $\eta \in \mathbb{R}$ and for all $k \geq 1$
$$\begin{array}{rcl} \Gamma(\eta + 2\pi) & = & \Gamma(\eta) \\ \gamma(\eta + 2\pi) & = & \gamma(\eta) \\ \Gamma(\eta)P_k(0) & = & P_k(\eta)\Gamma(\eta); \end{array}$$
4. $\eta \mapsto \gamma(\eta)$ is SOT-C^1.

Due to the spectral theorem, there is a Radon measure ν on $\mathbb{R}$ with compact support such that $\mathcal{H}$ is isomorphic to $L^2(\nu)$ and S, under this isomorphism, goes to the multiplicative operator by the function $s \in L^\infty(\nu)$. In the sequel we will identify $\mathcal{H}$ with $\mathrm{L}^2(\nu)$. Let $x_0 \notin \operatorname{supp} \nu$ and $\mu = \nu + \delta_{x_0}$.

Given $b \in \mathbb{R}$, let O_b be the open set in $\mathcal{M}$

$$O_b = \{(\phi, E, t) \in \mathcal{M} \ : \ (\phi, \gamma(t)\phi) - E > b\}$$

and $\Delta : O_b \to L^2(\mu)$ be the map defined by

$$\Delta(\phi, E, t)(x) = \left\{ \begin{array}{ll} (\Gamma(t)^{-1}\phi)(x) & x \neq x_0 \\ \sqrt{(\phi, \gamma(t)\phi) - E - b}\ \mathrm{e}^{-\mathrm{i}t} & x = x_0. \end{array} \right.$$

Theorem 1. *The classical Hamiltonian system $(\mathcal{M}, \Omega, h)$ is completely integrable on O_b with respect to the system of generalised action-angle variables Δ. Moreover $\Delta^*(\theta^{\mathrm{c}}) = \Theta - b\,dt$.*

We now show that the complete integrability of the adiabatic evolution provides a new geometric interpretation of the Berry phases. To obtain this result we use the methods of the reduction of phase spaces with symmetry as explained in Sec. 3.

Using Eq. (10), the momentum map J from O_b to $\mathrm{Lie}(G)^*$ is given by

$$J(\phi, E, t)(x) = \left\{ \begin{array}{ll} |(V(t)^{-1}\phi)(x)|^2 & x \neq x_0 \\ (\phi, V(t)SV(t)^{-1}\phi) + h(\phi, E, t) - b & x = x_0. \end{array} \right.$$

The level surfaces (invariant tori) are labelled by the functions $f \in L^1(\mu, \mathbb{R})$ and are explicitly given by

$$\mathcal{M}_f = \left\{ (\phi, E, t) \in O_b \ : \ \begin{array}{ll} |(V(t)^{-1}\phi)(x)|^2 = f(x) & x \neq x_0 \\ E = (\phi, \gamma(t)\phi) - b - f(x_0) & \end{array} \right\}$$

Using Eq. (11), we now compute the generalised action variables I_B. Let B be a Borel subset of $\mathbb{R}$, $x_0 \notin B$, then

$$I_B(\phi, E, t) = \int_B |(V(t)^{-1}\phi)(x)|^2 d\nu(x) \qquad (\phi, E, t) \in O_b$$

whereas, if $B = \{x_0\}$

$$I_0(\phi, E, t) = (\phi, V(t)SV(t)^{-1}\phi) + h(\phi, E, t) - b \quad (\phi, E, t) \in O_b. \tag{13}$$

If $\mathcal{H}$ is N-dimensional, ν is the counting measure on $\{1, \dots, N\}$. Let $B = \{i\}$ with $1 \leq i \leq N$, the corresponding action variable I_i is precisely one of the N first integrals (12) that characterise the *adiabatic intertwining condition* (5).

Let us look more accurately at the formula (13) that gives the action variable I_0 associated to x_0. This will lead to the integral representation of the Berry operator.

Let $(\phi, 0, 0) \in \mathcal{M}$ with $\phi \in \mathrm{range} P_k(0)$. If we choose $b < 0$, since $S \geq 0$, then $(\phi, 0, 0) \in O_b$. We denote by F the flow corresponding to the vector field $X_{\chi_{x_0}}$, by $\mathcal{C}_\phi$ the closed curve

$$[0, 2\pi] \ni \eta \mapsto F(\eta; \phi, 0, 0) \in \mathcal{M}.$$

A simple calculation shows that $F(\eta; \phi, 0, 0) = (\Gamma(\eta)\phi, 0, \eta)$. Finally, since $\Delta^*(\theta^c) = \Theta - b\, dt$ and, due to Eq. (6), $h(\phi, 0, 0) = 0$, we obtain, comparing Eq. (11) and Eq. (13),

Proposition 4. *The Berry phase operator S is completely characterised by the following two properties:*

$$\begin{aligned} SP_k(0) &= P_k(0)S \\ (\phi, S\phi) &= \frac{1}{2\pi} \int_{\mathcal{C}_\phi} \Theta \qquad \phi \in P_k(0)\mathcal{H}. \end{aligned}$$

The last relation shows that the action of S on the range of $P_k(0)$ is expressed as integral of the form Θ, defining the symplectic structure of $\mathcal{M}$, over the cycles $\mathcal{C}_\phi$ with $\phi \in \mathrm{range} P_k(0)$. The cycles $\mathcal{C}_\phi$ lie on the level surface $\mathcal{M}_{J(\phi,0,0)}$ and the pull-back of Θ on $\mathcal{M}_{J(\phi,0,0)}$ is closed. This shows that the Berry phase operator S is computed in terms of the cohomology of the invariant tori defined by the adiabatic evolution. This is the central result of the paper.

Moreover we notice that we also obtain an expression of S by means of an integral on $\mathcal{H}$. In fact, if $\mathcal{D}_\phi$ is the projection of $\mathcal{C}_\phi$ on $\mathcal{H}$, we get for all

$\phi \in P_k(0)\mathcal{H}$,

$$\begin{aligned}(\phi, S\phi) &= \frac{1}{2\pi}\int_{\mathcal{C}_m}(\theta - Edt)\\ &= \frac{1}{2\pi}\int_{\mathcal{D}_m}\theta - \frac{1}{2\pi}\int_0^{2\pi}E(\eta)d\eta\\ &= \frac{1}{2\pi}\int_{\mathcal{D}_m}\theta\end{aligned}$$

since $E(\eta) = 0$ for all η.

In particular if β is an eigenvalue of S, let ϕ be a corresponding eigenvector of S belonging to some $P_k(0)\mathcal{H}$ (this is always possible since S and $P_k(0)$ commute) then

$$2\pi\beta = \int_{\mathcal{D}_\phi}\theta,$$

where $\mathcal{D}_\phi$ is the closed curve

$$[0, 2\pi] \ni t \mapsto \Gamma(t)\phi \in \mathcal{H}.$$

The projection of $\mathcal{D}_\phi$ on the set of pure states is the quantum adiabatic cyclic evolution of the initial state $P[\phi]$ and $2\pi\beta$ is the corresponding Berry phase of the vector state ϕ.

5. Appendix

For reader's convenience we recall some standard facts on the differential calculus with respect to the strong operator topology that we are going to use freely in the sequel.

Lemma 1.

- *If* $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H})$ *is SOT-continuous, then* $t \mapsto \|A(t)\|$ *is bounded on any compact set of* $\mathbb{R}$.
- *If* $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H})$ *is SOT-*C^1*, then it is continuous in the operator norm topology.*
- *If* $\mathbb{R} \ni t \mapsto A(t), B(t) \in \mathcal{L}(\mathcal{H})$ *are SOT-*C^1*, then* $t \mapsto A(t)B(t)$ *is SOT-*C^1 *and*

$$(AB)'(t) = A'(t)B(t) + A(t)B'(t).$$

- *If* $\mathbb{R} \ni t \mapsto U(t) \in \mathcal{L}(\mathcal{H})$ *is SOT-*C^1 *and* $U(t)$ *is unitary, then* $t \mapsto U(t)^{-1}$ *is SOT-*C^1 *and*

$$(U^{-1})'(t) = -U(t)^{-1}U'(t)U(t)^{-1}.$$

In order to show Proposition 1 and Proposition 2 we need a technical lemma.

Lemma 2. *Let $\mathbb{R} \ni t \mapsto (P_k(t))_{k=1}^{\infty}$ be a family of projection valued measures (PVM) on $\mathcal{H}$ such that* H1, H2 *and* H3 *hold, then for all $t \in \mathbb{R}$ and $k \geq 1$ we have*

$$
\begin{aligned}
P_k'(t) &= \mathrm{i}[P_k(t), K(t)] && (14)\\
P_k(t)K(t)P_k(t) &= 0 && (15)\\
\mathrm{i}[P_k''(t), P_k(t)] &= K_1(t)P_k(t) + P_k(t)K_1(t) + \mathrm{i}\big[P_k(t), K(t)^2\big]. && (16)
\end{aligned}
$$

Proof. To simplify the notation, we drop out the dependence on the time t. Differentiating the relation $P_kP_n = \delta_{kn}P_k$, we obtain that

$$
\begin{aligned}
P_k'P_n &= -P_kP_n' \qquad k \neq n && (17)\\
P_k' &= P_kP_k' + P_k'P_k. && (18)
\end{aligned}
$$

Using these relations, we compute

$$
\begin{aligned}
\mathrm{i}[P_k, K] &= -\frac{1}{2}\text{s-}\sum_n [P_k, [P_n', P_n]] \\
&= \frac{1}{2}\text{s-}\sum_n \big(P_n'P_nP_k - P_nP_n'P_k - P_kP_n'P_n + P_kP_nP_n'\big) \\
&= \frac{1}{2}\left(P_k'P_k + P_kP_k' - \text{s-}\sum_n \big(P_nP_n'P_k + P_kP_n'P_n\big)\right) \\
&= \frac{1}{2}\left(P_k'P_k + P_kP_k' + \text{s-}\sum_{n\neq k} \big(P_n^2P_k' + P_k'P_n^2\big)\right) \\
&= \frac{1}{2}\text{s-}\sum_n \big(P_nP_k' + P_k'P_n\big) \\
&= P_k',
\end{aligned}
$$

so Eq. (14) is proven. Summing from 1 to n and passing to the SOT-limit, we obtain that

$$
\text{s-}\sum_k P_k' = 0. \tag{19}
$$

From Eq. (18) and Eq. (19), we deduce that

$$
\begin{aligned}
K &= \mathrm{i}\,\text{s-}\sum_k P_k'P_k && (20)\\
K &= -\mathrm{i}\,\text{s-}\sum_k P_kP_k'. && (21)
\end{aligned}
$$

Moreover by direct computation, we deduce Eq. (15) and

$$
P_kK_1P_k = 0. \tag{22}
$$

It remains to be shown Eq. (16). Differentiating Eq. (17) and Eq. (18), we have that

$$
P_i''P_j + 2P_i'P_j' + P_iP_j'' = \delta_{ij}P_i''.
$$

Multiplying on the right by P_j or on the left by P_i, we deduce that

$$\begin{aligned}
&(23) & P_i''P_j + 2P_i'P_j'P_j + P_iP_j''P_j &= 0 \quad i \neq j \\
&(24) & P_iP_i''P_j + 2P_iP_i'P_j' + P_iP_j'' &= 0 \quad i \neq j \\
&(25) & P_iP_i''P_i = -2P_i'P_i'P_i &= -2P_iP_i'P_i'.
\end{aligned}$$

In this way, we compute

$$\begin{aligned}
\mathrm{i}[P_k, K_1] &= -\frac{1}{2}\text{s-}\sum_n [P_k, [P_n'', P_n]] \\
&= \frac{1}{2}\text{s-}\sum_n \left(P_n''P_nP_k - P_nP_n''P_k - P_kP_n''P_n + P_kP_nP_n''\right) \\
&= \frac{1}{2}\left(P_k''P_k + P_kP_k'' - 2P_kP_k''P_k - \text{s-}\sum_{n\neq k}\left(P_nP_n''P_k + P_kP_n''P_n\right)\right),
\end{aligned}$$

using in the third addendum Eq. (25), in the fourth one Eq. (24) with $i = n$, $j = k$ and in the fifth one Eq. (23) with $i = k$, $j = n$, it follows that

$$\begin{aligned}
\mathrm{i}[P_k, K_1] &= \frac{1}{2}\left(P_k''P_k + P_kP_k'' + 2P_i'P_i'P_i + 2P_iP_i'P_i'\right) \\
&+ \frac{1}{2}\left(\text{s-}\sum_{n\neq k}\left(2P_nP_n'P_k' + P_nP_k'' + 2P_k'P_n'P_n + P_k''P_n\right)\right) \\
&= \frac{1}{2}\text{s-}\sum_n \left(P_nP_k'' + P_k''P_n + 2P_k'P_n'P_n + 2P_nP_n'P_k'\right),
\end{aligned}$$

eventually, taking into account Eq. (20) and Eq. (21), we deduce

$$\begin{aligned}
\mathrm{i}[P_k, K_1] &= \frac{1}{2}\left(P_k'' + P_k'' - 2\mathrm{i}P_k'K + 2\mathrm{i}KP_k'\right) \\
&= P_k'' - \mathrm{i}[P_k', K],
\end{aligned}$$

that is

$$P_k'' = \mathrm{i}[P_k', K] + \mathrm{i}[P_k, K_1]. \tag{26}$$

Using Eq. (26) and Eq. (14), we compute

$$\begin{aligned}
\mathrm{i}[P_k'', P_k] &= -\mathrm{i}[[[P_k, K], K], P_k] - [[P_k, K_1], P_k] \\
&= \mathrm{i}\left(P_kK^2 - P_kKP_kK - P_kKP_kK + P_kK^2P_k - P_kK^2P_k\right) \\
&+ \mathrm{i}\left(KP_kKP_k + KP_kKP_k - K^2P_k\right) - 2P_kK_1P_k + P_kK_1 + K_1P_k \\
&= \mathrm{i}\left[P_k, K^2\right] + K_1P_k + P_kK_1
\end{aligned}$$

where in the last equality we used Eq. (22) and Eq. (15). This ends the proof of the lemma.

□

Proof of proposition 1. Let f from $\mathbb{R} \times \mathcal{H}$ to $\mathcal{H}$ be the map

$$f(t, \phi) = -\mathrm{i}K(t)\phi \quad \phi \in \mathcal{H},\ t \in \mathbb{R};$$

since $K(t)$ is SOT-continuous by hypothesis H2, f is continuous and satisfies the Lipschitz condition in the variable ϕ locally in the variable t. Using the classical theorem of Cauchy-Lipschitz-Picard, we have that for all $\phi \in \mathcal{H}$, there is a unique C^1 map $\mathbb{R} \ni t \mapsto \psi_\phi(t) \in \mathcal{H}$, global solution of the Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\psi_\phi(t) & = & f(t, \psi_\phi(t)) \\ \psi_\phi(0) & = & \phi. \end{array} \right. \tag{27}$$

We define for all $t \in \mathbb{R}$, $V(t)$ from $\mathcal{H}$ to $\mathcal{H}$ as

$$V(t)\phi = \psi_\phi(t).$$

Since $\phi \mapsto f(t, \phi)$ is linear and $K(t)$ is selfadjoint, $V(t)$ is a linear, unitary operator. Moreover, by construction, $t \mapsto V(t)$ is SOT-C^1 and is the unique solution of the Cauchy problem (7), since Eq. (7) is equivalent to Eq. (27).

Now we show that Eq. (5) holds. Let $k \geq 1$, $\phi \in \mathcal{H}$ and ψ_1, ψ_2 be the maps from $\mathbb{R}$ to $\mathcal{H}$

$$\begin{array}{rcl} \psi_1(t) & = & P_k(t)V(t)\phi \\ \psi_2(t) & = & V(t)P_k(0)\phi. \end{array}$$

ψ_1 and ψ_2 are C^1 and satisfy $\psi_1(0) = \psi_2(0)$. Taking the derivative and using Eq. (14), we have that

$$\begin{array}{rcl} \dfrac{d\psi_1(t)}{dt} & = & \left(P_k'(t) - \mathrm{i}P_k(t)K(t)\right) V(t)\phi \\ & = & -\mathrm{i}K(t)P_k(t)V(t)\phi \\ & = & f(t, \psi_1(t)) \end{array}$$

and

$$\begin{array}{rcl} \dfrac{d\psi_2(t)}{dt} & = & -\mathrm{i}K(t)V(t)P_k(0)\phi \\ & = & f(t, \psi_2(t)). \end{array}$$

By unicity of the solution of the Cauchy problem (27), we have that $\psi_1(t) = \psi_2(t)$ for all $t \in \mathbb{R}$ and this proves Eq. (5).

The relation (6) is a simple consequence of Eq. (5), Eq. (7) and Eq. (15). The unicity of $V(t)$ is the content of the Proposition 2.

Observing that $f(t+2\pi, \cdot) = f(t, \cdot)$, we deduce that $V(t+2\pi) = V(t)V(2\pi)$.

Now we prove that $K(t)$ is SOT-C^1 and its derivative is $K_1(t)$. Since $K_1(t)$ is SOT-continuous by hypothesis H3, this amounts to show that for all $\phi \in \mathcal{H}$ and $t \in \mathbb{R}$

$$K(t)\phi = K(0)\phi + \int_0^t K_1(s)\phi \, ds.$$

Given $t \in \mathbb{R}$ and $\phi \in \mathcal{H}$, let f and f_n, $n \geq 1$, be the maps from $[0,t]$ to $\mathcal{H}$

$$\begin{aligned} f(s) &= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}[P_k''(s), P_k(s)]\phi \\ f_n(s) &= \frac{\mathrm{i}}{2}\sum_{k=1}^{n}[P_k''(s), P_k(s)]\phi. \end{aligned}$$

Using hypothesis H1, we have that f_n are continuous and the sequence $(f_n)_{n\geq 1}$ converges pointwise to the continuous map f by hypothesis H3. Moreover, using Eq. (16), we have that

$$\begin{aligned} 2\|f_n(s)\| &= \left\|\textstyle\sum_{k=1}^{n}\left(K_1(s)P_k(s) + P_k(s)K_1(s) + \mathrm{i}\big[P_k(s), K(s)^2\big]\right)\phi\right\| \\ &\leq \left\|K_1(s) - \mathrm{i}K(s)^2\right\|\|\textstyle\sum_{k=1}^{n} P_k(s)\phi\| \\ &+ \left\|\textstyle\sum_{k=1}^{n} P_k(s)\left(K_1(s) + \mathrm{i}K(s)^2\phi\right)\right\| \\ &\leq 2(\|K_1(s)\| + \|K(s)\|^2)\|\phi\| \\ &\leq C_{t,\phi}, \end{aligned}$$

where $C_{t,\phi}$ is a suitable constant depending on t and ϕ.

The above inequality shows that we can apply the dominated convergence theorem; in this way we obtain that

$$\begin{aligned} \int_0^t f(s)ds &= \lim_{n\to\infty}\int_0^t f_n(s)ds \\ &= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\int_0^t [P_k''(s), P_k(s)]ds \\ &= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}\int_0^t \frac{d}{ds}\left([P_k'(s), P_k(s)]\right)ds \\ &= \frac{\mathrm{i}}{2}\sum_{k=1}^{\infty}[P_k'(s), P_k(s)]\Big|_0^t \\ &= K(t)\phi - K(0)\phi. \end{aligned}$$

From Eq. (5), we have that the dimension of the range of $P_k(t)$ does not depend on t and this ends the proof of the proposition.

□

Proof of proposition 2. From Eq. (5), we have that

$$P_k(t) = \widehat{V}(t)P_k(0)\widehat{V}(t)^{-1},$$

so that $t \mapsto P_k(t)$ is SOT-C^2.

Let $\widehat{K}(t) = -\mathrm{i}\widehat{V}'(t)\widehat{V}(t)^{-1}$. Since $t \mapsto \widehat{V}(t)$ is SOT-C^2, $t \mapsto \widehat{K}(t)$ is SOT-C^1.

Moreover, differentiating Eq. (5), we obtain

$$\left(P_k'(t) - \mathrm{i}P_k(t)\widehat{K}(t)\right)\widehat{V}(t) = -\mathrm{i}\widehat{K}(t)\widehat{V}(t)P_k(0);$$

using again Eq. (5), we conclude that

$$P_k'(t) = \mathrm{i}\Big[P_k(t), \widehat{K}(t)\Big]. \tag{28}$$

Hence, we obtain that, for all $n \geq 1$,

$$\begin{aligned}
\frac{\mathrm{i}}{2}\sum_{k=1}^{n}[P_k'(t), P_k(t)] &= -\frac{1}{2}\sum_{k=1}^{n}\Big[\Big[P_k(t), \widehat{K}(t)\Big], P_k(t)\Big] \\
&= \frac{1}{2}\sum_{k=1}^{n}\Big(P_k(t)\widehat{K}(t) - 2P_k(t)\widehat{K}(t)P_k(t) + \widehat{K}(t)P_k(t)\Big) \\
&= \frac{1}{2}\sum_{k=1}^{n} P_k(t)\widehat{K}(t) + \widehat{K}(t)P_k(t),
\end{aligned}$$

where in the last equality we used Eq. (6). Passing to the SOT-limit, we have that hypothesis H2 holds and $K(t) = \widehat{K}(t)$; hence in the following we will not distinguish any longer between $K(t)$ and $\widehat{K}(t)$ and will use only the notation $K(t)$.

Differentiating Eq. (28), we obtain (dropping out the dependence on t)

$$\begin{aligned}
P_k'' &= \mathrm{i}[P_k', K] + \mathrm{i}[P_k, K'] \\
&= -[[P_k, K], K] + \mathrm{i}[P_k, K'] \\
&= 2KP_kK - P_kK^2 - K^2P_k + \mathrm{i}P_kK' - \mathrm{i}K'P_k. \qquad (29)
\end{aligned}$$

Moreover, differentiating Eq. (6) and using Eq. (28), we have that

$$\begin{aligned}
P_kK'P_k &= -P_k'KP_k - P_kKP_k' \\
&= -\mathrm{i}\left(P_kK^2P_k - KP_kKP_k + P_kKP_kK - P_kK^2P_k\right) \\
&= 0, \qquad (30)
\end{aligned}$$

since we have Eq. (6).

Using Eq. (29) and Eq. (28), we conclude that

$$\begin{aligned}
\mathrm{i}[P_k'', P_k] &= \mathrm{i}\left(-P_kK^2P_k - K^2P_k + P_kK^2 + P_kK^2P_k\right) + K'P_k + P_kK' \\
&= \mathrm{i}\big[P_k, K^2\big] + K'P_k + P_kK'
\end{aligned}$$

where we used Eq. (6) and Eq. (30).

Summing from 1 to n and passing to the SOT-limit, we have that

$$\frac{\mathrm{i}}{2}\text{s-}\sum_k [P_k''(t), P_k(t)] = K'(t),$$

so the hypothesis H3 holds.

Now both $t \mapsto \widehat{V}(t)$ and $t \mapsto V(t)$ satisfy the Cauchy problem (7) and $\widehat{V}(0) = V(0) = I$. Hence $V(t) = \widehat{V}(t)$ for all $t \in \mathbb{R}$ and this ends the proof. $\square$

Proof of proposition 3. The map h is well defined since $K(t)$ is 2π-periodic. By the fact that $K(t)$ is SOT-C^1, it follows that h is C^1. We define F from $\mathbb{R} \times \mathcal{M}$ to $\mathcal{M}$ as

$$F(\eta; \phi, E, t) = \big(V(t+\eta)V(t)^{-1}\phi, E(\eta), t+\eta\big)$$

where $E(\eta) = E + (V(t+\eta)V(t)^{-1}\phi, K(t+\eta)V(t+\eta)V(t)^{-1}\phi) - (\phi, K(t)\phi)$. It is immediate that F is a flow and, using the fact that $t \mapsto V(t)$ is continuous in the norm operator topology and SOT-C^1, we have that it is C^1. A simple calculation shows that

$$\frac{dF}{d\eta}(\eta; \phi, E, t) = X_h(F(\eta; \phi, E, t))$$

where X_h is the vector field corresponding to dh under the isomorphism defined by the symplectic 2-form Ω. The Hamilton equations easily follow. $\square$

Proof of theorem 1. We denote by i_ν and i_0 the immersions of $L^2(\nu) \simeq \mathcal{H}$ and $L^2(\delta_{x_0}) \simeq \mathbb{C}$ into $L^2(\mu)$ and P_ν and P_0 the corresponding projections. Hence

$$\Delta = i_\nu \circ \Delta_\nu + i_0 \circ \Delta_0$$

where

$$\begin{aligned} \Delta_\nu(\phi, E, t) &= \Gamma(t)^{-1}\phi \\ \Delta_0(\phi, E, t) &= \sqrt{(\phi, \gamma(t)\phi) - E - b}\ \mathrm{e}^{-\mathrm{i}t}. \end{aligned}$$

We split the proof in various steps.

1. Δ is C^1.
   Since $\Gamma(t)$ is continuous in norm and SOT-C^1, the map Δ_ν is C^1. Moreover γ is SOT-C^1 so that Δ_0 is C^1 on O_b. The thesis follows observing that i_ν and i_0 are obviously C$^\infty$.
2. Δ is bijective onto

   $$\Delta(O_b) = \{\psi \in L^2(\mu)\ :\ \psi(x_0) \neq 0\}.$$

   It is evident that $\Delta(\phi, E, t)(x_0) \neq 0$ for all $(\phi, E, t) \in O_b$. Define (with abuse of notation)

   $$\begin{aligned} \Delta^{-1} : \{\psi \in L^2(\mu)\ :\ \psi(x_0) \neq 0\} &\to \mathcal{M} \\ \psi &\mapsto (\phi(\psi), E(\psi), t(\psi)) \end{aligned}$$

   where

   $$\begin{aligned} t(\psi) &= -\arg\psi(x_0) \in \mathcal{T} \\ \phi(\psi) &= \Gamma(t(\psi))P_\nu\psi \\ E(\psi) &= (\phi(\psi), \gamma(t(\psi))\phi(\psi))_{\mathcal{H}} - |\psi(x_0)|^2 - b. \end{aligned}$$

   A simple calculation shows that $\Delta^{-1}(\psi) \in O_b$ for all ψ such that $\psi(x_0) \neq 0$ and that $\Delta^{-1} \circ \Delta$ is the identity on O_b and $\Delta \circ \Delta^{-1}$ is the identity on $\{\psi \in L^2(\mu)\ :\ \psi(x_0) \neq 0\}$.

3. Δ is a C^1 diffeomorphism from O_b to $\Delta(O_b)$, open in $L^2(\mu)$.
   $\Delta(O_b)$ is open since it is the inverse image of an open set with respect to the continuous function $\psi \mapsto \psi(x_0) = \int \overline{\chi_{x_0}}\psi \, d\mu$. By arguments similar to those of Δ, one verifies that Δ^{-1} is C^1.
4. $u(\Delta(O_b)) \subset \Delta(O_b)$ and $h \circ \Delta^{-1}(u\psi) = h \circ \Delta^{-1}(\psi)$ for all $\psi \in \Delta(O_b)$ and $u \in G$.
   The first property is immediate and, by direct computation, one has
   $$h \circ \Delta^{-1}(\psi) = b + |\psi(x_0)|^2 - \int_{\{x \neq x_0\}} \overline{\psi}(x)s(x)\psi(x)d\nu(x),$$
   that is clearly invariant for the action of G.
5. $\Delta^*(\theta^c) = \Theta - b\,dt$.
   One can easily compute that, if $(\phi, E, t) \in O_b$ and $(\psi, \epsilon, \tau) \in T_{(\phi,E,t)}(\mathcal{M})$, then
   $$T\Delta_{(\phi,E,t)}(\psi, \epsilon, \tau)(x) = \begin{cases} \Gamma(t)^{-1}(\psi + i\tau\gamma(t)\phi) & x \neq x_0 \\ e^{-it} \frac{2\Re e(\phi, \gamma(t)\psi) + \tau(\phi, \gamma'(t)\phi) - 2i\tau A - \epsilon}{2\sqrt{A}} & x = x_0 \end{cases}$$
   where $A = (\phi, \gamma(t)\phi) - E - b$. Now
   $$\begin{aligned} \Delta^*(\theta^c)_{(\phi,E,t)}(\psi, \epsilon, \tau) &= \theta^c_{\Delta(\phi,E,t)}\left(T\Delta_{(\phi,E,t)}(\psi, \epsilon, \tau)\right) \\ &= \Im m(\Gamma(t)^{-1}(\psi + i\tau\gamma(t)\phi), \Gamma(t)^{-1}\phi) \\ &+ \frac{1}{2}\Im m \left(\overline{2\Re e(\phi, \gamma(t)\psi) + \tau(\phi, \gamma'(t)\phi) - 2i\tau A - \epsilon}\right) \\ &= \Im m(\psi, \phi) - (E + b)\tau \\ &= (\Theta - b\,dt)_{(\phi,E,t)}(\psi, \epsilon, \tau). \end{aligned}$$
6. Δ is symplectic.
   It is a consequence of the previous point.

□

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