SQUARE-INTEGRABLE IMPRIMITIVITY SYSTEMS

G. CASSINELLI, E. DE VITO, AND A. LEVRERO

ABSTRACT. We give a definition of square-integrability for imprimitivity systems and we prove that the square-integrable ones share most of the properties of square-integrable representations of groups.

1. INTRODUCTION

In the first part of their seminal paper, [10], Duflo and Moore show the main properties of square -integrable representations for non-unimodular locally compact topological groups. By means of these general results, most of the properties shared by *continuous wavelet transforms*, *generalised Fourier operators* and *generalised coherent states* can be easily proved, [14], [15].

However, there are cases that require extensions of the above theory. For example, for classical coherent states associated with the quantum harmonic oscillator, one has to consider a representation of the Weyl-Heisenberg group that is square-integrable modulo a subgroup. To the best of our knowledge, this notion was introduced by Borel in the case of unimodular groups and central subgroups, [4], and it includes the definition of square-integrability for projective representations, [5]. In recent years, Ali extended this notion to arbitrary subgroups, considering only finite dimensional representations of such subgroups, [2].

Another example is the case of the Euclidean, Poincaré and Galilei groups, which do not have square-integrable representations at all. To treat these groups, a definition of square-integrability on homogeneous spaces was proposed by the use of sections, for a review see [1]. However, in this case, due to the dependence on the section, one loses the covariance properties of the corresponding wavelet transform as well as the existence of *orthogonality relations*.

Finally, the Gabor analysis, [11], [13], is a generalisation to an arbitrary locally compact abelian group G of the *short time Fourier transform* on the real line (this transform gives raise to the classical coherent states for a suitable choice of the window function). This theory is based on the properties of the *time-frequency shift operators* that define a projective square-integrable representation of the direct product $G \times G$.

In this paper we consider imprimitivity systems for a locally compact second countable topological group based on a locally compact second countable

Date: July 16, 1999.

topological space with a continuous action of the group. For these systems we propose a definition of square-integrability having three main features.

First of all, given a square-integrable imprimitivity system, there exists an isometry intertwining this system with a *canonical one*, playing the role of the left regular representation. Moreover, the existence of such an isometry is a sufficient condition for square-integrability. Finally, under some weak assumptions, the *discrete part* of the canonical imprimitivity system decomposes into its irreducible components by means of the existence of *orthogonality relations*. We will prove these results without the use of any theorem on the structure of imprimitivity systems.

The second feature is that, in the case of transitive imprimitivity systems, the Mackey inducing functor preserves the notion of square-integrability. Indeed, the square-integrability of such an imprimitivity system is equivalent to the square-integrability of the inducing representation of the stability subgroup.

Finally, recalling that, for groups that are semidirect products with normal abelian factor, there is a correspondence among representations and imprimitivity systems, a representation of such groups is square-integrable if and only if the corresponding imprimitivity system is square-integrable.

At the end of the paper we apply our theory to some examples. In particular, we show that the short time Fourier transform (as well as its generalisation in the framework of Gabor analysis) is the isometry associated with the imprimitivity system canonically defined by position and momentum operators for the one-dimensional quantum particle (or, in the context of signal analysis, by the time-frequency shift operators). This is an indication that our results provide the natural abstract framework for the extension of Gabor analysis to an arbitrary abelian group. The need for this general setup is witnessed, for instance, in [13].

In this paper we focus the attention on the abstract harmonic analysis viewpoint. Nevertheless, we believe that our results can be useful in the discretisation problems and their associated numerical algorithms arising in signal process and pattern recognition, see, as a source of references, the book containing papers [11] and [13]. Moreover, they can be used, for example, in the context of quantum mechanics on phase-space, [16], and in quantum tomography, [7].

2. Mathematical notations

In this section we introduce the mathematical notations we will use in the paper.

If X is a locally compact second countable (lcsc) topological space (in the following all the topological spaces are assumed to be Hausdörff), we let $C_c(X)$ be the vector space of the compactly supported continuous functions on X and $C_0(X)$ be the Banach space of the continuous functions vanishing at infinity with the sup norm $\|\cdot\|_{\mathrm{sup}}$. We consider $C_0(X)$ as a commutative C^*-algebra with respect to the pointwise multiplication $(f_1, f_2) \mapsto f_1 \cdot f_2$ and the complex conjugation $f \mapsto \overline{f}$. We denote by $\mathcal{B}(X)$ the σ-algebra of the Borel subsets of X. A *(positive) measure* is a positive linear form on $C_c(X)$ and a *(complex) bounded measure* is a continuous linear form on $C_0(X)$. As usual, we use the same symbol for the extensions of the above linear forms to $\mathcal{L}^1(X)$.

Given a lcsc topological group G with identity e, we denote by dg a left invariant Haar measure and by Δ_G its modular function. By *G-space*, we mean a lcsc topological space endowed with a continuous action of G. If X is such a space, we denote by $g[x]$ the action of $g \in G$ on the point $x \in X$, and by $G[x]$ the orbit at x. Given a map f defined on X, we let, for all $g \in G$,

$$f^g(x) = f(g^{-1}[x]) \qquad x \in X,$$

and, fixed $g \in G$, we define the operator l_g acting in $C_0(X)$ as $l_g f = f^g$. Obviously, l_g is a well-defined isometric $*$-homomorphism of $C_0(X)$ and

$$l_{g_1 g_2} = l_{g_1} l_{g_2} \qquad l_{g^{-1}} = l_g^{-1}, \qquad g_1, g_2, g \in G.$$

Moreover, given a measure ν on X, we denote by ν^g the measure on X

$$\int_X f(x) d\nu^g(x) = \int_X f^g(x) d\nu(x) \qquad f \in C_c(X).$$

A positive non-zero measure ν on X is said to be *quasi-invariant* if, for all $g \in G$, ν^g is equivalent to ν, *relatively invariant* if ν^g is proportional to ν, and *invariant* if $\nu^g = \nu$.

By *Hilbert space* we mean a complex separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ linear in the first argument. We denote by $\|\cdot\|$ the corresponding norm. We use the word *representation* (of a lcsc group G acting in a Hilbert space) to mean a unitary representation of G that is continuous with respect to the strong operator topology.

Let G be a lcsc group and X a G-space. We consider the topological product $G \times X$ as a G-space with respect to the following action of G,

$$g[(h, x)] := (gh, g[x]) \qquad g \in G,\ (h, x) \in G \times X.$$

We introduce the following notation. If ϕ_1 and ϕ_2 are two complex functions such that, for all $(g, x) \in G$, the function

$$G \ni h \mapsto \phi_1(h, x)\phi_2(h^{-1}[(g, x)]) \in \mathbb{C}$$

is integrable with respect to dg, then we define $\phi_1 \star \phi_2$ as the function on $G \times X$ given by

$$(\phi_1 \star \phi_2)(g, x) = \int_G \phi_1(h, x)\phi_2(h^{-1}[(g, x)])\, dh.$$

Moreover, a couple (M, U) is said to be an *imprimitivity system* (for G based on X) acting in a Hilbert space $\mathcal{H}$ if:

1. M is a non-degenerate *-representation of the C^*-algebra $C_0(X)$ in $\mathcal{H}$;
2. U is a representation of G in $\mathcal{H}$;

3. for all $f \in C_0(X)$ and $g \in G$
$$U_g M(f) U_g^{-1} = M(l_g f).$$

We denote by P^M the unique projection valued measure in $\mathcal{H}$ such that
$$M(f) = \int_X f(x) dP^M(x) \qquad f \in C_0(X),$$
where the integral is in the weak operator topology. Given an imprimitivity system (M, U) acting in a Hilbert space $\mathcal{H}$, according to [12], we define, for all $\phi \in C_c(G \times X)$, the operator on $\mathcal{H}$
$$\mathbb{U}(\phi) = \int_G M(\phi(g, \cdot)) U_g \, dg, \tag{1}$$
where the integral is in the strong operator topology.

Lemma 1. *The operator $\mathbb{U}(\phi)$ defined by Eq.* (1) *is well defined, bounded and $\|\mathbb{U}(\phi)\| \leq \int_G \|\phi(g, \cdot)\|_{\sup} dg$.*

Proof. Let $u \in \mathcal{H}$. Clearly, the map $g \mapsto \phi(g, \cdot) =: \Phi(g)$ is continuous from G to $C_0(X)$ and it has compact support. Since $f \mapsto M(f)$ is continuous from $C_0(X)$ in $\mathcal{L}(\mathcal{H})$ (with the operator norm), then $g \mapsto M(\Phi(g))$ is continuous from G to $\mathcal{L}(\mathcal{H})$. Moreover, the map $g \mapsto U_g u$ is continuous from G to $\mathcal{H}$, hence $g \mapsto M(\Phi(g)) U_g u$ is continuous from G to $\mathcal{H}$ and
$$\|M(\Phi(g)) U_g u\| \leq \|u\| \, \|\Phi(g)\|_{\sup}.$$
Then, the map $g \mapsto M(\Phi(g)) U_g u$ is dg-integrable, the operator $\mathbb{U}(\phi)$ is well defined and
$$\|\mathbb{U}(\phi) u\| \leq \|u\| \int_G \|\Phi(g)\|_{\sup} dg.$$
Form the above equation it follows that $\mathbb{U}(\phi)$ is bounded and $\|\mathbb{U}(\phi)\| \leq \int_G \|\phi(g, \cdot)\|_{\sup} dg$. □

Remark 1. Denote by $\mathbb{L}$ the space $C_c(G \times X)$ with the norm
$$\|\phi\| = \int_G \|\phi(g, \cdot)\|_{\sup} dg.$$
If $\phi_1, \phi_2 \in \mathbb{L}$, then one can easily check that $\phi_1 \star \phi_2$ is well defined and belongs to $\mathbb{L}$. Moreover, given $\phi \in \mathbb{L}$, the function
$$\phi^*(g, x) = \Delta_G(g^{-1}) \overline{\phi(g^{-1}, g^{-1}[x])} \qquad (g, x) \in G \times X,$$
is in $\mathbb{L}$. One can show, see, for example, Chapter 6 of [12], that, with respect to the above operations, $\mathbb{L}$ is an involutive normed algebra and the map $\phi \mapsto \mathbb{U}(\phi)$ is a $*$-representation of $\mathbb{L}$ in $\mathcal{H}$. This representation characterises completely the imprimitivity system (M, U). The completion $L^1(G, dx, C_0(X))$ of $\mathbb{L}$, with respect to the above norm, is an involutive Banach algebra and it is the analogous of the group algebra $L^1(G)$ for groups.

The following result is a version of Schur lemma for imprimitivity systems.

Lemma 2. *Let (M, U) and (N, V) be two imprimitivity systems acting in $\mathcal{H}$ and $\mathcal{K}$, respectively. Let T be a closed operator from $\mathcal{H}$ to $\mathcal{K}$ with a dense domain such that, for all $f \in C_0(X)$ and $g \in G$,*

$$N(f)V_gT \subset TM(f)U_g.$$

Suppose that (M, U) is irreducible, then T is a multiple of an isometry.

Proof. The proof is standard. Consider the selfadjoint operator T^*T. Fix $f \in C_0(X)$ and $g \in G$, then

$$\begin{aligned} M(f)U_gT^*T &= (M(l_{g^{-1}}\overline{f})U_{g^{-1}})^*T^*T \\ &\subset T^*(N(l_{g^{-1}}\overline{f})V_{g^{-1}})^*T \\ &= T^*N(f)V_gT \\ &\subset T^*TM(f)U_g. \end{aligned}$$

Moreover, let $E \mapsto Q(E)$ be the spectral measure of T^*T, then, see, for example, Th. 4.41 of [6], for all $E \in \mathcal{B}(\mathbb{R})$,

$$M(f)U_gQ(E) \subset Q(E)M(f)U_g.$$

Since $Q(E)$ is bounded the above relation is an equality. Since (M, U) is irreducible, it follows that $Q(E)$ is proportional to the identity. Hence $|T|$ is a scalar and one concludes using the polar decomposition of T. □

3. Square integrable imprimitivity system

In the following we give the definition of square-integrability for imprimitivity systems and we prove the main properties of this class of systems.

In this section, G is a lcsc topological group with a fixed left Haar measure dg and X is a G-space with a fixed quasi-invariant measure dx. We observe that quasi-invariant measures always exist, but, in general, they are not *canonical*, since two of them need not be proportional or equivalent.

First of all, we define a *canonical* measure on $G \times X$ in terms of the measures dg and dx. We denote by $dgdx$ the product measure on $G \times X$ and we let μ be the measure on $G \times X$ given by

$$\int_{G\times X} \phi(g, x)\, d\mu(g, x) = \int_{G\times X} \phi(g, g[x])\, dgdx \qquad \phi \in C_c(G \times X). \tag{2}$$

Lemma 3. *With the above notations, the measure μ is well-defined, invariant and equivalent to $dgdx$.*

Moreover, there is a measurable function $\lambda : G \times X \to]0, \infty[$, called the cocycle *of dx, such that*

1. *for dg-almost all $g \in G$,*
$$\int_X f(g^{-1}[x])\, dx = \int_X f(x)\lambda(g, x)\, dx \qquad f \in C_c(X);$$
2. *for $(dgdgdx)$-almost all $(g, h, x) \in G \times G \times X$*
$$\lambda(gh, x) = \lambda(g, h[x])\lambda(h, x); \tag{3}$$

3. *the function* $(g, x) \mapsto \lambda(g^{-1}, x)$ *is the density of* μ *with respect to* $dgdx$.

Finally, if dx' *is another quasi-invariant measure on* X *having density* α *with respect to* dx*, then, for all* $\phi \in C_c(G \times X)$,

$$\int_{G\times X} \phi(g, x) d\mu'(g, x) = \int_{G\times X} \phi(g, x)\alpha(g^{-1}[x])\, d\mu(g, x),$$

where μ' *is given by Eq.* (2)*, replacing* dx *with* dx'.

Proof. Define Ω as the map from $G\times X$ into itself given by $\Omega(g, x) = (g, g[x])$, then Ω is clearly a homeomorphism and μ is the image measure of the measure $dgdx$ with respect to Ω, so μ is well defined. We prove that it is invariant.

Let $\phi \in C_c(G \times X)$ and $g \in G$, then

$$\begin{aligned} \int_{G\times X} \phi^g(h, x)\, d\mu(h, x) &= \int_{G\times X} \phi(g^{-1}h, g^{-1}h[x])\, dhdx \\ &= \int_X \int_G \phi(h, h[x])\, dh\, dx \\ &= \int_{G\times X} \phi(h, x)\, d\mu(h, x). \end{aligned}$$

We prove that μ is equivalent to $dgdx$. To this aim, it is sufficient to prove that, given $E \in \mathcal{B}(G \times X)$, then E is $dgdx$-negligible if and only if $\Omega(E)$ is $dgdx$-negligible. The set E is negligible if and only if, for dg-almost all $g \in G$, the section E_g is dx-negligible. Since dx is equivalent to dx^g, this last condition is equivalent to the fact that, for dg-almost all $g \in G$, $g[E_g] = \Omega(E)_g$ is dx-negligible and, hence, if and only if $\Omega(E)$ is $dgdx$-negligible.

We now prove the second claim. By a standard result on G-spaces, see, for example, Th. 5.10 of [18], there is a positive and measurable function λ such that conditions in items 1.-2. above are satisfied. Let now ξ be the density of μ with respect to $dgdx$, which is a $dgdx$-locally integrable positive function, and choose $\phi \in C_c(G \times X)$ positive, then

$$\begin{aligned} \int_G \int_X \lambda(g^{-1}, x)\phi(g, x)\, dx\, dg &= \int_G \int_X \phi(g, g[x])\, dx\, dg \\ &= \int_{G\times X} \phi(g, g[x])\, dgdx \\ &= \int_{G\times X} \phi(g, x)\, d\mu(g, x) \\ &= \int_{G\times X} \phi(g, x)\xi(g, x)\, dgdx < \infty. \end{aligned}$$

From Fubini theorem, it follows that $(g, x) \mapsto \lambda(g^{-1}, x)$ is $dgdx$-locally integrable and, for $dgdx$-almost all $(g, x) \in G \times X$,

$$\lambda(g^{-1}, x) = \xi(g, x),$$

so condition in item 3. holds.

Let now dx' be a measure on X having density α with respect to dx, then, for all $\phi \in C_c(G \times X)$,

$$\begin{aligned}\int_{G\times X} \phi(g,x) d\mu'(g,x) &= \int_{G\times X} \phi(g, g[x]) dg dx' \\ &= \int_{G\times X} \phi(g, g[x]) \alpha(x) \, dg dx \\ &= \int_{G\times X} \phi(g, g[x]) \alpha(g^{-1}[x]) \, d\mu(g,x).\end{aligned}$$

So the last claim follows. □

The following definition of square-integrability for imprimitivity systems is the corner-stone of the paper.

Definition 1. *Let (M,U) be an irreducible imprimitivity system acting in a Hilbert space $\mathcal{H}$. The system (M,U) is said to be* square-integrable *with respect to dx if there are $u, v \in \mathcal{H}$, $u, v \neq 0$ and a function $c_{u,v}$ in $L^2(G \times X, \mu)$ such that, for all $\phi \in C_c(G \times X)$,*

$$\left\langle u, \mathbb{U}(\overline{\phi}) v \right\rangle = \int_{G\times X} c_{u,v}(g,x) \phi(g,x) \, d\mu(g,x). \tag{4}$$

For all $u, v \in \mathcal{H}$ we denote by $c_{u,v}$ the linear form on $C_c(G \times X)$

$$\phi \mapsto \left\langle u, \mathbb{U}(\overline{\phi}) v \right\rangle .$$

If there exists a locally μ-integrable function on $G \times X$ such that Eq. (4) holds we say that $c_{u,v}$ is a *locally μ-integrable function* and we use the same symbol $c_{u,v}$ to denote this function. We say that a vector $v \in \mathcal{H}$ is *admissible* if there is a non-zero vector $u \in \mathcal{H}$ such that $c_{u,v}$ is a function square-integrable with respect to μ. Clearly, (M,U) is square-integrable if and only if there is a non-zero admissible vector.

We will motivate our definition after the next lemma that characterises the square-integrability in terms of properties of M and U.

Lemma 4. *Let (M,U) be an imprimitivity system acting in $\mathcal{H}$. Given $u, v \in \mathcal{H}$, the following conditions are equivalent:*

1. *the linear form $c_{u,v}$ is a locally μ-integrable function;*
2. *for dg-almost all $g \in G$, the bounded measure on X*

$$f \mapsto \left\langle u, M(\overline{f}) U_g v \right\rangle$$

has a (complex) density $\eta_{u,v,g}$ with respect to dx.

If any of the above conditions is satisfied, then, for dg-almost all $g \in G$, there is a dx-negligible set A_g such that

$$\eta_{u,v,g}(x) = \lambda(g^{-1}, x) c_{u,v}(g,x) \qquad x \in X, \; x \notin A_g, \tag{5}$$

and $c_{u,v}$ belongs to $L^2(G\times X,\mu)$ if and only if

$$\int_G\int_X \lambda(g^{-1},x)^{-1}|\eta_{u,v,g}(x)|^2\,dx\,dg<\infty.$$

Proof. We prove that the first condition implies the second one. Since $c_{u,v}$ is locally μ-integrable and G is second countable, then, for almost all $g\in G$, the function $x\mapsto c_{u,v}(g,g[x])$ is locally dx-integrable and, for all $f\in C_c(X)$, the function $g\mapsto\int_X f(g[x])c_{u,v}(g,g[x])\,dx$ is locally dg-integrable.

Then, fixed $f\in C_c(X)$, for all $\varphi\in C_c(G)$, by Fubini theorem,

$$\begin{aligned}\int_G\varphi(g)\int_X f(g[x])c_{u,v}(g,g[x])\,dx\,dg &= \int_{G\times X}\varphi(g)f(x)c_{u,v}(g,x)\,d\mu(g,x)\\ &= \left\langle u,\mathbb{U}(\overline{f\otimes\varphi})v\right\rangle\\ &= \int_G\varphi(g)\left\langle u,M(\overline{f})U_gv\right\rangle dg.\end{aligned}$$

Hence, there is a dg-negligible set $A_f\subset G$, such that for all $g\notin A_f$,

$$\left\langle u,M(\overline{f})U_gv\right\rangle=\int_X f(g[x])c_{u,v}(g,g[x])\,dx$$

Since X is second countable, it follows that, for dg-almost all $g\in G$, the bounded measure $f\mapsto\left\langle u,M(\overline{f})U_gv\right\rangle$ has density $c_{u,v}(g,\cdot)$ with respect to $dx^{g^{-1}}$ and, taking into account that, by Lemma 3, for dg-almost all $g\in G$, $\lambda(g^{-1},\cdot)$ is the density of $dx^{g^{-1}}$ with respect to dx, the above bounded measure has density

$$\eta_{u,v,g}:=\lambda(g^{-1},\cdot)c_{u,v}(g,\cdot)$$

with respect to dx. The claim is proven and, in particular, Eq. (5) holds.

Conversely, assume condition 2. By definition, for all $\phi\in C_c(G\times X)$,

$$\left\langle u,\mathbb{U}(\overline{\phi})v\right\rangle=\int_G\int_X\phi(g,x)\eta_{u,v,g}(x)\,dx\,dg. \tag{6}$$

We claim that there is a μ-locally integrable function $c_{u,v}$ defined on $G\times X$ such that,

$$\left\langle u,\mathbb{U}(\overline{\phi})v\right\rangle=\int_{G\times X}\phi(g,x)c_{u,v}(g,x)\,d\mu(g,x).$$

Clearly, we can assume that $\eta_{u,v,g}$ is positive, so that $\phi\mapsto\left\langle u,\mathbb{U}(\overline{\phi})v\right\rangle$ is a positive measure on $G\times X$. We will prove that this measure has density with respect to μ. Let (ϕ_n) be a decreasing sequence of positive functions in $C_c(G\times X)$ such that $\lim_n\int_{G\times X}\phi_n d\mu=0$, then, by monotone convergence theorem and since $\phi_n\geq 0$, we have $\lim_n\phi_n=0$, μ-almost everywhere. Since μ is equivalent to $dgdx$, for dg-almost all $g\in G$,

$$\lim_n\phi_n(g,\cdot)=0\qquad dx\text{-a.e.},$$

and, hence,

$$\lim_n\left(\eta_{u,v,g}\phi_n(g,\cdot)\right)=0\qquad dx\text{-a.e.}.$$

Moreover, $\eta_{u,v,g}$ is in $\mathcal{L}^1(X, dx)$, being the density of a bounded measure; then $(\eta_{u,v,g}\phi_n(g,\cdot))_n$ is a positive decreasing sequence in $\mathcal{L}^1(X, dx)$, so, by monotone convergence theorem, for dg-almost all $g \in G$,

$$\lim_n \int_X \eta_{u,v,g}(x)\phi_n(g,x)\,dx = 0$$

and, by the same arguments,

$$\lim_n \int_G \int_X \eta_{u,v,g}(x)\phi_n(g,x)\,dx\,dg = 0,$$

where, for all $n \in \mathbb{N}$, the function $g \mapsto \int_X \eta_{u,v,g}(x)\phi_n(g,x)\,dx$ is dg-measurable, since it is equal dg-almost everywhere to the continuous function

$$g \mapsto \left\langle u, M(\overline{\phi_n(g,\cdot)})U_g v \right\rangle.$$

Hence, by Eq. (6), $\lim_n \left\langle u, \mathbb{U}(\overline{\phi_n})v \right\rangle = 0$. This shows that the measure $\phi \mapsto \left\langle u, \mathbb{U}(\overline{\phi})v \right\rangle$ has density with respect to μ, see, for example, Th. 6.5.3 of [17].

Let $c_{u,v}$ be such a density, which is a μ-locally integrable function, then, by definition, for all $\phi \in C_c(G \times X)$,

$$\left\langle u, \mathbb{U}(\overline{\phi})v \right\rangle = \int_{G\times X} \phi(g,x)c_{u,v}(g,x)\,d\mu(g,x).$$

Finally, assume that one of the two conditions holds, then, by Eq. (5) and Fubini theorem,

$$\int_G \int_X \lambda(g^{-1},x)^{-1}|\eta_{u,v,g}(x)|^2\,dx\,dg = \int_{G\times X} |c_{u,v}(g,x)|^2\,d\mu(g,x),$$

and the last claim is clear. □

The above definition of square-integrability depends only on the equivalence class of (M, U) and is motivated by the following observations.

Remark 2. With the notation of Remark 1, since $\phi \mapsto \mathbb{U}(\phi)$ is a *-representation of the involutive normed algebra $\mathbb{L}$, given $u \in \mathcal{H}$, the linear form Ω_u on $\mathbb{L}$

$$\phi \mapsto \langle \mathbb{U}(\phi)u, u\rangle$$

is of positive type, *i.e.* it satisfies $\langle \Omega_u, \phi^* \star \phi\rangle \geq 0$. It is well known, see, for example, theorem 6.28 of [12], that there is a one-to-one correspondence between cyclic imprimitivity systems and linear forms of positive type. It is natural to express the square-integrability in terms of such linear forms.

Remark 3. If X reduces to a single point, so that the imprimitivity system collapses into a representation of the group G, our definition is precisely the one given for groups. The main difference between the two notions is that, in case of groups, the Haar measure is *canonical* since it is the unique, up to a constant, invariant measure on G; whereas, in case of imprimitivity systems, the notion of square-integrability depends on the choice of the measure dx and, in the following, we will discuss carefully this point. In particular, given an imprimitivity system (M,U) square-integrable with respect to dx, Corollary 1 will prove that (M,U) is square-integrable with respect to any measure equivalent to dx. Moreover, Corollary 2 will show that there exists a *minimal* (with respect to the natural partial order among measures) quasi-invariant measure class where the measure dx can be chosen to test the square-integrability of (M,U). However, this measure class depends on the imprimitivity system, so that it is not *canonical* for X. Finally, there are many cases where there exists on X a *canonical* quasi-invariant measure or, at least, a *canonical* quasi-invariant measure class. For example.

1. If X is transitive, there is only one quasi-invariant measure class, see, for example, [12].
2. If G is a Lie group, X a manifold and the action on X is smooth, all the Lebesgue measures on X are quasi-invariant and equivalent among them, see, for example, [9].
3. If X has, by itself, a group structure compatible with its topology (so that X is a lcsc topological group) and the action of G preserves this group structure, then any left invariant Haar measure dx (unique up to a constant) of X is relatively invariant with respect to the action of G. Indeed, given $g \in G$, since $x \mapsto g[x]$ is a topological group isomorphism, for all $y \in X$ and $f \in C_c(X)$,

$$\begin{aligned} \int_X f^y(x) dx^g &= \int_X f(g^{-1}[g[y^{-1}]x])\, dx \\ & \quad (x \mapsto g[y]x) \\ &= \int_X f(g^{-1}[x])\, dx, \end{aligned}$$

then dx^g is a Haar measure of X and, hence, there is a strictly positive constant $\lambda(g)$ such that $dx^g = \lambda(g)dx$, *i.e.* dx is relatively invariant.

Remark 4. Suppose that X is the dual group of an abelian group A such that the action of G preserves the composition law of X. It is well known that there is a one-to-one correspondence between the imprimitivity systems (M,U) for G based on X and the representations V of the semidirect product $A \times' G$, where A is a G-space with respect to the dual action. Moreover, such correspondence preserves the irreducibility. By the third observation in Remark 3, any Haar measure dx of X is relatively invariant. With this choice, the square integrability of (M,U) is equivalent to the square-integrability of the corresponding representation V of $A \times' G$.

Indeed, since dx is relatively invariant, there is a positive character ρ of G such that $\lambda(g,x) = \rho(g)$ for all $g \in G$ (and $x \in X$). A simple calculation shows that a left invariant Haar measure of $A \times' G$ is $\rho(g)dadg$, where da is a Haar measure of A. Moreover, fixed $u, v \in \mathcal{H}$, for all $g \in G$, we denote by ν_g the bounded measure on X

$$f \mapsto \left\langle u, M(\overline{f})U_g v \right\rangle .$$

Then, since, for all $a \in A$, $V_a = \int_X \langle x, a\rangle \, dP^M(x)$ and, for all $g \in G$, $V_g = U_g$, we have that

$$\langle u, V_{ag} v\rangle = \int_X \overline{\langle x, a\rangle} d\nu_g(x) = \mathcal{F}(\nu_g)(a) \qquad a \in A,\ g \in G,$$

where $\mathcal{F}$ is the Fourier transform defined on the space of the bounded measures on X.

Let u, v be non-zero vectors in $\mathcal{H}$. By Fubini theorem, the continuous function $(a, g) \mapsto \langle u, V_{ag} v\rangle$ is square-integrable with respect to $\rho(g) da dg$ if and only if

$$\int_G \rho(g) \int_A |\langle u, V_{ag} v\rangle|^2 da\, dg = \int_G \rho(g) \int_A |\mathcal{F}(\nu_g)(a)|^2 da\, dg < \infty.$$

By a standard result of Fourier analysis (and Fubini theorem), this condition is equivalent to the fact that, for dg-almost all $g \in G$, the measure ν_g has a density η_g with respect to dx and

$$\int_G \rho(g) \int_X |\eta_g(x)|^2 \, dx\, dg < \infty.$$

The equivalence of the above conditions and the square-integrability of (M, U) is given by Lemma 4.

The properties of square-integrable representations can be extended to square-integrable imprimitivity systems. First of all, we define a canonical imprimitivity system playing the role of the left regular representation for groups.

Let (M^L, L) be the imprimitivity system acting in $L^2(G \times X, \mu)$ defined by

$$\begin{aligned} (M^L(f)\psi)(g, x) &= f(x)\psi(g, x) \qquad f \in C_0(X) \\ (L_h\psi)(g, x) &= \psi(h^{-1}g, h^{-1}[x]) \qquad h \in G, \end{aligned}$$

where $\psi \in L^2(G \times X, \mu)$ and the equalities hold for μ-almost all $(g, x) \in G \times X$. It is clear that (M^L, L) is an imprimitivity system.

The following theorem shows that a square-integrable imprimitivity system (M, U) acting in $\mathcal{H}$ defines an isometry from $\mathcal{H}$ to $L^2(G \times X, \mu)$ intertwining (M, U) with (M^L, L).

Theorem 1. *Let (M, U) be an imprimitivity system acting in $\mathcal{H}$ and square-integrable with respect to dx. Given an admissible vector $v \in \mathcal{H}$, $v \neq 0$, then*

1. *for all $u \in \mathcal{H}$ the linear form $c_{u,v}$ is a function in $L^2(G \times X, \mu)$;*
2. *the map W_v from $\mathcal{H}$ to $L^2(G \times X, \mu)$*

$$W_v u = c_{u,v}$$

   is a (non-zero) multiple of an isometry that intertwines (M, U) with (M^L, L);

3. *for all $\phi \in C_c(G \times X) \subset L^2(G \times X, \mu)$, $W_v^* \phi = \mathbb{U}(\phi) v$.*

Proof. We follow the proof of Duflo and Moore for representations, [10]. By definition of square-integrability and of admissible vector, there is a non-zero vector $u_0 \in \mathcal{H}$ such that $c_{u_0,v} \in L^2(G \times X, \mu)$. We claim that $c_{u_0,v} \neq 0$. Suppose the opposite, by definition of $\mathbb{U}$, it follows that, for dg-almost all $g \in G$ and all $f \in C_0(X)$, $\langle u, M(\overline{f})U_g v\rangle = 0$. By continuity, this relation holds for all $g \in G$ and, by irreducibility and $v \neq 0$, it follows that $u_0 = 0$. This is a contradiction and, hence, $c_{u_0,v} \neq 0$.

Define W_v as the operator in $L^2(G \times X, \mu)$ with domain

$$\{u \in \mathcal{H} \ : \ c_{u,v} \in L^2(G \times X, \mu)\}$$

and $W_v u = c_{u,v}$. The domain of W_v is not the null space since $c_{u_0,v}$ is square-integrable. We claim that W_v is a closed operator. Indeed, let (u_n) be a sequence in $\mathcal{H}$ converging to u such that $(W_v u_n)$ tends to ψ. Then, for all $\phi \in C_c(G \times X)$,

$$\begin{aligned}
\langle u, \mathbb{U}(\overline{\phi})v\rangle &= \lim \langle u_n, \mathbb{U}(\overline{\phi})v\rangle \\
&= \lim \int_{G\times X} (W_v u_n)(g,x)\phi(g,x)\, d\mu(g,x) \\
&= \int_{G\times X} \psi(g,x)\phi(g,x)\, d\mu(g,x).
\end{aligned}$$

So that $u \in \operatorname{dom} W_v$ and $W_v u = \psi$.

Moreover, we show that W_v intertwines (M, U) and (M^L, L). Indeed, let $f \in C_0(X)$ and $h \in G$, we have to prove that, given $u \in \operatorname{dom} W_v$, then $M(f)U_h u \in \operatorname{dom} W_v$ and $W_v M(f)U_h u = M^L(f)L_h W_v u$. Then, let $\phi \in C_c(G \times X)$

$$\begin{aligned}
\langle M(f)U_h u, \mathbb{U}(\overline{\phi})v\rangle &= \int_G \left\langle u, U_{h^{-1}} M(\overline{f\phi(g,\cdot)})U_g v\right\rangle dg \\
&= \int_G \left\langle u, M(\overline{l_{h^{-1}}(f\phi(g,\cdot))})U_{h^{-1}g} v\right\rangle dg \\
&= \int_G \left\langle u, M(\overline{l_{h^{-1}}(f\phi(hg,\cdot))})U_g v\right\rangle dg \\
&= \int_{G\times X} f(h[x])\phi(hg,h[x])c_{u,v}(g,x)\, d\mu(g,x) \\
&= \int_{G\times X} f(x)\phi(g,x)c_{u,v}(h^{-1}g,h^{-1}[x])\, d\mu(g,x) \\
&= \int_{G\times X} \phi(g,x)(M^L(f)L_h c_{u,v})(g,x)\, d\mu(g,x).
\end{aligned}$$

Since $\big(M^L(f)L_h c_{u,v}\big) \in L^2(G \times X, \mu)$, the claim follows.

Moreover, since the domain of W_v is invariant with respect to the action of (M, U) and it is not the null space, $\operatorname{dom} W_v$ is dense in $\mathcal{H}$. Hence, by Lemma 2, since (M, U) is irreducible, $\operatorname{dom} W_v = \mathcal{H}$ and W_v is a non-zero multiple of an isometry.

The claim in item 3 easily follows from the definition of W_v. $\square$

The above theorem shows that a square-integrable imprimitivity system is equivalent to the restriction of the canonical imprimitivity system (M^L, L) to an irreducible invariant closed subspace of $L^2(G \times X, \mu)$. The converse implication is showed by the following result.

Theorem 2. *Let $\mathcal{H}$ be a closed (non-null) subspace of $L^2(G \times X, \mu)$ that is invariant and irreducible with respect to the action of (M^L, L). Then, the restriction to $\mathcal{H}$ of (M^L, L) is square-integrable with respect to dx.*

Proof. Let Q be the orthogonal projection on $\mathcal{H}$. Since $\mathcal{H}$ is not the null space and $C_c(G \times X)$ is dense in $L^2(G \times X, dgdx)$, then there is $\Psi_2 \in C_c(G \times X)$ such that $Q\psi_2 \neq 0$, where $\psi_2 \in L^2(G \times X, \mu)$ is defined as

$$\psi_2(g,x) := \lambda(g^{-1}, x)^{-\frac{1}{2}}\Psi_2(g,x) \qquad (g,x) \in G \times X.$$

We claim that $Q\psi_2$ is an admissible vector for the restriction to $\mathcal{H}$ of (M^L, L). Indeed, given $\psi_1 \in \mathcal{H}$, we have to prove that the linear form $c_{\psi_1, Q\psi_2}$ is in $L^2(G \times X, \mu)$. Let $g \in G$ and $f \in C_0(X)$, then

$$\begin{aligned}
\langle \psi_1, M^L(\overline{f}) L_g Q\psi_2 \rangle &= \langle \psi_1, QM^L(\overline{f}) L_g \psi_2 \rangle \\
&= \langle Q\psi_1, M^L(\overline{f}) L_g \psi_2 \rangle \\
&= \langle \psi_1, M^L(\overline{f}) L_g \psi_2 \rangle \\
&= \int_{G \times X} \lambda(h^{-1}, x)\psi_1(h,x) f(x) \overline{\psi_2(g^{-1}h, g^{-1}[x])}\, dhdx \\
&= \int_X f(x)\eta_g(x)\, dx,
\end{aligned}$$

where η_g is defined for dx-almost all $x \in X$ as

$$\begin{aligned}
\eta_g(x) &:= \int_G \lambda(h^{-1}, x)\psi_1(h,x)\, \overline{\psi_2(g^{-1}h, g^{-1}[x])}\, dh \\
&= \int_G \lambda(h^{-1}g^{-1}, x)\psi_1(gh,x)\, \overline{\psi_2(h, g^{-1}[x])}\, dh \\
&=: \lambda(g^{-1}, x) \int_G \omega(g,x,h)\, dh.
\end{aligned}$$

According to Lemma 4, the linear form $c_{\psi_1, Q\psi_2} = c_{\psi_1, \psi_2}$ is a locally μ-integrable function Ω and for dg-almost all $g \in G$,

$$\Omega(g,x) = \lambda(g^{-1}, x)^{-1}\eta_g(x) = \int_G \omega(g,x,h)\, dh,$$

where, fixed $g \in G$, the equalities hold for dx-almost all $x \in X$. From Eq. (3), if follows that, for dg-almost all $h \in G$,

$$\lambda(g^{-1}, x) = \lambda(h^{-1}, hg^{-1}[x])\lambda(hg^{-1}, x) \qquad \mu\text{-a.e. } (g,x) \in G \times X. \tag{7}$$

Fixed $h \in G$ such that Eq. (7) holds, then

$$\begin{aligned}
\|\omega(\cdot,\cdot,h)\|^2_{L^2(G\times X,\mu)} &= \int_{G\times X} |\omega(g,x,h)|^2 \lambda(g^{-1},x)\,dgdx \\
&= \Delta_G(h)^{-1}\int_{G\times X}\Big\{|\psi_1(g,x)\overline{\psi_2(h,hg^{-1}[x])}|^2 \\
&\times \quad \lambda^2(g^{-1},x)\lambda(hg^{-1},x)^{-1}\Big\}\,dgdx \\
&= \Delta_G(h)^{-1}\int_{G\times X}\Big\{|\psi_1(g,x)\overline{\psi_2(h,hg^{-1}[x])}|^2 \\
&\times \quad \lambda(h^{-1},hg^{-1}[x])\Big\}\,d\mu(g,x) \\
&= \Delta_G(h)^{-1}\int_{G\times X}|\psi_1(g,x)\Psi_2(h,hg^{-1}[x])|^2\,d\mu(g,x),
\end{aligned}$$

where, by definition, $\Psi_2 \in C_c(G\times X)$. Then, by Minkowski inequality for integrals,

$$\begin{aligned}
\|\Omega\|_{L^2(G\times X,\mu)} &\le \int_G \|\omega(\cdot,\cdot,h)\|_{L^2(G\times X,\mu)}\,dh \\
&= \int_G \Delta_G(h)^{-\frac{1}{2}}\left\|\psi_1(g,x)\Psi_2(h,hg^{-1}[x])\right\|_{L^2(G\times X,\mu)}\,dh \\
&\le \|\psi_1\|_{L^2(G\times X,\mu)}\int_G \Delta_G(h)^{-\frac{1}{2}}\,\|\Psi_2(h,\cdot)\|_{\sup}\,dh.
\end{aligned}$$

Since $\Psi_2 \in C_c(G\times X)$, then $\Omega \in L^2(G\times X,\mu)$. So the thesis is proven. □

The following corollaries study how the square-integrability depends on the choice of the measure dx.

Corollary 1. *Let (M,U) be an imprimitivity system and dx' be a quasi-invariant measure on X having density with respect to dx. If (M,U) is square-integrable with respect to dx', then (M,U) is square integrable with respect to dx, too. In particular the square-integrability of (M,U) depends only on the equivalence class of dx.*

Proof. Denote with a prime all the objects defined replacing dx with dx' and let α be the density of dx' with respect to dx. Then, by Lemma 3, μ' has density $\beta(g,x) = \alpha(g^{-1}[x])$ with respect to μ and the map

$$\psi \mapsto \beta^{\frac{1}{2}}\psi$$

is an isometry from $L^2(G\times X,\mu')$ into $L^2(G\times X,\mu)$.

Due to the particular form of the density, this isometry intertwines $(M^{L'},L')$ and (M^L,L) . Since (M,U) is square-integrable with respect to dx', by Theorem 1, it is equivalent to a subsystem of $(M^{L'},L')$ and, so, to a subsystem of (M^L,L). Theorem 2 proves the claim. □

Nevertheless, by direct check, one can show that, passing from the measure dx' to dx, the set of admissible vectors can change.

Consider an imprimitivity system (M,U) acting in $\mathcal{H}$. The above result suggests that there could be a *minimal* quasi-invariant measure class such that (M,U) is square-integrable with respect to a measure belonging to this class. To this aim, we recall that, as a consequence of the spectral multiplicity theorem applied to M, there is a positive measure ν^M on X such that $\nu^M(E)=0$ if and only if $P^M(E)=0$; this measure is uniquely defined by M, up to an equivalence, and is quasi-invariant.

Corollary 2. *With the previous notation, let (M,U) be an imprimitivity system square-integrable with respect to dx, then ν^M has density with respect to dx.*

Moreover, if α is such a density and $Y=\{x\in X \ : \ \alpha(x)>0\}$, then, for all admissible vectors $v\in\mathcal{H}$ and all $u\in\mathcal{H}$,

$$(W_v u)(g,x)=\chi_Y(x)(W_v)(g,x) \qquad \mu\text{-a.e.}$$

Finally, the system (M,U) is square-integrable with respect to ν^M.

Proof. By means of Theorem 1, there is an admissible vector $v\in\mathcal{H}$ such that the operator W_v is an isometry intertwining P^M with P^L. Let $E\in\mathcal{B}(X)$ be dx-negligible, then, taking into account that $P^L(E)$ is the multiplicative operator by the characteristic function of the set $G\times E$, then $P^L(E)=0$ and, hence, $0=P^L(E)W_v=W_vP^M(E)$. Then, $P^M(E)=0$ and, hence, $\nu^M(E)=0$. This implies that ν^M has density with respect to dx.

Fix now an admissible vector $v\in\mathcal{H}$. With the same arguments, $P^L(Y)W_v=W_vP^M(Y)=W_v$ since the complement of Y is ν^M-negligible and the second claim follows.

Finally, we prove the square-integrability of (M,U) with respect to ν^M. Let $v\in\mathcal{H}$ be an admissible non-zero vector and $u\in\mathcal{H}$ non-zero. By means of Corollary 1, we can always suppose that α is the characteristic function of Y so that, taking into account the previous result

$$(W_v u)(g,x)=\alpha(x)(W_v u)(g,x) \qquad \mu\text{-a.e.}$$

We claim that for μ-almost all $(g,x)\in G\times X$, $\alpha(g^{-1}[x])=\alpha(x)$. Indeed, taking into account that $\mu'=\lambda'(g^{-1},x)dg\nu^M(x)$, $\mu=\lambda(g^{-1},x)dgdx$ and $\mu'=\alpha(g^{-1}[x])\mu$, it follows that, for μ-almost all $(g,x)\in G\times X$

$$\lambda'(g^{-1},x)\alpha(x)=\lambda(g^{-1},x)\alpha(g^{-1}[x]).$$

The claim follows since λ and λ' are strictly positive and α takes values 0 and 1. Fix now $\phi\in C_c(G\times X)$, then

$$\begin{aligned}
\int_{G\times X}(W_v u)(g,x)\phi(g,x)d\mu'(g,x) &= \int_{G\times X}(W_v u)(g,x)\phi(g,x)\alpha(g^{-1}[x])\,d\mu(g,x)\\
&= \int_{G\times X}\alpha(x)(W_v u)(g,x)\phi(g,x)\,d\mu(g,x)\\
&= \int_{G\times X}(W_v u)(g,x)\phi(g,x)\,d\mu(g,x)\\
&= \left\langle u,\mathbb{U}(\overline{\phi})v\right\rangle.
\end{aligned}$$

The thesis is now clear. □

From the above two corollaries it follows that (M,U) is square-integrable with respect to dx if and only if $\nu^M \ll dx$ and (M,U) is square-integrable with respect to ν^M. Since the equivalence class of ν^M is uniquely defined by M, the square-integrability of (M,U) with respect to ν^M is a property *intrinsic* to the imprimitivity system. However, in some cases, there is on X a *natural* quasi-invariant measure, which is independent of the imprimitivity system, and it is useful to study the square-integrability with respect to such a measure.

Remark 5. With the above notations, let (M,U) be square-integrable with respect to dx. Taking into account that (M,U) is irreducible, one can obtain an explicit form of the operator W_v. Indeed, by irreducibility there is a Hilbert space $\mathcal{K}$ such that, up to a unitary equivalence,

$$\begin{aligned} \mathcal{H} &= L^2(X,\nu^M,\mathcal{K}) \\ M(f)u &= fu, \end{aligned}$$

where $f \in C_0(X)$ and $u \in \mathcal{H}$. Moreover, denoted by λ_{ν^M} the cocycle of ν^M, there is a measurable function Π from $G \times X$ to the unitary group of $\mathcal{K}$ such that, for dg-almost all $g \in G$,

$$(U_g u)(x) = \lambda_{\nu^M}(g^{-1},x)^{\frac{1}{2}}\Pi(g,g^{-1}[x])u(g^{-1}[x]),$$

for $u \in \mathcal{H}$ and for dx-almost all $x \in X$, see, for example, Th. 6.10 of [18]. Then, given $u,v \in \mathcal{H}$ with v admissible, it easily follows from Lemma 4 that

$$(W_v u)(g,x) = \frac{\alpha(x)\lambda_{\nu^M}(g^{-1},x)^{\frac{1}{2}}}{\lambda(g^{-1},x)} \left\langle u(x), \Pi(g,g^{-1}[x])v(g^{-1}[x])\right\rangle_{\mathcal{K}} \qquad \mu\text{-a.e..}$$

The next goal is to prove the existence of *orthogonality relations*. To this aim, given an imprimitivity system (M,U) acting in $\mathcal{H}$ and square-integrable with respect to dx, we let

$$\mathcal{E}_{(M,U)} := \overline{\operatorname{span}}\{W_v u \;:\; u,v \in \mathcal{H}, \text{ such that } v \text{ is admissible}\}.$$

The following properties are immediate.

Lemma 5. *Let (M,U) be an imprimitivity system square-integrable with respect to dx. With the above notations, $\mathcal{E}_{(M,U)}$ is a closed subspace of $L^2(G \times X,\mu)$ invariant with respect to the action of (M^L,L).*

Moreover, let (M',U') be another imprimitivity system square-integrable with respect to dx. If (M,U) and (M',U') are equivalent, then $\mathcal{E}_{(M,U)} = \mathcal{E}_{(M',U')}$, whereas, if they are not equivalent, $\mathcal{E}_{(M,U)} \perp \mathcal{E}_{(M',U')}$.

Proof. The first statement is consequence of Theorem 1. Assume now that (M, U) and (M', U') are equivalent and let J be the unitary operator intertwining them. If $\mathcal{H}$ and $\mathcal{H}'$ are the Hilbert spaces where the two imprimitivity systems act, then, given $u', v' \in \mathcal{H}'$, v' admissible, for all $\phi \in C_c(G \times X)$

$$\begin{aligned}\int_{G\times X} \phi(g,x)(W'_{v'}u')(g,x)\, d\mu(g,x) &= \int_G \left\langle u', M'(\phi(g,\cdot))U'_g v'\right\rangle dg \\ &= \int_G \left\langle J^{-1}u', M(\phi(g,\cdot))U_g J^{-1}v'\right\rangle dg.\end{aligned}$$

Then $v = J^{-1}v'$ is admissible and $W'_{v'}u' = W_v u$ where $u = J^{-1}u'$.

If (M, U) and (M', U') are not equivalent, given $u, v \in \mathcal{H}$ with v admissible and $u', v' \in \mathcal{H}'$ with v' admissible, then

$$\left\langle W'_{v'}u', W_v u\right\rangle = \left\langle W_v^* W'_{v'}u', u\right\rangle = 0,$$

since, from Theorem 1, $W_v^* W'_{v'}$ intertwines (M', U') with (M, U), so it is the null operator and the claim follows. □

The second step is to decompose each $\mathcal{E}_{(M,U)}$ into its irreducible components. We are able to obtain this result only with a weak assumption on the measure dx.

Theorem 3. *Assume that, for all $g \in G$, $\lambda(g, \cdot)$ is locally dx-essentially bounded on X. Let (M, U) be an imprimitivity system acting in $\mathcal{H}$ and square-integrable with respect to dx, then there exists a unique positive self-adjoint operator C, said the* normalising operator *of (M, U), such that*

1. *the vector $v \in \mathcal{H}$ is admissible for (M, U) if and only if $v \in \operatorname{dom} C$;*
2. *for all $u_1, u_2 \in \mathcal{H}$, $v_1, v_2 \in \operatorname{dom} C$*

$$\langle W_{v_1}u_1, W_{v_2}u_2\rangle_{L^2(G\times X,\mu)} = \langle u_1, u_2\rangle_{\mathcal{H}} \langle Cv_2, Cv_1\rangle_{\mathcal{H}}.$$

To prove the above theorem we need some preliminary results. We define T to be the operator acting on $L^2(G \times X, \mu)$ as

$$(T\psi)(g,x) = \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1},x)}}\psi(g^{-1}, g^{-1}[x]),$$

where $\psi \in L^2(G \times X, \mu)$ and for μ-almost all $(g, x) \in G \times X$.

Lemma 6. *The operator T is well-defined, unitary and $T^2 = I$.*

Proof. Let $\psi \in L^2(G \times X, \mu)$. Then, $T\psi$ is clearly μ-measurable and

$$\begin{aligned}\int_{G\times X} |(T\psi)(g,x)|^2\, d\mu(g,x) &= \int_{G\times X} \frac{|\psi(g^{-1}, g^{-1}[x])|^2}{\Delta_G(g)\lambda(g^{-1},x)}\lambda(g^{-1},x)\, dgdx \\ &= \int_{G\times X} |\psi(g,x)|^2 \lambda(g^{-1},x)\, dgdx \\ &= \int_{G\times X} |\psi(g,x)|^2\, d\mu(g,x).\end{aligned}$$

Moreover, for μ-almost all $(g,x) \in G \times X$

$$\begin{aligned}(T^2\psi)(g,x) &= \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1},x)}}(T\psi)(g^{-1},g^{-1}[x]) \\ &= \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1},x)}}\sqrt{\frac{1}{\Delta_G(g^{-1})\lambda(g,g^{-1}[x])}}\psi(g,x) \\ &= \psi(g,x),\end{aligned}$$

where we used Eq. (3). □

We let (M^R, R) be the imprimitivity system acting in $L^2(G \times X, \mu)$ defined by

$$\begin{aligned}R_g &:= TL_gT \quad g \in G \\ M^R(f) &:= TM^L(f)T \quad f \in C_0(X).\end{aligned}$$

Explicitly, if $g \in G$ and $f \in C_0(X)$,

$$\begin{aligned}(R_g\psi)(h,x) &= \sqrt{\Delta_G(g)\lambda(g^{-1},h^{-1}[x])}\psi(hg,x) \\ (M^R(f)\psi)(h,x) &= f(h^{-1}[x])\psi(h,x),\end{aligned}$$

for all $\psi \in L^2(G \times X, \mu)$ and for μ-almost all $(h,x) \in G \times X$. The above imprimitivity system plays the role of the right regular representation.

We are now ready to prove the theorem.

Proof of Theorem 3. We follow the proof of [10]. Let $\mathcal{B} \subset B$ be the subspace of admissible vectors and, for all $u \in \mathcal{H}$, $u \neq 0$, define B_u as the operator from $\mathcal{B}$ to $L^2(G \times X, \mu)$ given by

$$B_u v = \overline{c_{u,v}}.$$

By Theorem 1, the domain of B_u is the set of $v \in \mathcal{H}$ such that $c_{u,v}$ is square-integrable and it is not the null space by definition of square integrability.

Since $\lambda(g,\cdot)$ is defined up to a dx-negligible set, due to the hypothesis on λ, we can always assume $\lambda(g,\cdot)$ to be locally bounded. Hence, for all $g \in G$ and $f \in C_c(X)$, let $S_g(f)$ be the function on X given by

$$(S_g(f))(x) = \sqrt{\Delta_G(g^{-1})\lambda(g^{-1},x)}f(x).$$

It is clear that $S_g(f)$ is a measurable bounded function and

$$M^R(S_g(f)) := \int_X S_g(f)(x)dP^R(x)$$

is a well defined bounded operator on $L^2(G \times X, \mu)$. We claim that, given $f \in C_c(X)$ and $g \in G$,

$$M^R(S_g(f))R_gB_u \subset B_uM(f)U_g. \tag{8}$$

Indeed, let $v \in \operatorname{dom} B_u$. By Lemma 4, there is a dg-negligible set A, such that for all $h \in G$, $h \notin A$, the measure

$$f_1 \mapsto \langle u, M(\overline{f_1})U_hv\rangle$$

has a density $\lambda(h^{-1},\cdot)c_{u,v}(h,\cdot)$ with respect to dx.

Let now $f \in C_c(X)$ and $g \in G$, then for all $h \notin Ag^{-1}$,

$$\begin{aligned}
\left\langle u, M(\overline{f_1})U_h M(f)U_g v\right\rangle &= \left\langle u, M(\overline{f_1}l_h f)U_{hg}v\right\rangle \\
&= \int_X f_1(x)\overline{f}(h^{-1}[x])c_{u,v}(hg,x)\lambda(g^{-1}h^{-1},x)\,dx \\
&= \int_X \big\{f_1(x)\overline{f}(h^{-1}[x])c_{u,v}(hg,x)\ \times \\
&\qquad \times\ \lambda(g^{-1},h^{-1}[x])\lambda(h^{-1},x)\big\}\,dx \\
&= \int_X f_1(x)\left(M^R(S_g(\overline{f}))R_g c_{u,v}\right)(h,x)\lambda(h^{-1},x)\,dx \\
&= \int_X f_1(x)\left(\overline{M^R(S_g(f))R_g\overline{c_{u,v}}}\right)(h,x)\lambda(h^{-1},x)\,dx.
\end{aligned}$$

Using Lemma 4 with the fact that Ag^{-1} is dg-negligible, it follows that $M(f)U_g v \in \operatorname{dom} B_u$ and $B_u M(f)U_g v = M^R(S_g(f))R_g B_u v$, so the claim is proven.

Hence, since (M,U) is irreducible, $\mathcal{B}$ is dense in $\mathcal{H}$ and, as in the proof of Theorem 1, we can prove that B_u is a closed operator and $B_u \neq 0$.

Fix $u_0 \in \mathcal{H}$ such that $\|u_0\| = 1$ and define $C = \sqrt{B_{u_0}^* B_{u_0}}$, so that $\operatorname{dom} C = \mathcal{B}$ and, for all $v_1, v_2 \in \mathcal{B}$,

$$\langle B_{u_0}v_2, B_{u_0}v_1\rangle = \langle Cv_2, Cv_1\rangle\,.$$

Let now $u_1, u_2 \in \mathcal{H}$ and $v_1, v_2 \in \mathcal{B}$. Then, by Theorem 1, we have that

$$\begin{aligned}
\langle c_{u_1,v_1}, c_{u_2,v_2}\rangle &= \langle W_{v_1}u_1, W_{v_2}u_2\rangle \\
&= \left\langle W_{v_2}^* W_{v_1}u_1, u_2\right\rangle \\
&= \epsilon_{v_2,v_1}\langle u_1, u_2\rangle\,,
\end{aligned}$$

where $\epsilon_{v_2,v_1} \in \mathbb{C}$ since $W_{v_2}^* W_{v_1}$ is in the commuting ring of (M,U) so that it is a multiple of a scalar.

With the choice $u_1 = u_2 = u_0$ in the above equation, one obtains that

$$\begin{aligned}
\epsilon_{v_2,v_1} &= \langle c_{u_0,v_1}, c_{u_0,v_2}\rangle \\
&= \langle B_{u_0}v_2, B_{u_0}v_1\rangle \\
&= \langle Cv_2, Cv_1\rangle\,.
\end{aligned}$$

The unicity of C is evident and this ends the proof. □

The condition on the cocycle λ given in the previous proposition clearly holds if dx is relatively invariant or, more generally, if the cocycle λ is continuous on $G \times X$ (in this case dx is said to be *strongly quasi-invariant*). This happens, for instance, when X is transitive, see, for example, [12]. Moreover, we use the above condition only in order to prove that the domain of C, which coincides with set of admissible vectors, is a dense subspace of $\mathcal{H}$.

Comparing the orthogonality relations for representations with the ones for imprimitivity systems, we see that C^{-2} plays the role of the formal degree operator. The main difference is that the formal degree operator is semi-invariant with respect to the action of the representation, see Theorem 3 of [10], whereas our Theorem 3 does not give any information about the covariance properties of the normalising operator C with respect to the action of the imprimitivity system.

However, when dx is relatively invariant (so that $\lambda(g,x) = \lambda(g)$) we have the following result. We denote by χ the character of G given by

$$\chi(g) = \sqrt{\Delta_G(g)\lambda(g)}$$

Corollary 3. *Assume the measure dx is relatively invariant. Let (M,U) be an imprimitivity system acting in $\mathcal{H}$ and square-integrable with respect to dx and let C be the corresponding normalising operator, then*

$$\begin{array}{rcll} M(f)C & \subset & CM(f) & f \in C_0(X) \\ U_g C U_g^{-1} & = & \chi(g)\ C & g \in G. \end{array}$$

Moreover, there is a unique isometry σ from the Hilbert space $\mathcal{L}^2(\mathcal{H})$ of Hilbert-Schmidt operators in $\mathcal{H}$ into $L^2(G \times X, \mu)$ such that for all $u \in \mathcal{H}$ and $v \in \operatorname{dom} C^{-1}$

$$\sigma(v^* \otimes u) = W_{C^{-1}v} u.$$

Finally,

1. *the range of σ is $\mathcal{E}_{(M,U)}$;*
2. *for all $f_1, f_2 \in C_0(X)$, $g_1, g_2 \in G$ and $A \in \mathcal{L}^2(\mathcal{H})$,*

$$\sigma\left(M(f_1)U_{g_1}AU_{g_2}^{-1}M(f_2)\right) = M^L(f_1)M^R(f_2)L_{g_1}R_{g_2}\sigma(A); \tag{9}$$

3. *for all $\phi \in C_c(G \times X)$ and $v \in \operatorname{dom} C^{-1}$*

$$\sigma^*(\phi)v = \mathbb{U}(\phi)C^{-1}v.$$

Remark 6. With respect to Eq. (9), notice that, since λ does not depend on x, the imprimitivity system (M^L, L) commutes with (M^R, R). Hence, an imprimitivity system for $G \times G$ based on $X \times X$ is canonically defined on $L^2(G \times X, \mu)$.

Proof of Corollary 3. We use the same notation of the proof of Theorem 3. Fixed $u_0 \in \mathcal{H}$, $\|u_0\| = 1$, one has that $B_{u_0}^* B_{u_0} = C^2$. Taking into account Eq. (8) and the fact that $S_g(f) = \chi(g^{-1})f$, one has, for all $f \in C_c(X)$ and $g \in G$,

$$\chi(g^{-1})^2 M(f)U_g C^2 \subset C^2 M(f) U_g.$$

In the above relation, let $g = e$ and $f \in C_c(X)$, then, as a consequence of the spectral theorem, for all $f \in C_c(X)$,

$$M(f)C \subset CM(f).$$

Clearly, the above equation holds for all $f \in C_0(X)$. Fix now $g \in G$, let (f_n) be a sequence in $C_c(X)$ converging to 1 pointwisely and boundedly, then, for all $u, v \in \operatorname{dom} C^2$, taking into account that $\lim_n M(f_n)u = u$,

$$\begin{aligned}
\langle C^2 u, U_g v\rangle &= \lim_n \langle C^2 u, M(f_n) U_g v\rangle \\
&= \chi(g^{-1})^2 \lim_n \langle u, M(f_n) U_g C^2 v\rangle \\
&= \chi(g^{-1})^2 \langle u, U_g C^2 v\rangle .
\end{aligned}$$

Hence, U_g leaves $\operatorname{dom} C^2$ invariant and $C^2 U_g = \chi(g^{-1})^2 U_g C^2$. Finally, the spectral theorem proves that $C = \chi(g^{-1}) U_g C U_{g^{-1}}$.

Since $\|W_{C^{-1}v} u\|_{L^2(G\times X,\mu)} = \|u\|\,\|v\|$, then σ is a well-defined isometry on $\mathcal{L}^2(\mathcal{H})$. The fact that it is onto $\mathcal{E}_{(M,U)}$ follows by the definition of $\mathcal{E}_{(M,U)}$. The other properties are consequences of the orthogonality relations and the covariance relation of C, as in the case of square-integrable representations, see Lemma 4 of [10]. □

As in the case of square-integrable representations, a reproducing formula follows from the orthogonality relations.

Corollary 4. *Assume that, for all $g \in G$, $\lambda(g, \cdot)$ is locally, dx-essentially bounded on X and let (M, U) be an imprimitivity system square-integrable with respect to dx. Let C be the normalising operator. Let $v, v_1, v_2 \in \operatorname{dom} C$ and $u \in \mathcal{H}$, then*

$$\langle Cv_2, Cv_1\rangle\, W_v u = W_{v_1} u \star W_v v_2.$$

Proof. First of all, we claim that, fixed $u, v \in \operatorname{dom} C$ and $g \in G$, for μ-almost all $(h, x) \in G \times X$,

$$\overline{W_v u(g^{-1}[(h,x)])}\lambda(h^{-1}, x) = W_u v(h^{-1}[(g,x)])\lambda(g^{-1}, x).$$

Indeed, let $f \in C_0(X)$. Since $v \in \operatorname{dom} C$, by Lemma 4, there is a dg-negligible set Ω, depending on u, v, but not on f, such that, for all $g^{-1}h \notin \Omega$,

$$\begin{aligned}
\langle u, M(l_{g^{-1}} f) U_{g^{-1}h} v\rangle &= \int_X \overline{f(g[x])}(W_v u)(g^{-1}h, x)\lambda(h^{-1}g, x)\, dx \\
&= \int_X \overline{f(x)}(W_v u)(g^{-1}[(h,x)])\lambda(h^{-1}g, g^{-1}[x])\lambda(g^{-1}, x)\, dx \\
&= \int_X \overline{f(x)}(W_v u)(g^{-1}[(h,x)])\lambda(h^{-1}, x)\, dx,
\end{aligned}$$

due to Eq. (3). Interchanging the role of $g \leftrightarrow h$ and $u \leftrightarrow v$, since $u \in \operatorname{dom} C$, for all $h^{-1}g \notin \Omega'$, where Ω' is dg-negligible, then

$$\langle v, M(l_{h^{-1}}\overline{f}) U_{h^{-1}g} u\rangle = \int_X f(x)(W_u v)(h^{-1}[(g,x)])\lambda(g^{-1}, x)\, dx.$$

On the other hand, for all $g, h \in G$,

$$\begin{aligned}\langle v, M(l_{h^{-1}}\overline{f})U_{h^{-1}g}u\rangle &= \langle U_h v, M(\overline{f})U_g u\rangle \\ &= \overline{\langle M(\overline{f})U_g u, U_h v\rangle} \\ &= \overline{\langle u, M(l_{g^{-1}}f)U_{g^{-1}h}v\rangle}.\end{aligned}$$

So, fixed $g \in G$, for almost all $h \notin g\Omega \cup g\Omega'^{-1}$, one has that

$$\int_X f(x)\overline{(W_v u)(g^{-1}[(h,x)])}\lambda(h^{-1},x)\,dx = \int_X f(x)(W_u v)(h^{-1}[(g,x)])\lambda(g^{-1},x)\,dx.$$

This relation holds for all $f \in C_0(X)$, hence, for all $h \notin g\Omega \cup g\Omega'^{-1}$ and for all $x \notin Y_h$, where Y_h is a dx-negligible set, one has that

$$\overline{(W_v u)(g^{-1}[(h,x)])}\lambda(h^{-1},x) = (W_u v)(h^{-1}[(g,x)])\lambda(g^{-1},x).$$

Since the two sides are μ-measurable functions and $g\Omega\cup g\Omega'^{-1}$ is dg-negligible, the claim follows.

Let now $u \in \mathcal{H}$, $v, v_1, v_2 \in \operatorname{dom} C$. Fixed $g \in G$, for all $f \in C_0(X)$, one has that, by the orthogonality relations,

$$\begin{aligned}\langle u, M(\overline{f})U_g v\rangle \langle Cv_2, Cv_1\rangle &= \langle W_{v_1}u, W_{v_2}M(\overline{f})U_g v\rangle \\ &= \langle W_{v_1}u, M^L(\overline{f})L_g W_{v_2}v\rangle \\ &= \int_{G\times X} \Big\{(W_{v_1}u)(h,x)f(x)\overline{(W_{v_2}v)(g^{-1}[(h,x)])} \times \\ &\qquad \times \quad \lambda(h^{-1},x)\Big\}\,dhdx \\ &= \int_{G\times X} \big\{f(x)\lambda(g^{-1},x)\,\times \\ &\qquad \times \quad (W_{v_1}u)(h,x)(W_v v_2)(h^{-1}[(g,x)])\big\}\,dhdx \\ &= \int_X \big\{f(x)\lambda(g^{-1},x)\,\times \\ &\qquad \times \int_G (W_{v_1}u)(h,x)(W_v v_2)(h^{-1}[(g,x)])\,dh\Big\}\,dx, \\ &= \int_X f(x)\lambda(g^{-1},x)(W_{v_1}u \star W_v v_2)(g,x)\,dx\end{aligned}$$

where we used the claim stated at the beginning of the proof. Moreover, Fubini theorem assures that $W_{v_1}u \star W_v v_2$ is well defined. By Lemma 4 and the above relation, it follows that, for almost all $g \in G$,

$$\langle Cv_2, Cv_1\rangle\,(W_v u)(g,\cdot) = (W_{v_1}u \star W_v v_2)(g,\cdot).$$

The thesis is now clear. □

4. Transitive imprimitivity systems

In this section we study the square-integrability of imprimitivity systems in the case that X is transitive. With this assumption, on X there is only one quasi-invariant measure class, so that the square-integrability does not depend on the measure and we can always choose (and we do) the measure dx to be strongly quasi-invariant and the corresponding cocycle λ to be a continuous function on $G\times X$, see, for example, [12]. The importance of transitive imprimitivity systems is twofold. First of all, their square-integrability can be characterised in terms of square-integrability of a representation of the stability subgroup; this latter representation is uniquely defined by the imprimitivity system by means of the inducing functor of Mackey. On the other hand, if X is not transitive, but its orbits with respect to the action of G are locally closed, any irreducible imprimitivity system based on X is completely defined in terms of an imprimitivity system based on an orbit of X and, hence, the study of square-integrability can be done by the use of the results about the transitive case.

Hence, we assume now that X is a transitive space and we fix $x_0 \in X$, so that $X = G[x_0]$. We let H be the stability subgroup of G at x_0. If (M, U) is an imprimitivity system, by the Mackey theorem there is, up to unitary equivalence, a unique representation m of H such that (M, U) is equivalent to the imprimitivity system (M^m, U^m) induced by m. We denote such representation m by $\mathrm{Res}_H^G(U)$.

Remark 7. A realisation of (M^m, U^m) is the following one. Let c be a *regular* section, [18], from X to G such that $c(x_0) = e$, and $\mathcal{K}$ the Hilbert space where m acts. Then (M^m, U^m) is given by

$$\begin{aligned} \mathcal{H} &= L^2(X, dx, \mathcal{K}) \\ (M^m(f)u)(x) &= f(x)u(x) \\ (U^m_g u)(x) &= \lambda^{\frac{1}{2}}(g^{-1}, x)m(c(x)^{-1}gc(g^{-1}[x]))u(g^{-1}[x]), \end{aligned}$$

where $u \in \mathcal{H}$, $f \in C_0(X)$, $g \in G$ and the equalities hold for dx-almost all $x \in X$.

The next theorem characterises the square-integrability of transitive imprimitivity systems.

Theorem 4. *Assume that there is $x_0 \in X$ such that $X = G[x_0]$. Let dx be strongly-quasi invariant and let (M, U) be an irreducible imprimitivity system. The following conditions are equivalent:*

1. *(M, U) is a square-integrable imprimitivity system with respect to dx;*
2. *$Res_H^G(U)$ is a square-integrable representation of the stability subgroup H of G at x_0.*

Moreover, if m is a square-integrable representation of H and (M^m, U^m) is the imprimitivity system given in Remark 7, then the corresponding normalising operator C is given by

$$\begin{aligned} \operatorname{dom} C &= \{v \in \mathcal{H} : v(x) \in \operatorname{dom} K_m^{-\frac{1}{2}},\ \int_X \gamma(x) \left\| K_m^{-\frac{1}{2}} v(x) \right\|^2 dx < \infty\} \\ (Cu)(x) &= \gamma(x)^{\frac{1}{2}} K_m^{-\frac{1}{2}} v(x), \end{aligned}$$

where $\gamma(x) = \frac{1}{\Delta_G(c(x))\lambda(c(x),x_0)}$ and K_m is the formal degree of m for a suitable choice of the Haar measure of H.

Proof. Since the square integrability depends only on the equivalence class of (M, U), we can alway assume that (M, U) is of the form (M^m, U^m), where $m = \operatorname{Res}_H^G(U)$ is an irreducible representation of H. Moreover, we recall that the cocycle λ satisfies

$$\lambda(h, x_0) = \frac{\Delta_H(h)}{\Delta_G(h)} \qquad h \in H,$$

where Δ_H is the modular function of H, see, for example, [12].

First of all, we prove the equivalence between the two conditions in the statement of the theorem.

Let $f \in C_0(X)$, $g \in G$ and $u, v \in \mathcal{H}$, then

$$\left\langle u, M(\overline{f})U_g v\right\rangle = \int_X f(x)\lambda^{\frac{1}{2}}(g^{-1}, x) \left\langle u(x), m(c(x)^{-1}gc(g^{-1}[x]))v(g^{-1}[x])\right\rangle dx.$$

so that, for all $g \in G$ the measure $f \mapsto \left\langle u, M(\overline{f})U_g v\right\rangle$ has density

$$\lambda^{\frac{1}{2}}(g^{-1}, x) \left\langle u(x), m(c(x)^{-1}gc(g^{-1}[x]))v(g^{-1}[x])\right\rangle .$$

Moreover, let $I_{u,v} := \int_G \int_X \left|\left\langle u(x), m(c(x)^{-1}gc(g^{-1}[x]))v(g^{-1}[x])\right\rangle\right|^2 dx\, dg$, where $I \in [0, \infty]$.

By Lemma 4, the system (M, U) is square integrable, *i.e.* condition 1 of the theorem holds, if and only if there exist $u, v \in \mathcal{H}$, such that $0 < I_{u,v} < \infty$.

Assume the existence of such vectors u and v. Since

$$(g, x) \mapsto \left|\left\langle u(x), m(c(x)^{-1}gc(g^{-1}[x]))v(g^{-1}[x])\right\rangle\right|^2$$

is $dgdx$-measurable and positive, due to Fubini theorem,

$$\begin{aligned} I_{u,v} &= \int_X \int_G \left|\left\langle u(x), m(c(x)^{-1}gc(g^{-1}[x]))v(g^{-1}[x])\right\rangle\right|^2 dg\, dx \\ &= \int_X \int_G \left|\left\langle u(x), m(gc(g^{-1}[x_0]))v(g^{-1}[x_0])\right\rangle\right|^2 dg\, dx \\ &= \int_X \int_G \left|\left\langle u(x), m(g^{-1}c(g[x_0]))v(g[x_0])\right\rangle\right|^2 \Delta_G(g)^{-1}\, dg\, dx. \end{aligned}$$

The Mackey-Bruhat formula implies that, if we identify G with $X \times H$ by means of $g = c(y)h$, where $y \in X$, $h \in H$, then

$$dg = \frac{\Delta_G(h)}{\Delta_H(h)\lambda(c(y), x_0)} dx dh,$$

where dh is a suitable Haar measure on H. Hence,

$$\begin{aligned} I_{u,v} &= \int_X \int_X \int_H \left|\left\langle u(x), m(h^{-1})\right) v(y) \right\rangle\right|^2 \frac{1}{\Delta_G(c(y))\lambda(c(y), x_0)\Delta_H(h)} dh\, dy\, dx \\ &= \int_X \int_X \gamma(y) \int_H \left|\langle u(x), m(h)) v(y) \rangle\right|^2 dh\, dy\, dx, \end{aligned}$$

where $\gamma(x) = \frac{1}{\Delta_G(c(y))\lambda(c(y), x_0)}$ and the map γ is bounded on the compact sets, due to the regularity of c.

Obviously, $0 < I_{u,v} < \infty$ if and only if

a) for dx-almost all $x \in X$ and, fixed $x \in X$, for dx-almost all $y \in X$,

$$0 < \int_H \left|\langle u(x), m(h)) v(y) \rangle\right|^2 dh < \infty.$$

This condition is equivalent to the fact that m is square-integrable and, due to the orthogonality relations for representations, we have that

$$\begin{aligned} v(y) &\in \operatorname{dom} K_m^{-\frac{1}{2}} \\ \int_H \left|\langle u(x), m(h)) v(y) \rangle\right|^2 dh &= \|u(x)\|^2 \left\| K_m^{-\frac{1}{2}} v(y) \right\|^2, \end{aligned}$$

where K_m is the formal degree of m;

b) for dx-almost all $x \in X$,

$$0 < \|u(x)\|^2 \int_X \gamma(y) \left\| K_m^{-\frac{1}{2}} v(y) \right\|^2 dy < \infty.$$

This is equivalent to

$$0 < \int_X \gamma(y) \left\| K_m^{-\frac{1}{2}} v(y) \right\|^2 dy < \infty,$$

and $u \neq 0$;

c)

$$0 < \int_X \|u(x)\|^2 dx < \infty.$$

Hence, the fact that $0 < I_{u,v} < \infty$ implies that m is square-integrable.

Conversely, assume that m is square-integrable. Choose any $u \in \mathcal{H}$, $u \neq 0$, and define

$$v(x) = \chi_K v_0,$$

where K is a compact non-negligible subset of X and $v_0 \in \operatorname{dom} K_m^{\frac{1}{2}}$, $v_0 \neq 0$. Then, conditions a), b) and c) hold, so that $0 < I_{u,v} < \infty$. This prove the equivalence between the two condition.

The form of the normalising operator C follows from condition b) above when $v_1 = v_2 = v$ and by polarisation identity in the arbitrary case. □

Finally, we consider the case when X is not transitive, but, nevertheless, its orbits are locally closed. With this assumption, every orbit Y is a lcsc transitive G-space and we say that M *lives* on Y if $P^M(Y) = I$. From a theorem of Glimm, see, for example, Proposition 6.6 of [12], given an irreducible imprimitivity system (M, U), there exist an orbit $Y = G[x_0]$ and an irreducible representation m of H, the stability subgroup at x_0, such that M lives on Y and U is equivalent to the induced representation $\mathrm{Ind}_H^G(m)$ by m from H to G. Moreover, we define the measure dx^Y as the restriction of dx to Y and (M^Y, U^Y) as the imprimitivity system for G based on Y given by

$$\begin{aligned} M^Y(f) &= \int_Y f(x) dP^M(x) \\ U_g^Y &= U_g, \end{aligned}$$

where $f \in C_0(Y)$ and $g \in G$. Notice that (M^Y, U^Y) acts in the same Hilbert space of (M, U) and, since $P^M(Y) = I$, (M^Y, U^Y) is irreducible. The following corollary characterises the square-integrability of (M, U) in terms of the square-integrability of (M^Y, U^Y), which is an imprimitivity system based on a transitive space.

Corollary 5. *Assume that the orbits of X are locally closed and let (M, U) be an irreducible imprimitivity system acting in $\mathcal{H}$. Denote by Y the orbit where M lives and by m the representation of H, the stability subgroup at $x_0 \in Y$, such that U is equivalent to $Ind_H^G(m)$.*

With this notations, the following conditions are equivalent

1. *(M, U) is square-integrable with respect to dx;*
2. *Y is not negligible with respect to dx and m is a square-integrable representation of H.*
3. *Y is not negligible and (M^Y, U^Y) is square-integrable with respect to dx^Y.*

If any of the above equivalent conditions is satisfied and dx^Y is strongly quasi-invariant, then $v \in \mathcal{H}$ is an admissible vector for (M, U) if and only if $v \in \mathrm{dom}\, C$, where C is the normalising operator of (M^Y, U^Y). Moreover, if $u_1, u_2 \in \mathcal{H}$ and $v_1, v_2 \in \mathrm{dom}\, C$,

$$\langle W_{v_1} u_1, W_{v_2} u_2 \rangle = \langle u_1, u_2 \rangle \langle C v_2, C v_1 \rangle .$$

Proof. We prove the equivalence between the first and third condition. If (M, U) is square-integrable, with the notation of Corollary 2, the measure ν^M has density with respect to dx and, since $P^M(Y) = I$, $\nu^M(Y) > 0$, then Y is not dx-negligible. Hence, from now on, we assume that this condition is satisfied.

Let $u, v \in \mathcal{H}$ and, given $g \in G$, let ω_g be the bounded measure on X

$$f \mapsto \langle u, M(\overline{f})U_g v\rangle$$

and ω_g^Y the measure on Y

$$\widehat{f} \mapsto \left\langle u, M^Y(\overline{\widehat{f}})U_g^Y v\right\rangle.$$

By definition of (M^Y, U^Y), ω_g^Y is the restriction to Y of the measure ω_g. On the other hand, taking into account that ω_g has density with respect to ν^M, the complement of Y is ω_g-negligible, so that ω_g is the *natural* extension of ω_g^Y.

By Lemma 4, the linear form $c_{u,v}$ on $C_c(G \times X)$ is a locally μ-integrable function if and only if, for dg-almost all $g \in G$, ω_g has density η_g with respect to dx. This last condition, taking into account that, for dx-almost all $x \notin Y$, $\eta_g(x) = 0$, is equivalent to the fact that ω^Y has density $(\eta_g)_{|Y}$ with respect to dx^Y. Moreover, if one of the above equivalent conditions is satisfied, then, for μ-almost all $(g, x) \in G \times X$, $x \notin Y$,

$$c_{u,v}(g, x) = 0,$$

and, for μ-almost all $(g, x) \in G \times Y$,

$$c_{u,v}(g, x) = c_{u,v}^Y(g, x),$$

where the apex ^{Y} refers to the quantities defined in terms of (M^Y, U^Y).

Taking into account the above relations and Lemma 4, if follows that $c_{u,v} \in L^2(G \times X, \mu)$ if and only if $c_{u,v}^Y \in L^2(G \times Y, \mu^Y)$. Hence, the vector v is admissible for (M, U) if and only if it is so for (M^Y, U^Y) and, in this case, for μ-almost all $(g, x) \in G \times Y$

$$(W_v u)(g, x) = (W_v^Y u)(g, x)$$

and, for μ-almost all $(g, x) \in G \times X$, $x \notin Y$,

$$(W_v u)(g, x) = 0.$$

The equivalence of the first and third condition is clear as well as, by means of Theorem 3 and Theorem 4, the last statement of the theorem.

The equivalence of the second and third condition is a restatement of the second part of Theorem 4, taking into account that $m = \mathrm{Res}_H^G(U^Y)$. □

5. Imprimitivity systems based on an abelian group

In the case of square-integrable representations of a locally compact group G, the Hilbert space carrying the representation is canonically embedded in $L^2(G, dg)$ as a subspace of continuous functions. This is no longer true in the case of square-integrable imprimitivity systems. However, when X is an abelian group, we can regularise the image of the operator W_v by means of Fourier transform.

In the following, we assume the G-space X to be an abelian lcsc group, the measure dx to be a Haar measure of X (considering X as an abelian group), and dx to be relatively invariant with respect to the action of G. We denote by $\widehat{X}$ the dual group of X, by $d\hat{x}$ a Haar measure of $\widehat{X}$ and by $\mathcal{F}$ the Fourier-Plancherel operator from $L^2(X, dx)$ onto $L^2(\widehat{X}, d\hat{x})$ (we normalise $d\hat{x}$ in such a way that $\mathcal{F}$ be unitary). Since dx is relatively invariant, the corresponding cocycle λ depends only on g, so we regard λ as a function on G. Moreover, we let $\widehat{\mu}$ be the measure on $G \times \widehat{X}$ given by

$$\int_{G\times\widehat{X}} \phi(g,\hat{x}) d\widehat{\mu}(g,\hat{x}) = \int_{G\times\widehat{X}} \phi(g,\hat{x})\lambda(g^{-1})\, dg d\hat{x} \qquad \phi \in C_c(G \times \widehat{X}).$$

Hence, we define J as the unitary operator from $L^2(G \times X, \mu)$ onto $L^2(G \times \widehat{X}, \widehat{\mu})$ given by

$$(J\psi)(g,\hat{x}) = \left(\mathcal{F}\psi(g,\cdot)\right)(\hat{x}) \qquad g \in G,\ \hat{x} \in \widehat{X},\ \psi \in L^2(G \times X, \mu).$$

Finally, let (M, U) be an imprimitivity system square-integrable with respect to dx. For all $\hat{x} \in \widehat{X}$, let $V_{\hat{x}} = \int_X \langle x, \hat{x}\rangle\, dP^M(x)$, where the integral is in the weak operator topology. Moreover, we denote by C the corresponding normalising operator. Hence, given $v \in \operatorname{dom} C$, we define $\widehat{W_v} = JW_v$.

Theorem 5. *Assume that X is an abelian lcsc group and its Haar measure dx is relatively invariant. Let (M, U) be an imprimitivity system acting in $\mathcal{H}$. If (M, U) is square-integrable with respect to dx, then*

1. *given $v \in \operatorname{dom} C$, for all $u \in \mathcal{H}$,*
$$(\widehat{W_v}u)(g,\hat{x}) = \lambda(g)\, \langle u, V_{\hat{x}} U_g v\rangle \qquad \widehat{\mu}\text{-a.e.},$$
in particular, $\widehat{W_v}u$ has a continuous representative in $\mathcal{L}^2(G \times \widehat{X}, \widehat{\mu})$;
2. *for all $u_1, u_2 \in \mathcal{H}$ and $v_1, v_2 \in \operatorname{dom} C$,*
$$\left\langle \widehat{W_{v_1}}u_1, \widehat{W_{v_2}}u_2 \right\rangle_{L^2(G\times\widehat{X},\widehat{\mu})} = \langle u_1, u_2\rangle \langle Cv_2, Cv_1\rangle .$$

Proof. We prove the first item. For dg-almost all $g \in G$,

$$\begin{aligned}
\langle u, V_{\hat{x}} U_g v\rangle &= \int_X \overline{\langle x, \hat{x}\rangle}\, \langle u, dP^M(x) U_g v\rangle \\
&= \int_X \overline{\langle x, \hat{x}\rangle} \lambda(g^{-1}) (W_v u)(g, x)\, dx \\
&= \lambda(g^{-1}) \left(\mathcal{F}(W_v u)(g,\cdot)\right)(\hat{x}) \\
&= \lambda(g^{-1}) (\widehat{W_v}u)(g,\hat{x}),
\end{aligned}$$

where we used the fact that $(W_v u)(g,\cdot) \in L^1(X, dx) \cap L^2(X, dx)$. The claim now easily follows.

The second item is consequence of the orthogonality relations and the fact that J is unitary. □

Remark 8. In the previous theorem, we do not assume that the action of G preserves the group law of X. Compare with item 3 of Remark 3. In paper [3], there is a partial overlap with the results contained in the above theorem.

6. Examples

In this section we give some examples of square integrable imprimitivity systems. The first one clarifies the contacts between our construction and the Gabor analysis on abelian groups, whereas the other two examples are simple toy-models. Some more examples, in a slightly different framework, can be found in [3], where we discuss the relation between square-integrability of imprimitivity systems and square-integrability of representations on quotient spaces.

6.1. **Short time Fourier transform.** Consider a lcsc abelian group G acting on itself by translation, so that $X = G$ and the action is transitive and free. Let dg be a Haar measure of G. With the choice $dx := dg$, the measure μ on $G \times G$ of Lemma 3 is the product measure $dgdg$.

Let now (M, U) be the *Weyl-Heisenberg imprimitivity system* for G based on G explicitly given by

$$\begin{aligned} \mathcal{H} &= L^2(G, dg) \\ M(f)u &= fu \\ U_a u &= u^a, \end{aligned}$$

where $f \in C_0(G)$, $a \in G$ and $u \in L^2(G, dg)$. In Gabor analysis, U is the time-shift operator and P^M is the projection valued measure associated with the frequency-shift operator, [13].

Taking into account that the action is transitive and free, one has that (M, U) is irreducible and, as a consequence of Theorem 4, it is square integrable with respect to dg. Moreover, the normalising operator C is the identity , so that any vector is admissible. Explicitly, fixed $v \in \mathcal{H}$, W_v is an operator from $L^2(G, dg)$ into $L^2(G \times G, dgdg)$ given by

$$(W_v u)(a, x) = u(x)\overline{v(x-a)} \qquad dgdg\text{-a.e.}.$$

Let $\hat{G}$ be the dual group of G and $d\hat{g}$ the Haar measure of $\hat{G}$ such that the Fourier-Plancherel transform is unitary. Using Theorem 5, $\widehat{W}_v$ is an operator from $L^2(G, dg)$ into $L^2(G \times \hat{G}, dgd\hat{g})$ given by

$$(\widehat{W}_v u)(a, \omega) = \int_X u(x)\overline{v(x-a)\omega(x)}\, dx \qquad \forall\ (a, \omega) \in G \times \hat{G}$$

and $\widehat{W}_u$ is nothing but the short time Fourier transform, see [13] and, in the case $G = \mathbb{R}$, for example, Chap. 2.7 of [8], where it is called *windowed Fourier transform.*

Notice that, since (M, U) is the unique, up to an equivalence, cyclic imprimitivity system for G based on G, then $\mathcal{E}_{(M,U)}$ is equal to $L^2(G\times G, dgdg)$ and any orthonormal basis (e_n) of $L^2(G, dg)$ gives rise to a decomposition into irreducible subspaces of $L^2(G \times \hat{G}, dgd\hat{g})$ as

$$L^2(G \times \hat{G}, dgd\hat{g}) = \bigoplus_n \widehat{W}_{e_n}\mathcal{H}.$$

6.2. **Vector valued Euclidean transforms.** Given $n \in \mathbb{N}$, let G be the Euclidean group $\mathbb{R}^n \times' SO(n)$ acting on $X = \mathbb{R}^n$ in a natural way, so that the action on X is transitive. The Haar measure of G is $dg = d^n x dR$ where $d^n x$ is the Lebesgue measure on $\mathbb{R}^n$ and dR is the normalised Haar measure of $SO(n)$. The Lebesgue measure $d^n x$ is invariant with respect to the action of G and, with the choice $dx = d^n x$t the measure μ on $G \times X$ is $d^n x dR d^n x$.

Given an irreducible representation τ of $SO(n)$ acting in a finite dimensional vector space $\mathcal{K}$, let (M, U) be the imprimitivity system for G based on X given by

$$\begin{aligned} \mathcal{H} &= L^2(\mathbb{R}^n, d^n x, \mathcal{K}) \\ M(f)u &= fu \\ (U_{(\vec{a},R)}u)(\vec{x}) &= \tau(R)u(R^{-1}(\vec{x} - \vec{a})) \qquad d^n x\text{-a.e.}, \end{aligned}$$

where $f \in C_0(\mathbb{R}^n)$, $(\vec{a}, R) \in G$ and $u \in L^2(\mathbb{R}^n, dx^n, \mathcal{K})$. Since τ is irreducible (M, U) is irreducible. Applying Theorem 4, since $SO(n)$ is compact, (M, U) is square integrable with respect to $d^n x$ and the normalising operator is the identity. Explicitly, fixed $v \in \mathcal{H}$,

$$\begin{aligned} (W_v u)(\vec{a}, R, \vec{x}) &= \left\langle u(\vec{x}), \tau(R)v(R^{-1}(\vec{x} - \vec{a}))\right\rangle_{\mathcal{K}} \\ (\widehat{W}_v u)(\vec{a}, R, \vec{p}) &= \int \left\langle u(\vec{x}), \tau(R)v(R^{-1}(\vec{x} - \vec{a}))\right\rangle_{\mathcal{K}} e^{-i\vec{x}\cdot\vec{p}}\, d^n x, \end{aligned}$$

where $(\vec{a}, R) \in \mathbb{R}^n \times' SO(n)$, $\vec{x}, \vec{p} \in \mathbb{R}^n$ and $u \in \mathcal{H}$.

Notice that, since $SO(n)$ admits many inequivalent irreducible representations, $\mathcal{E}_{(M,U)}$ is a proper closed subspace of $L^2(G \times X, \mu)$.

6.3. **Spin transforms.** Consider the compact group $SU(2)$ with the normalised Haar measure dh. The group $SU(2)$ acts on the three-dimensional sphere $S^2 \subset \mathbb{R}^3$ by means of the covering homomorphism δ from $SU(2)$ onto $SO(3)$. The action on S^2 is transitive and an invariant measure on it is the area element $d\Omega$. With this choice, $\mu = dh d\Omega$.

Given j such that $2j$ is in $\mathbb{N}$, let D^j be the unique, up to an equivalence, irreducible representation of $SU(2)$ acting on $\mathbb{C}^{2j+1}$. We define (M^j, U^j) to be the imprimitivity system for $SU(2)$ based on S^2 and acting in $L^2(S^2, d\Omega, \mathbb{C}^{2j+1})$ as

$$\begin{aligned} (M^j(f)u)(\vec{x}) &= f(\vec{x})u(\vec{x}) \qquad f \in C_0(S^2) \\ (U^j_h u)(\vec{x}) &= D^j(h)u(\delta(h)^{-1}\vec{x}) \qquad h \in SU(2), \end{aligned}$$

for $u \in L^2(S^2, d\Omega, \mathbb{C}^{2j+1})$ and for $d\Omega$-almost all $\vec{x} \in S^2$.

The system (M^j, U^j) is not irreducible and, by the Mackey theorem, its irreducible components are

$$L^2(S^2, d\Omega, \mathbb{C}^{2j+1}) = \bigoplus_{m=-j}^{j} L^2(S^2, d\Omega)_m,$$

where the index m refers to the fact that the subspace $L^2(S^2, d\Omega)_m$ is unitarily equivalent to $L^2(S^2, d\Omega)$ carrying the representation of $SU(2)$ induced by the character of the torus $z \mapsto z^{2m}$.

Clearly, the restriction to $L^2(S^2, d\Omega)_m$ of (M^j, U^j) is square-integrable with respect to $d\Omega$ and the corresponding normalising operator C^m is proportional to the identity. Fix an admissible vector $v_m \in L^2(S^2, d\Omega)_m$ such that $\|C^m v_m\| = 1$, then the corresponding isometry $W^m_{v_m}$ from $L^2(S^2, d\Omega)_m$ in $L^2(SU(2) \times S^2, dhd\Omega)$ is given by

$$(W^m_{v_m} u)(h, \vec{x}) = \left\langle u(\vec{x}), D^j(h) v_m(\delta(h)^{-1}\vec{x}) \right\rangle_{\mathbb{C}^{2j+1}} ,$$

for $u \in L^2(S^2, d\Omega)_m$ and for $dhd\Omega$-almost all $(h, \vec{x}) \in SU(2) \times S^2$.

Moreover, we can define an isometry from $L^2(S^2, d\Omega, \mathbb{C}^{2j+1})$ into $L^2(SU(2) \times S^2, dhd\Omega)$ as $W^j = \oplus_{m=-j}^{j} W^m_{v_m} P_m$, where P_m is the projection onto $L^2(S^2, d\Omega)_m$. Clearly we have that, for all $u \in L^2(S^2, d\Omega, \mathbb{C}^{2j+1})$,

$$(W^j u)(h, \vec{x}) = \sum_{m=-j}^{j} \left\langle u(\vec{x}), D^j(h) v_m(\delta(h)^{-1}\vec{x}) \right\rangle_{\mathbb{C}^{2j+1}} ,$$

for $dhd\Omega$-almost all $(h, \vec{x}) \in SU(2) \times S^2$.

References

[1] Ali, S.T., Antoine, J.P., Gazeau, J.P., Mueller, U.A.: Coherent states and their generalizations: a mathematical overview. Rev. Math. Phys. **7**, 1013-1104 (1995).

[2] Ali, S.T.: A general theorem on square-integrability: vector coherent states. J. Math. Phys. **39**, 3954-3964 (1999).

[3] Aniello, P., Cassinelli, G., De Vito, E., Levrero, A.: Frames associated with imprimitivity systems, to appear in J. Math. Phys (1999).

[4] Borel, A.: *Representations de groupes localment compacts*, Lectures Notes in Mathematics, No. **256**. Berlin: Springer-Verlag, 1972.

[5] Christensen, O.: Atomic decomposition via projective group representations". Rocky Mountain J. Math., **26**, 1289-1312 (1996).

[6] Conway, J.: *A course in Functional Analysis*, second edition. Berlin: Springer-Verlag, 1985.

[7] D'Ariano, M.: Group theorethical quantum tomography, to appear in Acta Physica Slovaca (1999).

[8] Daubechies, I.: *Ten lectures on wavelets*. Philadelphia: SIAM, 1992.

[9] Dieudonné, J.: *Elements d'Analyse*. Vol. **3**, Paris: Gauthiers-Villars, 1970.

[10] Duflo, M., Moore, C.C.: On the regular representations of a nonunimodular locally compact group. J. Funct. Anal. **21**, 209-243 (1976).

[11] Feichtinger, H.G., Kozek, W.: Operator quantisation on LCA groups. In Feichtinger, H.G., Strohmer, T. (eds.) *Gabor Analysis and Algorithms: Theory and Applications*, pp. 233-266. Berlin: Birkhäuser, 1998.

[12] Folland, G.B.: *A course in abstract harmonic analysis*. Boca Raton: CRC Press, 1995.

[13] Gröchenig, K.: Gabor Analysis on LCA groups. In Feichtinger, H.G., Strohmer, T. (eds.) *Gabor Analysis and Algorithms: Theory and Applications*, pp. 211-231. Berlin: Birkhäuser, 1998.

[14] Grossmann, A., Morlet, J., Paul, T.: Transforms associated to square integrable group representations I. General results. J. Math. Phys. **26**, 2473-2479 (1985).

[15] Grossmann, A., Morlet, J., Paul, T.: Integral transforms associated to square integrable representations II. Examples. Ann. Inst. H. Poincaré **45**, 293-309 (1986).

[16] Lahti, P., Pellonpää, J.-P., Ylinen, K.: Operator integrals and phase space observables. J. Math. Phys. **40**, 2181-2189 (1999).

[17] Pedersen, G.: *Analysis now.* Berlin: Springer-Verlag, 1989.

[18] Varadarajan, V.S.: *Geometry of Quantum Theory*, second edition. Berlin: Springer-Verlag, 1985.

Gianni Cassinelli, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: `cassinelli@ge.infn.it`

Ernesto De Vito, Dipartimento di Matematica, Università di Modena, Via Campi 213/B, 41100 Modena, Italy and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: `devito@unimo.it`

Alberto Levrero, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: `levrero@ge.infn.it`