COORBIT SPACES WITH VOICE IN A FRÉCHET SPACE

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ABSTRACT. We set up a new general coorbit space theory for reproducing representations of a locally compact second countable group G that are not necessarily irreducible nor integrable. Our basic assumption is that the kernel associated with the voice transform belongs to a Fréchet space $\mathcal{T}$ of functions on G, which generalizes the classical choice $\mathcal{T} = L^1_w(G)$. Our basic example is $\mathcal{T} = \bigcap_{p\in(1,+\infty)} L^p(G)$, or a weighted versions of it. By means of this choice it is possible to treat, for instance, Paley-Wiener spaces and coorbit spaces related to Shannon wavelets and *Schrödingerlets*.

1. INTRODUCTION

One of the central problems in applied mathematics is the analysis of signals. Usually signals are modelled by functions in suitable functions spaces (e.g., L^2 or Sobolev spaces) and they might be given explicitly or implicitly as the solution of an operator equation. In most applications, the signal is transformed via a mapping into a suitable parameter space where it is easier to extract the information of interest. By discretization, one obtains suitable building blocks that give rise to a discrete representation of the signal and can be used to decompose, compress and process the signal. Over the years, many different transforms have been derived in response to particular problems, including the wavelet and Gabor transforms. Representation theory, however, gives a general approach to construct continuous transforms for L^2-functions, and coorbit space theory allows both to extend these transforms to more general function spaces and to provide discrete systems. Indeed, it was shown that virtually all well-known transforms used in signal analysis can be derived from this general setting. In this sense, coorbit space theory serves as a common thread in the jungle of all possible signal transformations. Nevertheless, as will be explained below, the classical coorbit space setting relies on specific assumptions that might be hard to verify in practice. The purpose of this paper is to investigate how to weaken these basic assumptions with the goal of extending the applicability of this framework to a much larger class of problems.

Coorbit space theory was originally introduced by H. Feichtinger and K. Gröchenig in a series of papers in 1988-89 [1, 2, 3, 4]. By means of this theory, given a square integrable representation, it is possible to construct in an efficient and systematic way a full scale of smoothness spaces where the smoothness of a function is measured by the decay of the so-called *voice transform*. For any unitary representation π of a locally compact topological group G on a Hilbert space $\mathcal{H}$ and a fixed $u \in \mathcal{H}$, the voice transform V is the map assigning to $v \in \mathcal{H}$ the corresponding transform $x \mapsto Vv(x) = \langle v, \pi(x)u\rangle$ as x ranges in G. Evidently, Vv is a function on the group. Since $\mathcal{H}$ is often a Hilbert space of functions,

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the voice transform connects two function spaces, that is, it maps signals to functions on the group.

Coorbit space theory has been very successful in many ways and has given rise to a wealth of results, enabling to derive a very large family of smoothness spaces as coorbit spaces, including both classical functions spaces and new ones. In particular, the classical Besov spaces are derived as coorbit spaces where smoothness is measured by the decay of the wavelet transform, i.e., the voice transform associated with the affine group. Similarly, the well-established class of modulation spaces corresponds to the family of coorbit spaces where smoothness is measured by the decay of the Gabor transform, i.e., the voice transform associated with the Weyl-Heisenberg group (cf. [1, 2, 3, 5, 6]). As another example, let us mentions the α–modulation spaces [7] which can be interpreted as coorbit spaces related to group representations modulo quotients [8]. Another advantage of coorbit space theory is to provide atomic decompositions and Banach frames for the coorbit spaces, through a procedure which generates discrete function systems by discretization of the group representation. This is important since it provides a way to understand the properties of discrete signal representations through the group theoretic properties of their corresponding continuous voice transforms.

In recent years, a new generation of multiscale transforms has emerged in applied harmonic analysis, such as the shearlet and curvelet transforms, which were introduced to overcome the limitations of the traditional multiscale framework in multi-dimensional setting with high efficiency [9, 10]. Recent results have shown that the continuous shearlet transform, in particular, stems from a square integrable group representation of the so-called *full shearlet group* [11, 12, 13]. By applying the coorbit space theory to this setup, it is possible to define some useful anisotropic smoothness spaces via the decay properties of the shearlet transform and to relate these spaces to other well-known function spaces [14, 15, 16, 17]. This has stimulated the investigation of a larger class of group representations, primarily those arising from the restriction of the metaplectic representation to a class of triangular subgroups of the symplectic group [18]. This class includes many known cases of interest in signal analysis and gives rise to several new examples, such as the Schrödingerlets that we discuss in this paper. Yet another potential extension of this framework is the general context of the so-called mock metaplectic representations, introduced in [19]. However, the classical coorbit space theory appears to be too restrictive to deal with this more general class of group representations and the corresponding voice transforms and function spaces.

Let us recall that the classical coorbit space theory *à la* Feichtinger-Gröchenig makes the following two assumptions:
(FG1) The *kernel* $K = Vu$, that is, the voice transform of the admissible vector itself, is an absolutely integrable function on the group[1].
(FG2) The representation is assumed to be irreducible.
A major part of this paper is concerned with replacing (FG1) by some weaker condition. The fundamental concepts will be presented in Sections 2 and 3. First of all, let us mention that the problem of removing the integrability condition has already been addressed by J. Christensen and G. Ólafsson in [20, 21]. Classically, the reservoir of test functions is obtained by taking the functions whose voice transform is in $L^1(G)$, hence it is a Banach space in a natural way. In the papers [20, 21], the reservoir is a fixed Fréchet space $\mathcal{S}$ densely embedded into $\mathcal{H}$. The basic example is the set of C^∞ vectors for the

[1]For simplicity, in this introduction we use unweighted versions of $L^p(G)$.

representation. The approach that we consider in this paper is indeed similar to [21], but features an important new datum that we call the *target space* $\mathcal{T}$. This is a Fréchet space of functions on G and plays the role of $L^1(G)$ in the classical setup. In our theory, the reservoir $\mathcal{S}$ is the set of functions whose voice transform is in $\mathcal{T}$. In Section 4, we provide a new model for the target space, namely

$$\mathcal{T} = \bigcap_{p>1} L^p(G). \tag{1}$$

For this choice we are able to produce concrete cases arising from triangular subgroups of $Sp(2,\mathbb{R})$, notably the case of the so-called *Schrödingerlets*, discussed in Section 4.3. As a toy example, in Section 4.2 we also consider the case of the band-limited functions[2].

As for assumption (FG2), most of the classical coorbit space theory can be carried out also in the reducible case. In the irreducible case, it is possible to show that the construction of the coorbit spaces is independent of the choice of the admissible vector. Exactly this property is lost in the reducible case, as showed by an important example in [22]. Neither in [21] nor in our setting irreducibility is needed. In Section 3.3 we give a reasonable description of admissible vectors leading to the same coorbit spaces. Finally, in Section 5 we present a detailed account of the extent to which the classical L^1 theory can be developed without the assumption of irreducibility.

Let us briefly describe in some detail the main features of our approach. The starting ingredients are a unitary reproducing representation π of a locally compact second countable group G on a separable Hilbert space $\mathcal{H}$ and an admissible vector $u \in \mathcal{H}$. Next, the Fréchet spaces $\mathcal{S}$ and $\mathcal{T}$ come into play, and their roles in the theory can be described by the following very basic conceptual picture

$$\begin{tikzcd} \mathcal{S} \arrow[r, "i"] & \mathcal{H} \arrow[d, "V"] \\ \mathcal{T} \arrow[r, "j"] & L^0(G) \end{tikzcd}$$

where $L^0(G)$ denotes the space of measurable functions on G. Thus, $\mathcal{S}$ and $\mathcal{T}$ are Fréchet spaces that embed into $\mathcal{H}$ and $L^0(G)$, respectively, and should therefore be thought of as signals and functions on the group, respectively. The space $\mathcal{T}$ is a free choice, as long as one can embed it continuously into $L^0(G)$ in such a way that some basic properties are satisfied (Assumptions 1 and 2). The space $\mathcal{S}$ will serve as the reservoir of *test functions* and it is defined as the subset of $\mathcal{H}$ consisting of those vectors whose voice transform belongs to $j(\mathcal{T})$. Test functions are modeled in terms of their voice transforms and the latter ones constitute the true degree of freedom in the construction, the target space $\mathcal{T}$.

Once the basic structures are laid out, one then follows the lines of coorbit space theory and defines first the distributions $\mathcal{S}'$ and then coorbits associated to Banach spaces of functions.

Technically speaking, our theory is determined by the data set $(G, \mathcal{H}, \pi, u, \mathcal{T}, Y)$, where:

- G is a locally compact second countable topological group;
- $\mathcal{H}$ is a separable Hilbert space;
- π is a continuous unitary reproducing representation of G on $\mathcal{H}$;
- $u \in \mathcal{H}$ is an admissible vector;
- $\mathcal{T}$ is a Fréchet space continuously embedded via j into $L^0(G)$;

[2]Notice that the sinc function is in every L^p with $p > 1$ but not in L^1.

- Y is a Banach space continuously embedded in $L^0(G)$ and left invariant.

The main ancillary objects attached to them are:

- the voice transform $V_2 : \mathcal{H} \to L^2(G)$;
- the reproducing kernel $K = V_2 u$;
- the reproducing kernel space $\mathcal{M} = \{f \in L^0(G) \mid f * K = f\}$;
- the space of test functions $\mathcal{S} = \{v \in \mathcal{H} \mid V_2 v \in j(\mathcal{T})\}$;
- the space of distributions $\mathcal{S}'$;
- the extended voice transform $V_e : \mathcal{S}' \to C(G)$;
- the coorbit space $\mathrm{Co}(Y) = \{T \in \mathcal{S}' \mid V_e T \in Y\}$.

These data are assumed to satisfy several assumptions that are made both for the target space $\mathcal{T}$ and for the model space Y. Assumption 1 asks that the kernel K is in $\mathcal{T}$, which substitutes the classical integrability condition $K \in L^1(G)$, and that $Kf \in L^1(G)$ whenever $f \in \mathcal{T}$. This second requirement has a twin version for Y, namely Assumption 5, and it is trivially satisfied in the case $\mathcal{T} = L^1(G)$ because K is bounded. Assumption 2 and Assumption 6 ask that the product of any voice transform and any "good" function in $\mathcal{T} \cap \mathcal{M}$ (or in $Y \cap \mathcal{M}$) is in $L^1(G)$.

Assumption 3 ensures that the extended voice transform is injective. This is a necessary condition to reconstruct a distribution from its voice transform. Finally, Assumption 4 requires that the reproducing formula extends to all distributions. In particular, we prove in Proposition 3.3 that Assumption 4 holds true if $\mathcal{T}$ is reflexive and $V_e\mathcal{S}' \subset \mathcal{T}'$.

Our theory is succesful in the sense that: it provides a workable substitute for the classical integrability condition $K \in L^1(G)$; it contains the classical coorbit space theory even for non irreducible representations; it applies to several interesting examples; it is consistent with the recent theory developed in [20, 23].

2. Fréchet spaces of functions

In this section, we recall some properties of the Fréchet spaces that are relevant to the main objects of our theory, namely the *target space* $\mathcal{T}$ and the space $\mathcal{S}$ of *test signals* that will be defined in the next section. We introduce abstract spaces E and F. The space E must be interpreted as modeling a subspace of the Hilbert space $\mathcal{H}$, hence of signals, but possibly with a different topology. Its properties will be used primarily for the test space $\mathcal{S}$, but also for $\mathcal{H}$ itself. Similarly, the space F should be thought of as an abstract model of a Fréchet space of functions on the group. The results proved for F will be primarily applied to the target space $\mathcal{T}$, which in many examples is a genuine Fréchet space but not a Banach space, but will also be useful for $F = L^1(G)$, $F = L^2(G)$ and, most notably, for $F = Y$, the space of functions used to define coorbit spaces. From this point of view, our theory indicates that it is possible to develop a useful coorbit space theory assuming that Y is a Fréchet space rather than the more common choice of a Banach space. This further extension, however, is beyond the scope of this article, and we content ourselves with the classical case in which Y is Banach space.

2.1. **Background.**

We now introduce the basic notation and recall some elementary properties. Further technical results are recalled in Section 6.1.

Throughout this paper, G, denotes a fixed locally compact second countable group with a left Haar measure β and Δ is its modular function. We write $\int_G f(x)dx$ instead of $\int_G f(x)\, d\beta(x)$ and denote the classical spaces of complex functions on G as follows:

$L^0(G)$	β-measurable functions,
$L^p(G)$	p-integrable functions with respect to β, $p \in [1, +\infty)$,
$L^\infty(G)$	β-essentially bounded functions,
$L^1_{\rm loc}(G)$	locally β-integrable functions,
$C(G)$	continuous functions,
$C_0(G)$	continuous functions going to zero at infinity,
$C_c(G)$	compactly supported continuous functions.

The space $L^0(G)$ is a metrizable complete topological vector space with respect to the topology of convergence in measure (see Section 6.1). The norm of $f \in L^p(G)$ and the scalar product between $f, g \in L^2(G)$ are denoted by $\|f\|_p$ and $\langle f, g\rangle_2$, respectively. The space $L^1_{\rm loc}(G)$ is a Fréchet space with respect to the topology defined by the family of semi-norms

$$f \mapsto \int_{\mathcal{K}} |f(x)| dx,$$

where $\mathcal{K}$ runs over the compact subsets of G (see Section 6.1).

We denote by λ and ρ the left and right regular representations of G on $L^0(G)$, namely

$$\lambda(x)f\,(y) = f(x^{-1}y)$$
$$\rho(x)f\,(y) = f(yx)$$

for all $x \in G$, all $f \in L^0(G)$ and almost all $y \in G$. Both λ and ρ leave $L^1_{\rm loc}(G)$ and each $L^p(G)$ invariant, and λ is equicontinuous both on $L^1_{\rm loc}(G)$ and on each $L^p(G)$. In Section 6.2 we recall the main properties of the representations acting on Fréchet spaces. For general background on representations the reader is referred to [24].

For all $f \in L^0(G)$ we denote by $\check{f}$ the element in $L^0(G)$ given by

$$\check{f}(x) = f(x^{-1})$$

for almost all $x \in G$ (see Section 6.1). Given two functions $f, g \in L^0(G)$, we say that the convolution $f * g$ exists if for almost all $x \in G$ the function $f\lambda(x)\check{g}$ is in $L^1(G)$. We write

$$f * g(x) = \int_G f(y)\lambda(x)\check{g}(y)\,dy = \int_G f(y)g(y^{-1}x)\,dy \qquad \text{a.e. } x \in G$$

and we have that $f * g \in L^0(G)$ (see Section 6.3).

2.2. **Voice transform.** In what follows, π denotes a fixed strongly continuous unitary representation of G acting on the separable Hilbert space $\mathcal{H}$ and u a fixed vector in $\mathcal{H}$. We stress that π is not assumed to be irreducible, nor, at this stage, reproducing. As it is customary, the *voice transform* associated to the these data is the map

$$V : \mathcal{H} \to L^\infty(G) \cap C(G), \qquad Vv(x) = \langle v, \pi(x)u\rangle_{\mathcal{H}}.$$

It intertwines π and λ, that is

$$V\pi(x) = \lambda(x)V \tag{2}$$

for all $x \in G$. The corresponding kernel is given by

$$K : G \to \mathbb{C}, \qquad K(x) = Vu(x) = \langle u, \pi(x)u\rangle_{\mathcal{H}}. \tag{3}$$

It enjoys the basic symmetry property

$$\overline{K} = \check{K}. \tag{4}$$

For all $f \in L^1(G)$ the *Fourier transform* $\pi(f)$ is the bounded operator on $\mathcal{H}$ defined by

$$\langle \pi(f)w, v\rangle_{\mathcal{H}} = \int_G f(x)\langle \pi(x)w, v\rangle_{\mathcal{H}}\, dx$$

for all $w, v \in \mathcal{H}$. Note that, with the choice $w = u$, we get

$$\langle \pi(f)u, v\rangle_{\mathcal{H}} = \int_G f(x)\overline{Vv(x)}\, dx. \tag{5}$$

2.3. **Functions on the group.** In this section, we consider a space F of functions on G and we study the properties of the convolution operator $f \mapsto f * K$ from F into $C(G)$. In particular, we introduce the subspace $\mathcal{M}^F$ of those functions which are left fixed by the convolution operator. In the theory of reproducing representations, F is the Hilbert space $L^2(G)$; in the theory developed by H. Feichtinger and K. Gröchenig it is a weighted version of $L^1(G)$; in our setting it is the target space $\mathcal{T}$.

We assume that F is a Fréchet space with a continuous embedding $j : F \to L^0(G)$. With slight abuse of notation, given $f \in F$, we denote by $f(\cdot)$ a β-measurable function such that for almost every $x \in G$, $f(x) = j(f)(x)$. Further, we assume that there exist

i) a continuous involution $f \mapsto \overline{f}$ on F such that $j(\overline{f}) = \overline{j(f)}$ (so that $\overline{f}(x) = \overline{f(x)}$);
ii) a continuous representation ℓ of G acting on F for which

$$j(\ell(x)f) = \lambda(x)j(f) \qquad f \in F,\ x \in G,$$

so that $(\ell(x)f)(y) = f(x^{-1}y)$ and $\overline{\ell(x)f} = \ell(x)\overline{f}$.

Standard examples of spaces satisfying the above assumptions are the L^p-spaces or their weighted versions. Other important examples are the space of C^∞ functions, if G is a Lie group, or the space of rapidly decreasing functions, whenever this notion makes sense.

The space $j(F)$ is a subspace of $L^0(G)$, stable under complex conjugation and λ-invariant. Clearly, we could identify F with $j(F)$ avoiding the cumbersome map j. However, we want to stress that F has its own topology, which is not necessarily the topology of $j(F)$, that is, the relative topology as a topological subspace of $L^0(G)$. In order to clarify the role of the two topologies, we shall not identify F with $j(F)$.

Since j is continuous from F into $L^0(G)$, for any sequence (f_n) converging to an element f in F, there exists a subsequence $(f_{n_k})_k$ such that $(f_{n_k}(x))_k$ converges to $f(x)$ for almost all $x \in G$ (see (72) for details).

We denote by F' the topological dual of F. For each $T \in F'$, the map $f \mapsto T(\overline{f})$ defines a continuous anti-linear function on F, which we denote by $\langle T, \cdot\rangle_F$. The map $T \mapsto \langle T, \cdot\rangle_F$ is a linear isomorphism of F' onto the anti-dual $F^\wedge$, the space of anti-linear continuous forms on F. In what follows, we identify $F^\wedge$ with F'. Observe that the map $(T, f) \mapsto \langle T, f\rangle_F$ is a sequilinear form on $F' \times F$, linear in the first entry and anti-linear in the second.

The Köthe dual of F is defined by

$$F^\# = \{g \in L^0(G) \mid gj(f) \in L^1(G),\ \text{for all } f \in F\}.$$

It is closed under complex conjugation and its elements can be regarded as anti-linear forms on F, as shown by the next lemma. Here and below, we fix a countable fundamental system $\{q_i\}_i$ of saturated[3] semi-norms in F.

[3]A family is saturated if the maximum of any finite subset is in the family.

Lemma 2.1. *Given $g \in F^{\#}$, the map*

$$F \ni f \mapsto \langle g, f\rangle_F = \int_G g(x)\overline{f(x)}\,dx \in \mathbb{C} \tag{6}$$

is a continuous anti-linear form, that is, an element of F', which we denote again by g. The representation λ leaves $F^{\#}$ invariant and for all $x \in G$

$$\langle \lambda(x)g, f\rangle_F = \langle g, \ell(x^{-1})f\rangle_F. \tag{7}$$

Finally, there exist a constant $C > 0$ and a semi-norm q_k in the fundamental saturated family $\{q_i\}_i$ such that for all $f \in F$

$$\int_G |f(x)||g(x)|\,dx \leqslant Cq_k(f). \tag{8}$$

Proof. By definition, for each $f \in F$ the function $g\,j(f)$ is in $L^1(G)$. We claim that the linear map

$$\mathcal{L} : F \to L^1(G), \qquad \mathcal{L}f = g\,j(f)$$

is continuous. Since both F and $L^1(G)$ are separable metrizable vector spaces, by the closed graph theorem (Corollary 5 of Chapter I.3.3 of [25]) it is enough to show that the graph of $\mathcal{L}$ is sequentially closed in $F \times L^1(G)$. Take a sequence $(f_n)_n$ in F converging to f in F and such that $(\mathcal{L}f_n)_n$ converges to φ in $L^1(G)$. Since both F and $L^1(G)$ are continuously embedded in $L^0(G)$, possibly passing to a subsequence, we can assume that both $(f_n(x))_n$ and $(\mathcal{L}f_n(x))_n$ converge to $f(x)$ and $\varphi(x)$, respectively, for almost every x. Hence for almost all $x \in G$

$$\mathcal{L}f\,(x) = g(x)f(x) = \lim_{n\to+\infty} g(x)f_n(x) = \lim_{n\to+\infty} \mathcal{L}f_n(x) = \varphi(x),$$

that is, $\mathcal{L}f = \varphi$ in $L^1(G)$. Hence $\mathcal{L}$ is continuous, as well as the anti-linear form

$$f \mapsto \int_G \mathcal{L}\overline{f}(x)dx = \int_G g(x)\overline{f(x)}\,dx = \langle g, f\rangle_F.$$

We now prove that λ leaves $F^{\#}$ invariant. Indeed, given $x \in G$ and $f \in F$

$$\int_G |\lambda(x)g\,(y)f(y)|dy = \int_G |g(x^{-1}y)f(y)|dy = \int_G |g(y)f(xy)|dy = \int_G |g(y)\ell(x^{-1})f\,(y)|dy < +\infty,$$

where the last integral is finite since $\ell(x^{-1})f \in F$. Hence $\lambda(x)g \in F^{\#}$. The same string of equalities gives (7). Finally, the last formula follows directly from the continuity of $\mathcal{L}$. ☐

In general, $F^{\#}$ is a proper subset of F' as the following example clarifies. Take $F = L^p(G)$ with $p \in [1, +\infty]$. Then $L^p(G)^{\#} = L^{p'}(G)$, where p' is the dual exponent of p, so that $L^p(G)^{\#} = L^p(G)'$ for all $p < +\infty$, but of course $L^\infty(G)^{\#} = L^1(G) \subsetneq L^\infty(G)'$.

The next proposition shows that, if the kernel K belongs to the Köthe dual of F, then for all $f \in F$ the convolution $j(f) * K$ exists. Furthermore, we introduce the subspace $\mathcal{M}^F \subset F$ whose elements are those reproduced by convolution with K on the right. In the following statement $C(G)$ is endowed with the topology of the compact[4] convergence.

Proposition 2.2. *Assume that $K \in F^{\#}$. Then:*

a) for all $f \in F$, $j(f) * K$ exists everywhere, it is a continuous function and, for all $x \in G$,

$$j(f) * K(x) = \langle \lambda(x)\check{K}, \overline{f}\rangle_F = \langle \check{K}, \overline{\ell(x^{-1})f}\rangle_F; \tag{9}$$

b) the map $f \mapsto j(f) * K$ is continuous from F to $C(G)$;

[4]Short for: uniform convergence on compact sets.

c) the set

$$\mathcal{M}^F = \{f \in F \mid j(f) * K = j(f)\} \tag{10}$$

is an ℓ-invariant closed subspace of F, and therefore it is a Fréchet space respect to the relative topology.

Proof. Notice that, in view of (4), $K \in F^{\#}$ if and only if $\check{K} \in F^{\#}$. Since λ leaves $F^{\#}$ invariant (Lemma 2.1), for all $x \in G$ we have $\lambda(x)\check{K} \in F^{\#}$ and, for all $f \in F$,

$$\langle \lambda(x)\check{K}, \overline{f}\rangle_F = \int_G f(y)\lambda(x)\check{K}\,(y)\,dy = \int_G f(y)K(y^{-1}x)\,dy = j(f) * K\,(x).$$

Hence $j(f) * K(x)$ exists and the first equality of (9) holds true. The change of variables $y \mapsto xy$ proves the second equality of (9). Since the involution and $x \mapsto \ell(x^{-1})f$ are continuous, (9) implies that $j(f) * K$ is a continuous function.

To prove b), fix a compact subset $\mathcal{K} \subset G$. By (9), since $\check{K} \in F^{\#} \subset F'$, there exist two semi-norms q_j, q_k in the fundamental saturated system $\{q_i\}_i$ and constants C and C' such that

$$\sup_{x\in\mathcal{K}} |j(f) * K\,(x)| \leqslant C \sup_{x\in\mathcal{K}^{-1}} q_j(\ell(x)f) \leqslant Cq_k(f),$$

where the last inequality follows from the fact that $\ell(\mathcal{K}^{-1})$ is equicontinuous since $\mathcal{K}^{-1}$ is compact (see Section 6.2).

As for c), since F is a metrizable vector space, it is sufficient to prove that $\mathcal{M}^F$ is sequentially closed. Take a sequence $(f_n)_n$ in $\mathcal{M}^F$ converging to $f \in F$. Possibly passing to a subsequence, we can assume that there exists a negligible set N such that for all $x \notin N$ $(f_n(x))_n$ converges to $f(x)$. Furthermore, possibly changing N, we can also assume that, for all $n \in \mathbb{N}$ and $x \notin N$, $j(f_n) * K\,(x) = f_n(x)$. Hence, given $x \notin N$, by b) we have

$$j(f) * K\,(x) = \lim_n j(f_n) * K\,(x) = \lim_n f_n(x) = f(x).$$

Hence $j(f) * K = j(f)$ in $L^0(G)$, that is $f \in \mathcal{M}^F$. Finally, given $x \in G$ and $f \in \mathcal{M}^F$, by (77b) in the appendix

$$j(\ell(x)f) = \lambda(x)j(f) = \lambda(x)(j(f) * K) = \lambda(x)j(f) * K = j(\ell(x)f) * K,$$

that is $\ell(x)f \in \mathcal{M}^F$. □

In what follows, for each $f \in \mathcal{M}^F$, we choose the continuous everywhere defined function $j(f) * K$ as representative of f, so that for all $x \in G$

$$j(f) * K\,(x) = f(x). \tag{11}$$

2.4. **Extension of the voice transform and Fourier transform.** We are interested in extending the voice transform V from $\mathcal{H}$ to some bigger space, namely the dual of a Fréchet space E which is continuously embedded into $\mathcal{H}$, in such a way that the duality relation (5) still holds true. In the classical coorbit space theory and in our setting, E is the space of test functions $\mathcal{S}$, which in [21] is the basic object on which the theory is developed.

We fix a Fréchet space E together with a continuous representation τ of G acting on E and a continuous embedding $i : E \to \mathcal{H}$ intertwining τ and π. As above, we identify the dual E' and the anti-dual $E^\wedge$. We are interested in the transpose ${}^t i : \mathcal{H} \to E'$, defined as usual by

$$\langle {}^t i(w), v\rangle_E = \langle w, i(v)\rangle_{\mathcal{H}}.$$

We assume that $u \in i(E)$ and, with slight abuse of notation, we regard u as an element both in E and in $\mathcal{H}$. Hence, we define the *extended voice transform* by

$$V_e : E' \to C(G), \qquad V_e T = \langle T, \tau(\cdot)u \rangle_E, \tag{12}$$

which intertwines the contragredient representation ${}^t\tau$ with the left regular representation λ.

Hereafter we establish some useful properties of the abstract space E, that will be applied to the space $\mathcal{S}$ of test functions in Section 3.1.

A basic requirement on V_e is that it must be injective. In the next lemma some standard equivalent conditions are established.

Lemma 2.3. *The following facts are equivalent:*

i) the map V_e is injective;
ii) the set $\tau(G)u$ is total in E;
iii) the set $\tau(G)u$ is total in[5] *E_{weak}.*

Proof. Define D as the closure in E of the linear span of $\tau(G)u$. Since the linear span of $\tau(G)u$ is convex, D is convex (Proposition 14 Chapter II.2.6 of [25]). Then D is a closed convex subset, and so it is also weakly closed (Proposition 1 Chapter IV.1.1 of [25]). Therefore D is also the closure in E_{weak} of the linear span of $\tau(G)u$. This proves the equivalence between ii) and iii).

Next, assume ii), that is $D = E$. If $T \in E'$ is such that $V_e T = 0$, then $\langle T, v\rangle_E = 0$ for all $v \in D = E$, hence $T = 0$. Therefore ii) implies i). Finally, suppose $D \subsetneq E$. Then, as a consequence of the Hahn–Banach theorem ([26], Proposition 2, p.180), there exist $T \in E'$ and $v_0 \in E \backslash D$ such that $\langle T, v\rangle_E = 0$ for all $v \in D$ and $\langle T, v_0\rangle_E \neq 0$, so that $V_e T = 0$ but $T \neq 0$. Therefore i) implies ii). □

By definition, u is a cyclic vector for the representation τ if condition ii) holds true. Notice that the cyclicity for π does not imply the cyclicity for τ, since the topology of E is finer than the topology of $\mathcal{H}$.

The topological dual of E comes now into play and will be denoted by E'. When topological properties are involved, E' is understood to have the topology of the convergence on the bounded subsets of E. We will write E'_s to stress when E' is rather thought with the topology of the simple convergence (compare Section 6.4).

The following proposition shows that, for any function $f \in L^0(G)$ satisfying a suitable integrability condition, it is possible to define an element in E' which plays the role of the Fourier transform of f at u. It is useful to compare our assumption (13) with condition (R3) in [21].

Proposition 2.4. *Take $f \in L^0(G)$ and assume that*

$$f\, V i(v) \in L^1(G) \qquad \text{for all } v \in E. \tag{13}$$

Then there exists a unique $\pi(f)u \in E'$ such that, for all $v \in E$,

$$\langle \pi(f)u, v\rangle_E = \int_G f(x)\langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\, dx = \int_G f(x)\overline{V i(v)(x)}\, dx. \tag{14}$$

For any such f, we have

$$V_e \pi(f)u = f * K. \tag{15}$$

[5] Evidently, E_{weak} is just E endowed with the weak topology.

Finally, assume that $K * K$ exists, is equal to K and that $f * (|K| * |K|)$ exists. Then

$$V_e\pi(f)u * K = V_e\pi(f)u. \tag{16}$$

Proof. Define the map $\Psi : G \to E'_s$ by $\Phi(x) = f(x)\,{}^t\!i(\pi(x)u)$. Since for all $v \in E$

$$\langle \Psi(x), v\rangle_E = f(x)\langle \pi(x)u, i(v)\rangle_{\mathcal{H}} = f(x)\overline{Vi(v)}(x),$$

by (13) we know that the map Ψ is scalarly β-integrable. Since E is a Fréchet space, then it satisfies the (GDF) property. Then Theorem 6.4 applies, showing that the scalar integral $\int \Psi(x)dx$ exists and belongs to E' (see Section 6.4). We set $\pi(f)u = \int \Psi(x)dx$ and, by definition of scalar integral, (14) holds true for all $v \in E$. Also, for all $x \in G$,

$$V_e\pi(f)u\,(x) = \int_G f(y)\langle \pi(y)u, i(\tau(x)u)\rangle_{\mathcal{H}}\,dy = \int_G f(y)\langle \pi(y)u, \pi(x)u\rangle_{\mathcal{H}}\,dy = (f * K)(x).$$

Finally, under the ongoing assumptions, (77d) in the appendix implies that $(f * K) * K = f * (K * K) = f * K$, so that (16) is a direct consequence of (15). $\square$

If (13) is satisfied, we say that the Fourier transform of f at u exists in E' or, simply, that $\pi(f)u \in E'$ exists. Condition (13) is actually both necessary and sufficient to define $\pi(f)u$ as an element of E'. The next lemma ensures that the voice transform is reproduced by convolution.

Lemma 2.5. *Assume that the extended voice transform is injective and take $T \in E'$. The following assertions are equivalent:*

a) $V_eT * K$ exists and satisfies the reproducing formula

$$V_eT * K = V_eT; \tag{17}$$

b) for all $x \in G$, the map $y \mapsto \langle T, \tau(y)u\rangle_E\langle \pi(y)u, \pi(x)u\rangle_{\mathcal{H}}$ is in $L^1(G)$ and

$$\int_G \langle T, \tau(y)u\rangle_E\langle \pi(y)u, \pi(x)u\rangle_{\mathcal{H}}dy = \langle T, \tau(x)u\rangle_E.$$

If the Fourier transform of V_eT at u exists in E', i.e. *the map $x \mapsto V_eT(x)\,{}^t\!i(\pi(x)u)$ is scalarly integrable, then a) and b) are also equivalent to each of the following assertions:*

c) $\pi(V_eT)u = T$;

d) the reconstruction formula

$$T = \int_G \langle T, \tau(x)u\rangle_E\,{}^t\!i(\pi(x)u)dx. \tag{18}$$

holds true weakly.

Proof. The equivalence between a) and b) is just the definition of V_e and K. Taking into account that V_e is injective, the equivalence between a) and c) follows from (15) with $f = V_eT$. The equivalence between c) and d) is just the definition of scalar integral. $\square$

In the next proposition we assume that the Fourier transform $\pi(f)u$ exists in E' for all $f \in F$, where F is a Fréchet space satisfying the assumptions of Section 2.3. With slight abuse of notation, we write $\pi(f)u$ instead of $\pi(j(f))u$. We define the coorbit space

$$\mathrm{Co}(E', F) = \{T \in E' \mid V_eT \in j(F)\}.$$

Proposition 2.6. *Take $E \xhookrightarrow{i} \mathcal{H}$, $F \xhookrightarrow{j} L^0(G)$ and $K(x) = \langle u, \pi(x)u\rangle_{\mathcal{H}}$ as above. Assume that for all $f \in F$ and all $v \in E$*

$$j(f)Vi(v) \in L^1(G), \tag{19}$$

which may be rephrased as $V(i(E)) \subset F^{\#}$. Then:

a) the Fourier transform of any $f \in F$ at u exists in E', so does the convolution $j(f) * K$, and

$$V_e\pi(f)u = j(f) * K; \tag{20}$$

b) the space $\mathcal{M}^F$, defined by (10)*, is an ℓ-invariant closed subspace of F,* $\mathrm{Co}(E', F)$ *is a ${}^t\!\tau$-invariant subspace of E', and V_e intertwines ${}^t\!\tau$ with λ;*

c) the map $(f, v) \mapsto \langle \pi(f)u, v\rangle_E$ is continuous from $F \times E$ into $\mathbb{C}$;

d) if V_e is injective and the reproducing formula (17) *holds for all $T \in \mathrm{Co}(E', F)$, then*

$$V_e \,\mathrm{Co}(E', F) = j(\mathcal{M}^F), \tag{21a}$$

$$\{\pi(f)u \mid f \in \mathcal{M}^F\} = \mathrm{Co}(E', F), \tag{21b}$$

$$V_e\pi(f)u = j(f), \qquad f \in \mathcal{M}^F, \tag{21c}$$

$$\pi(V_eT)u = T, \qquad T \in \mathrm{Co}(E', F). \tag{21d}$$

Hence, V_e is a bijection of $\mathrm{Co}(E', F)$ *onto $j(\mathcal{M}^F)$ and therefore it induces a bijection, denoted again by V_e, from* $\mathrm{Co}(E', F)$ *onto $\mathcal{M}^F$, whose inverse is the Fourier transform at u.*

Proof. Item a) is a direct consequence of Proposition 2.4. Item b) is due to Proposition 2.2. The invariance property of $\mathrm{Co}(E', F)$ is a consequence of the fact that $V_e\,{}^t\!\tau(x) = \lambda(x)V_e$ for all $x \in G$.

As for c), since F and E are Fréchet spaces it is enough to show that $(f, v) \mapsto \langle \pi(f)u, v\rangle_E$ is separately continuous. Clearly, given $f \in F$, the map $v \mapsto \langle \pi(f)u, v\rangle_E$ is continuous since $\pi(f)u \in E'$. On the other hand, given v in E, the hypothesis (19) states that $Vi(v) \in F^{\#}$. Lemma 2.1 shows that $Vi(v) \in F'$ where the identification is given by (6), namely

$$\langle Vi(v), f\rangle_F = \int_G Vi(v)(x)\overline{f(x)}\,dx = \overline{\int_G f(x)\langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\,dx} = \overline{\langle \pi(f)u, v\rangle_E},$$

so that $f \mapsto \langle \pi(f)u, v\rangle_E$ is continuous.

Finally, we prove d). The definition of $\mathcal{M}^F$ and (20) imply (21c). Given $T \in \mathrm{Co}(E'; F)$, by definition $V_eT \in j(F)$ and, hence, the convolution $V_eT * K$ exists. Furthermore, by assumption $V_eT * K = V_eT$. Hence, condition a) of Lemma 2.5 is satisfied and this implies that $\pi(V_eT)u = T$, which is (21d). To prove (21a) and (21b), observe that the reproducing formula implies that $V_e \,\mathrm{Co}(E'; F) \subset j(\mathcal{M}^F)$ and equality (21d) that $\mathrm{Co}(E', F) \subset \{\pi(f)u \mid f \in \mathcal{M}^F\}$. Furthermore, since $\mathcal{M}^F \subset F$, implies $\{\pi(f)u \mid f \in \mathcal{M}^F\} \subset \mathrm{Co}(E', F)$ and, hence, $j(\mathcal{M}^F) \subset V_e \,\mathrm{Co}(E', F)$. □

The above result is an adaptation of Theorem 2.3 of [21]. Conditions (R3) and (R4) in [21] are replaced by (19) and the reproducing property (17), respectively.

Under all the assumptions of Proposition 2.6, in particular the conditions of item d), the space $\mathrm{Co}(E', F)$ has a natural topology that makes it a Fréchet space.

Corollary 2.7. *The space* $\mathrm{Co}(E', F)$ *is a Fréchet space with respect to any of the following equivalent topologies:*

a) the topology induced by the family of semi-norms $\{q_i(V_e(\cdot))\}_i$, where $\{q_i\}_i$ is any fundamental family of semi-norms of F;
b) the initial topology induced from the topology of F by the restriction of the voice V_e;
c) the final topology induced from the topology of F, restricted to $\mathcal{M}^F$, by the Fourier transform at u.

Proof. By Proposition 2.6, V_e is a bijection from $\mathrm{Co}(E', F)$ onto $\mathcal{M}^F$ whose inverse is the Fourier transform at u, the initial and final topologies on $\mathrm{Co}(E', F)$ coincide and they realize $\mathrm{Co}(E', F)$ as a Fréchet space (isomorphic to $\mathcal{M}^F$). The equivalence between a) and b) is a standard result (see remark before Example 4 Ch. 2.11 of [26]). The equivalence between b) and c) follows from the fact that V_e is a bijection. Since $\mathcal{M}^F$ is a closed subspace of a Fréchet space, then both $\mathcal{M}^F$ and $\mathrm{Co}(E', F)$ are (isomorphic) Fréchet spaces. □

2.5. **Reproducing representations: the standard setup.** In this section, we further assume that π is a *reproducing* representation and that the vector u is an admissible vector for π. This means that the voice transform V maps $\mathcal{H}$ into $L^2(G)$ and that for all $v \in \mathcal{H}$

$$\|v\|_{\mathcal{H}} = \|Vv\|_2. \tag{22}$$

To stress that the voice transform is an isometry of $\mathcal{H}$ into $L^2(G)$, we write it with the suffix 2:

$$V_2 : \mathcal{H} \to L^2(G), \qquad V_2 v(x) = \langle v, \pi(x)u \rangle_{\mathcal{H}}.$$

Recalling that $K = V_2 u$ and (4), we have

$$\check{K} = \overline{K} \in L^2(G). \tag{23}$$

In the following proposition, some consequences of the assumption that π is reproducing are drawn. The results are well known for irreducible representations [27, 28] and their extensions to non-irreducible representations are taken for granted in many papers. We provide a proof based on Proposition 2.6.

Proposition 2.8. *Suppose that π is a reproducing representation of G on $\mathcal{H}$ and that $u \in \mathcal{H}$ is an admissible vector. Then:*

a) for every $f \in L^2(G)$, the Fourier transform of f at u exists in $\mathcal{H}$ and for all $v \in \mathcal{H}$

$$\langle \pi(f)u, v \rangle_{\mathcal{H}} = \langle f, V_2 v \rangle_2;$$

b) for every $f \in L^2(G)$ the convolution $f * K$ exists and

$$V_2 \pi(f) u = f * K, \tag{24}$$

where both sides belong to $C_0(G)$ and, for every $v \in \mathcal{H}$,

$$V_2 v * K = V_2 v; \tag{25}$$

in particular, $K = K * K$.

c) the space

$$\mathcal{M}^2 = \{ f \in L^2(G) \mid f * K = f \}$$

is a λ-invariant closed subspace of $L^2(G)$ and

$$V_2 \mathcal{H} = \mathcal{M}^2; \tag{26}$$

$$V_2 \pi(f) u = f, \qquad \text{for all } f \in \mathcal{M}^2; \tag{27}$$

$$\pi(V_2 v) u = v, \qquad \text{for all } v \in \mathcal{H}. \tag{28}$$

Hence, the voice transform V_2 is a unitary map from $\mathcal{H}$ onto $\mathcal{M}^2$ whose inverse is given by the map $f \mapsto \pi(f)u$.

Proof. We first prove (25). By (76c) with $p = q = 2$, the convolution $V_2 v * K$ exists and is in $C_0(G)$ because $\check{K} \in L^2(G)$ by (23). Furthermore, given $x \in G$, for all $y \in G$

$$V_2v(y)\,\lambda(x)\check{K}(y) = V_2v(y)\,\overline{\lambda(x)K(y)} = V_2v(y)\,\overline{(V_2\pi(x)u)(y)}.$$

Integrating with respect to y, we obtain

$$V_2v * K = \langle V_2v, V_2\pi(x)u\rangle_2 = \langle v, \pi(x)u\rangle_{\mathcal{H}} = V_2v.$$

To prove the remaining statements, we apply Proposition 2.6 with $F = L^2(G)$ and $E = E' = \mathcal{H}$, with the understanding that i and j are the canonical inclusions, $\lambda = \ell$, $\pi = \tau$ and $V = V_2$. Observe that (19) is satisfied since $V_2\mathcal{H} \subset L^2(G) = L^2(G)^\#$ and, by (25), the reproducing formula (17) holds for every $v \in \mathcal{H}$, regarded as anti-linear form on $\mathcal{H}$. Furthermore, by (76c) in the appendix with $p = q = 2$, for all $f \in L^2(G)$ the function $f * K$ is in $C_0(G)$, taking (23) into account. □

3. Main results

In this section, we assume that the representation π is reproducing and that the vector $u \in \mathcal{H}$ is admissible, as in Section 2.5. We will construct a coorbit space theory based on the choice of a suitable target space $\mathcal{T}$ embedded in $L^0(G)$.

3.1. **The space of test functions and distributions.** We choose a Fréchet space $\mathcal{T}$ with

i) a continuous embedding $j : \mathcal{T} \to L^0(G)$;
ii) a continuous representation ℓ of G acting on $\mathcal{T}$ such that $j\ell(x) = \lambda(x)j$ for all $x \in G$;
iii) a continuous involution $f \mapsto \overline{f}$ such that $\overline{j(f)} = j(\overline{f})$,

so that $\mathcal{T}$ enjoys all the properties of the space F in Section 2, from which we adopt the notations. In particular, as in (10), we put

$$\mathcal{M}^{\mathcal{T}} = \{f \in \mathcal{T} \mid j(f) * K = j(f)\}.$$

The classical theory corresponds to the choice $\mathcal{T} = L^1(G)$, or a weighted version of it. The following assumptions are at the root of our construction and are trivially satisfied for $L^1(G)$.

Assumption 1. The kernel K is in $j(\mathcal{T})$ and $j(f)K \in L^1(G)$ for all $f \in \mathcal{T}$, *i.e* $K \in j(\mathcal{T}) \cap \mathcal{T}^\#$.

Assumption 2. For all $f \in \mathcal{M}^{\mathcal{T}}$ and all $v \in \mathcal{H}$ we have $j(f)V_2v \in L^1(G)$, *i.e* $V_2\mathcal{H} \subset (\mathcal{M}^{\mathcal{T}})^\#$.

Assumption 3. The linear space spanned by the orbit $\{\ell(x)K \mid x \in G\}$ is dense in $\mathcal{M}^{\mathcal{T}}$.

By Proposition 2.2, Assumption 1 implies that for all $f \in \mathcal{T}$ the convolution $j(f) * K$ exists and $\mathcal{M}^{\mathcal{T}}$ is a closed ℓ-invariant subspace of $\mathcal{T}$, so that $\operatorname{span}\{\ell(x)K \mid x \in G\}$ is a subspace of $\mathcal{M}^{\mathcal{T}}$. Assumption 3 is formulated with a slight abuse of notation, regarding K as an element of $\mathcal{T}$. It is a strengthening of Assumption 1 because it is equivalent to the requirement that K is actually a cyclic vector for the representation ℓ restricted to $\mathcal{M}^{\mathcal{T}}$.

Assumptions 1 and 2 should be compared with hypotheses (R2) and (R3) of [21]. In our approach they are needed to define the test space, as in the classical setting, whereas in [21] the test space is given a-priori. We are now in a position to define the space of *test signals*, namely

$$\mathcal{S} = \{v \in \mathcal{H} \mid V_2 v \in j(\mathcal{T})\}. \tag{29}$$

We define the *restricted voice transform* $V_0 : \mathcal{S} \to \mathcal{T}$ as the unique map satisfying $jV_0 = V_2 i$, that is, for all $v \in \mathcal{S}$ and $x \in G$ we put

$$(V_0 v)(x) = \langle i(v), \pi(x)u \rangle_{\mathcal{H}}$$

where $i : \mathcal{S} \to \mathcal{H}$ is the canonical inclusion. It is by means of V_0 that we topologize $\mathcal{S}$: we endow $\mathcal{S}$ with the initial topology induced by V_0. As it will be shown in Theorem 3.1 below, this is just an explicit description of the topology that $\mathcal{S}$ naturally inherits as coorbit space, because $\mathcal{S} = \mathrm{Co}(\mathcal{H}, \mathcal{T})$. Observe that Assumption 1 implies that $u \in \mathcal{S}$, since $K \in j(\mathcal{T})$.

Theorem 3.1. *The space $\mathcal{S}$ is a Fréchet space isomorphic to $\mathcal{M}^{\mathcal{T}}$ via V_2, and $j(\mathcal{M}^{\mathcal{T}}) \subset L^2(G)$. The canonical embedding $i : \mathcal{S} \to \mathcal{H}$ is continuous and has dense range. The transpose ${}^t i : \mathcal{H}_s \to \mathcal{S}'_s$ is continuous, injective and has dense range. The representation π leaves $\mathcal{S}$ invariant, its restriction τ to $\mathcal{S}$ is a continuous representation of G acting on $\mathcal{S}$ and u is a cyclic vector of τ.*

Proof. We first prove that $\mathcal{S}$ is a Fréchet space. Let $E = \mathcal{H}$ and $F = \mathcal{T}$. By the properties i), ii) and iii) stated at the beginning of this section, we are in the general setting of Section 2.4. Observe that $\mathcal{H}' = \mathcal{H}$, $V_e = V_2$ and clearly $\mathcal{S} = \mathrm{Co}(\mathcal{H}, \mathcal{T})$. Furthermore, the fact that π is reproducing implies that V_2 is injective and, by (25) in Proposition 2.8, the reproducing formula holds true for all $v \in \mathcal{H}$. Hence $V_2 v \in j(\mathcal{M}^{\mathcal{T}})$ for all $v \in \mathcal{S}$, and we actually get $\mathcal{S} = \mathrm{Co}(\mathcal{H}, \mathcal{M}^{\mathcal{T}})$. We can apply Corollary 2.7 because the hypotheses of Proposition 2.6 that imply it are both satisfied: (19) is just Assumption 2 and, as already noticed, the reproducing property holds for all $v \in \mathcal{S}$ because π is reproducing. Hence $\mathcal{S}$ is a Fréchet space and V_0 induces a topological linear isomorphism from $\mathcal{S}$ onto $\mathcal{M}^{\mathcal{T}}$. Since $V_2\mathcal{H} \subset L^2(G)$, clearly $j(\mathcal{M}^{\mathcal{T}}) \subset L^2(G)$.

Since $\mathcal{S}$ and $\mathcal{M}^{\mathcal{T}}$ are isomorphic, in order to show that i is continuous it is enough to prove that j is continuous from $\mathcal{M}^{\mathcal{T}}$ into $L^2(G)$. Both are Fréchet spaces, hence it is sufficient to show that $j : \mathcal{M}^{\mathcal{T}} \to L^2(G)$ has sequentially closed graph. If $(f_n)_n$ is a sequence in $\mathcal{M}^{\mathcal{T}}$ converging to f in $\mathcal{M}^{\mathcal{T}}$ and $(j(f_n))_n$ converges to φ in $L^2(G)$, then possibly passing to a subsequence, we can assume that $(f_n(x))_n$ converges for almost all $x \in G$. Hence $\varphi(x) = f(x)$ almost everywhere.

Item b) of Proposition 2.6 gives that π leaves $\mathcal{S}$ invariant. Since for all $x \in G$ and $v \in \mathcal{S}$

$$V_0 \tau(x) v = \ell(x) V_0 v,$$

the restriction τ is a continuous representation on $\mathcal{S}$ because ℓ is a continuous representation on $\mathcal{T}$. The fact that π is reproducing implies that $\mathrm{span}\{\pi(x)u \mid x \in G\} \subset \mathcal{S}$ is dense in $\mathcal{H}$, so that i has dense range.

Finally, since $V_0 \tau(x) u = \ell(x) K$ for all $x \in G$, Assumption 3 is another way of saying that u is a cyclic vector for τ. As for the properties of ${}^t i$, Corollary 3 Chapter II.6.3 of [25] shows that i is continuous from $\mathcal{S}_{weak}$ into $\mathcal{H}_{weak}$. Hence, Corollary of Proposition 5 Chapter II.6.4 of [25] gives that ${}^t i$ is continuous from $\mathcal{H}_s = \mathcal{H}_{weak}$ into $\mathcal{S}'_s$ and ${}^t({}^t i) = i$. Finally, Corollary 2 Chapter II.6.4 of [25] shows that since i is injective and has dense range, ${}^t i$ has the same properties. □

As shown in the above proof, all the hypotheses of Corollary 2.7 are satisfied. This implies that, whenever a fundamental family $\{q_i\}_i$ of semi-norms of $\mathcal{T}$ is given, then $\{q_i(V_0(\cdot))\}_i$ is a fundamental family of semi-norms of $\mathcal{S}$. This is yet another way to get a direct handle on its topology when a family of seminorms of $\mathcal{T}$ is known.

We regard the dual $\mathcal{S}'$ of $\mathcal{S}$ as the space of *distributions* and we define the *extended voice transform* on it by setting for all $T \in \mathcal{S}'$

$$V_e : \mathcal{S}' \to C(G), \qquad V_e T = \langle T, \tau(\cdot)u\rangle_{\mathcal{S}}. \tag{30}$$

The definition works because $\mathcal{S}$ is τ-invariant, $u \in \mathcal{S}$ and τ is a continuous representation. The following theorem states the main properties of V_0 and V_e. We recall that the contragredient representations ${}^t\tau$ and ${}^t\ell$ are continuous representations acting on $\mathcal{S}'$ and $\mathcal{T}'$, respectively, where the dual spaces are endowed with the topology of the convergence on compact subsets (see Proposition 3 Chapter VIII.2.3 of [29]). Furthermore, since π is a reproducing representation, Proposition 2.8 ensures that for all $f \in L^2(G)$ the Fourier transform of f at u exists in $\mathcal{H}$.

Theorem 3.2. *The restricted voice transform V_0 is an injective strict morphism*[6] *from $\mathcal{S}$ into $\mathcal{T}$ with image $\mathcal{M}^{\mathcal{T}}$. For all $f \in \mathcal{M}^{\mathcal{T}}$, we have*

$$\pi(f)u \in \mathcal{S}, \qquad V_0\pi(f)u = f$$

and, for all $v \in \mathcal{S}$, we have

$$\pi(V_0 v)u = v.$$

Furthermore, V_0 intertwines τ and ℓ and its transpose ${}^tV_0 : \mathcal{T}'_s \to \mathcal{S}'_s$ is a surjective continuous map, intertwining the representations ${}^t\ell$ and ${}^t\tau$.

The extended voice transform V_e intertwines ${}^t\tau$ with λ, is injective and continuous from $\mathcal{S}'$ to $C(G)$, where both spaces are endowed with the topology of compact convergence. Finally, for all $\Phi \in \mathcal{T}^{\#} \subset \mathcal{T}'$, we have

$$V_e\, {}^tV_0\Phi = \Phi * K. \tag{31}$$

Proof. By Theorem 3.1, V_0 induces a topological linear isomorphism from $\mathcal{S}$ onto $\mathcal{M}^{\mathcal{T}}$, which is a closed subspace of $\mathcal{T}$. Corollary 1, Chapter II.4.2 of [25] gives that ${}^tV_0$ is surjective. By Corollary of Proposition 5, Chapter II.6.4 of [25], the map ${}^tV_0$ is continuous when both $\mathcal{T}'$ and $\mathcal{S}'$ are equipped with the topology of the simple convergence.

Since π is reproducing and $j(f) \in L^2(G)$ for all $f \in \mathcal{M}^{\mathcal{T}}$, Proposition 2.8 shows that $V_2\pi(j(f))u = j(f) \in j(\mathcal{T})$. Hence, by definition of $\mathcal{S}$, $\pi(j(f))u \in \mathcal{S}$ and the construction of V_0 gives that $V_0\pi(f)u = f$, where, with slight abuse of notation, $\pi(f)u$ is the Fourier transform of $j(f)$ at u. Take now $v \in \mathcal{S}$. Since $V_0 v \in \mathcal{M}^{\mathcal{T}}$, again Proposition 2.8 yields $\pi(V_0 v)u = v$.

The intertwining property is straightforward: for any $x, y \in G$

$$\left(V_e\, {}^t\tau(x)T\right)(y) = \langle T, \tau(x^{-1})\tau(y)u\rangle_{\mathcal{S}} = V_eT(x^{-1}y).$$

Injectivity is due to the fact that u is cyclic for τ. To prove that V_e is continuous, fix a compact subset Q of G. Since $x \mapsto \tau(x)u$ is continuous, the set $A = \tau(Q)u$ is compact in $\mathcal{S}$, and $T \mapsto \sup_{v\in A}|\langle T, v\rangle_{\mathcal{S}}|$ is continuous on $\mathcal{S}'$. Finally, take $\Phi \in \mathcal{T}^{\#}$. Then for all $x \in G$

$$V_e\, {}^tV_0\Phi(x) = \langle \Phi, V_0\tau(x)u\rangle_{\mathcal{T}} = \int_G \Phi(y)\overline{\langle \pi(x)u, \pi(y)u\rangle_{\mathcal{H}}}dy = (\Phi * V_2u)(x). \qquad \square$$

[6]A strict morphism is a continuous linear map whose image is closed.

We add a remark on the finer topological properties of V_e. If B is a bounded subset of $\mathcal{S}'$ or, equivalently, of $\mathcal{S}'_s$, the restriction of V_e to B, endowed with the topology of $\mathcal{S}'_s$, into $C(G)$, with the topology of the compact convergence, is continuous. Indeed, since $\mathcal{S}$ is a Fréchet space, then it is barrelled (Corollary of Proposition 2 Chapter III.4.2 of [25]). Hence

$$\text{strongly bounded} \Leftrightarrow \text{weakly bounded} \Leftrightarrow \text{equicontinuous}.$$

(Scolium and Definition 2 Chapter III.4.2 of [25]). Proposition 5 Chapter III.3.4 of [25] implies that on B the topology of the simple convergence is equivalent to the topology of precompact subsets. Hence, for any compact subset $\mathcal{K}$ of G, since $x \mapsto \tau(x)u$ is continuous, the set $A = \tau(\mathcal{K})u$ is compact in $\mathcal{S}$, hence precompact and, by the above reasoning $B \ni T \mapsto \sup_{v\in A}|\langle T, v\rangle_{\mathcal{S}}|$ is continuous with respect to the topology of the simple convergence.

The next assumption requires that the reproducing formula holds for any distribution in $\mathcal{S}'$.

Assumption 4. For all $T \in \mathcal{S}'$, $K\, V_eT \in L^1(G)$ and $V_eT * K = V_eT$.

Since the representation ${}^t\tau$ leaves $\mathcal{S}'$ invariant and V_e intertwines ${}^t\tau$ with λ, the requirement $K\, V_eT \in L^1(G)$ implies that $V_eT * K$ exists. Furthermore, if $V_e\mathcal{S}'$ is contained in $\mathcal{T}^\#$, then $K\, V_eT \in L^1(G)$ holds for all $T \in \mathcal{S}'$ since $K \in \mathcal{T}$.

In the two propositions that follow, we give sufficient conditions implying Assumption 4.

Proposition 3.3. *Assume that $\mathcal{M}^{\mathcal{T}}$ is a reflexive space and $K\, V_eT \in L^1(G)$ for all $T \in \mathcal{S}'$. Then the reproducing formula $V_eT * K = V_eT$ holds true for all $T \in \mathcal{S}'$.*

Proof. Since $\mathcal{S}$ and $\mathcal{M}^{\mathcal{T}}$ are isomorphic (Theorem 3.1), then also $\mathcal{S}$ is a reflexive space. Regard $\mathcal{S}$ as the dual of $\mathcal{S}'$, which has the property (GDF) by Proposition 3 Chapter 6. Appendix No.2 of [30]. The assumption implies that the map $x \mapsto \tau(x)u\langle\pi(y)u, u\rangle_{\mathcal{H}}$ is scalarly integrable from G to $\mathcal{S}_s$, hence Theorem 6.4 shows that there exists $v_u \in \mathcal{S}$ such that

$$\langle T, v_u\rangle_{\mathcal{S}} = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}}\langle\pi(x)u, u\rangle_{\mathcal{H}}dx.$$

By Theorem 3.1, $\mathcal{H}$ is dense in $\mathcal{S}'_s$ and by (25) $v_u = u$, which means that

$$\langle T, u\rangle_{\mathcal{S}} = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}}\langle\pi(x)u, u\rangle_{\mathcal{H}}dx.$$

Given $y \in G$, by applying the above equality to ${}^t\tau(y^{-1})T$, we get

$$\begin{aligned} V_eT\,(y) &= \langle T, \tau(y)u\rangle_{\mathcal{S}} = \langle {}^t\tau(y^{-1})T, u\rangle_{\mathcal{S}} \\ &= \int \langle {}^t\tau(y^{-1})T, \tau(x)u\rangle_{\mathcal{S}}\langle\pi(x)u, u\rangle_{\mathcal{H}}\, dx \\ &= \int \langle T, \tau(yx)u\rangle_{\mathcal{S}}\langle\pi(x)u, u\rangle_{\mathcal{H}}\, dx \\ &= \int \langle T, \tau(x)u\rangle_{\mathcal{S}}\langle u, \pi(x^{-1}y)u\rangle_{\mathcal{H}}\, dx, \end{aligned}$$

where the last line is due to the change of variable $x \mapsto y^{-1}x$ and the fact that π is a unitary representation. Hence the convolution $VT * K$ exists and is equal to V_eT. □

The property $K\, V_e T \in L^1(G)$ for all $T \in \mathcal{S}'$ means that the map $x \mapsto \tau(x)u\langle \pi(y)u, u\rangle_{\mathcal{H}}$ is scalarly integrable from G to $\mathcal{S}$, *i.e.*, there exists a linear map $\omega : \mathcal{S}' \to \mathbb{C}$ such that

$$\omega(T) = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}} \langle \pi(x)u, u\rangle_{\mathcal{H}} dx.$$

Furthermore, since $\mathcal{H}$ is continuously embedded in $\mathcal{S}'$ and π is a reproducing representation, for all $w \in \mathcal{H}$ we have

$$\omega({}^t\imath(w)) = \langle w, u\rangle_{\mathcal{H}}.$$

By Theorem 3.1, the map ${}^t\imath$ has a dense image in $\mathcal{S}'_s$. However, ω is continuous with respect to the weak-$*$ topology of $\mathcal{S}'$ if and only if $\omega \in \mathcal{S}$. In the setting of reproducing representations, the requirement that the reproducing formula holds for all distributions is equivalent to assuming that $\omega \in \mathcal{S}$ and, in this case, ω is precisely u. The hypothesis $\omega \in \mathcal{S}$ is precisely property (R4) in [21]. Furthermore, if $\mathcal{S}$ is a Banach space, as in the classical setting, and if the map $x \mapsto \tau(x)u\langle \pi(y)u, u\rangle_{\mathcal{H}}$ is Bochner-integrable, then it is scalarly integrable and, clearly, ω is always in $\mathcal{S}$.

Here is another sufficient condition.

Proposition 3.4. *Assume that $\mathcal{T}^\# = \mathcal{T}'$ and suppose that $|K| \ast |K|$ exists and belongs to $\mathcal{T}$. Then $V_e T \ast K = V_e T$ for all $T \in \mathcal{S}'$.*

Proof. By Theorem 3.1, ${}^tV_0$ is surjective, so that if $\mathcal{T}' = \mathcal{T}^\#$, then for any $T \in \mathcal{S}'$ there exists $\Phi \in \mathcal{T}^\#$ such that ${}^tV_0\Phi = T$. Furthermore, if $|K| \ast |K|$ exists and belongs to $\mathcal{T}$, then

$$\begin{aligned}\int_{G\times G} |\Phi(zx)\langle \pi(x)u, \pi(y)u\rangle_{\mathcal{H}} \langle \pi(y)u, u\rangle_{\mathcal{H}}| dx\, dy &= \int_G |\Phi(zx)| \left(\int_G |K(y)||K(y^{-1}x)| dy \right) dx \\ &= \int_G |\lambda(z^{-1})\Phi(x)| (|K| \ast |K|)(x)\, dx\end{aligned}$$

and, since $|\lambda(\underbrace{z^{-1})\Phi|} \in \mathcal{T}^\#$, the last integral is finite for all $z \in G$. By (77a) and (23), we have that $|K| \ast |K| = |K| \ast |K|$, hence Fubini theorem implies that the convolution $|\Phi| \ast (|K| \ast |K|)$ exists, and (77d) in the appendix shows

$$(\Phi \ast K) \ast K = \Phi \ast (K \ast K).$$

Finally, (31) and (25) give

$$V_e T \ast K = (\Phi \ast K) \ast K = \Phi \ast (K \ast K) = \Phi \ast K = V_e T. \qquad \square$$

3.2. **Coorbit spaces.** We now fix a Banach space Y, with norm $\|\cdot\|_Y$, continuously embedded in $L^0(G)$ and λ-invariant. In order to be consistent with the current literature, we do not indicate the explicit embedding as we did for the other spaces. The results in this section hold true under the weaker assumption that Y is a Fréchet space. However, we do not need this generality because the main example that we are interested in is the case when Y is a weighted L^p space for a fixed value of p.

The *coorbit space* of Y is

$$\mathrm{Co}(Y) = \{T \in \mathcal{S}' \mid V_e T \in Y\} \tag{32}$$

endowed with the norm

$$\|T\|_{\mathrm{Co}(Y)} = \|V_e T\|_Y. \tag{33}$$

Since V_e is a linear injective map, $\|\cdot\|_{\mathrm{Co}(Y)}$ is clearly a norm. We will prove below that $\mathrm{Co}(Y)$ is in fact a Banach space.

Just as for the target space $\mathcal{T}$, the two basic assumptions for the space Y may be formulated in terms of Köthe duals and have to do with the kernel K (compare Assumption 5 below with Assumption 1) and with the image of the voice transform $V_2\mathcal{H}$ (compare Assumption 6 below with Assumption 2). They should also be compared with the corresponding assumptions made in [20, 23, 21]. As above, we write

$$\mathcal{M}^Y = \{f \in Y \mid f * K = f\}.$$

Assumption 5. For all $f \in Y$, we have $fK \in L^1(G)$, that is, $K \in Y^{\#}$.

Assumption 6. For all $f \in \mathcal{M}^Y$ and all $v \in \mathcal{S}$, we have $fV_0v \in L^1(G)$, *i.e.*, $V_0\mathcal{S} \subset (\mathcal{M}^Y)^{\#}$.

By Proposition 2.2 applied to $F = Y$, Assumption 5 implies that $\mathcal{M}^Y$ is a λ-invariant closed subspace of Y. Furthermore, by Proposition 2.4 with $E = \mathcal{S}$, Assumption 6 implies that for all $f \in \mathcal{M}^Y$ the Fourier transform of f at u exists in $\mathcal{S}'$.

In the following proposition we list the main properties of $\mathrm{Co}(Y)$.

Proposition 3.5. *The space* $\mathrm{Co}(Y)$ *is a Banach space invariant under the action of the representation* ${}^t\tau$. *The extended voice transform is an isometry from* $\mathrm{Co}(Y)$ *onto* $\mathcal{M}^Y$ *and its inverse is the Fourier transform at* u. *Therefore*

$$\begin{aligned}
&V_e\,\mathrm{Co}(Y) = \mathcal{M}^Y,\\
&\{\pi(f)u \mid f \in \mathcal{M}^Y\} = \mathrm{Co}(Y),\\
&V_e\pi(f)u = f, \qquad f \in \mathcal{M}^Y,\\
&\pi(V_eT)u = T, \qquad T \in \mathrm{Co}(Y).
\end{aligned}$$

Proof. The proposition is a restatement of Proposition 2.6 and Corollary 2.7 with $E = \mathcal{S}$ and $F = \mathcal{M}^T$. The hypothesis (19) is Assumption 6 and the hypothesis in item d) of Proposition 2.6 is satisfied by Assumption 3 and Assumption 4. □

As in the classical setting, we have the following canonical identification.

Corollary 3.6. *The Hilbert space* $L^2(G)$ *satisfies Assumptions* 5 *and* 6, *and* $\mathrm{Co}(L^2(G)) = \mathcal{H}$.

Proof. Since π is a reproducing representation, Assumptions 5, and 6 are clearly satisfied, and $\mathcal{H} \subset \mathrm{Co}(L^2(G))$. Take now $T \in \mathrm{Co}(L^2(G))$. By Proposition 3.5 $T = \pi(V_eT)u$. However, since $V_2T \in L^2(G)$, by Proposition 2.8 $\pi(V_eT)u \in \mathcal{H}$. □

Even though $\mathcal{T}$ is not a Banach space, the space

$$\mathrm{Co}(\mathcal{T}) = \{T \in \mathcal{S}' \mid V_eT \in \mathcal{T}\}$$

is well defined and, under Assumption 4, Corollary 3.6 and the definition of $\mathcal{S}$ imply that, as in the classical setting, $\mathrm{Co}(\mathcal{T}) = \mathcal{S}$. The above identification suggests to characterize the space

$$\mathrm{Co}(\mathcal{T}') = \{T \in \mathcal{S}' \mid V_eT \in \mathcal{T}^{\#}\} \subset \mathcal{S}'.$$

The equality $\mathrm{Co}(\mathcal{T}') = \mathcal{S}'$ is equivalent to require that $j(f)V_eT \in L^1(G)$ for all $f \in \mathcal{T}$ and $T \in \mathcal{S}'$, that is, $V_e\mathcal{S}' \subset \mathcal{T}^{\#}$, which is in general stronger than Assumption 4.

Let us compare our approach with the theory developed by J. Christensen and G. Ólafsson in [20, 23, 21]. Assumptions 1÷6 ensure that the test space $\mathcal{S}$ defined by (29) satisfies the properties (R1)÷(R4), and some of our claims can be directly deduced by the results contained in [21] (for example, compare Theorem 2.3 of [21] with our Proposition 3.5). In our setting, which is somehow parallel to the classical L^1 case, we first introduce the target space $\mathcal{T}$, which is independent of the reproducing representation, and then we define the test space $\mathcal{S}$ as the set of vectors for which the voice transform belongs to $\mathcal{T}$. The introduction of the target space $\mathcal{T}$ makes our construction closer to the classical approach by H. Feichtinger and K. Gröchenig, and Assumptions 1, 2, 3 and 4 involve only the target space $\mathcal{T}$ without any reference to the model space Y. Moreover, our proofs mainly rely on the theory of weak integrals *à la Dunford-Pettis*, which allows us to state our hypotheses as integrability conditions, rather than a continuity requirement as in [21].

Assumption 4 requires that the reproducing formula $V_eT * K = V_eT$ holds for all $T \in \mathcal{S}'$. However, in the proof of Proposition 3.5, the reproducing formula is needed only for the distributions in $\mathrm{Co}(Y)$ (compare with item d) of Proposition 2.6). The following lemma shows some equivalent conditions, weaker than Assumption 4, under which Proposition 3.5 remains true.

Lemma 3.7. *Take $\mathcal{T}$ and Y such that Assumptions $1, 2, 3$ and Assumptions $5, 6$ hold true. Then the following facts are equivalent:*

a) for all $T \in \mathrm{Co}(Y)$, $V_eT \in \mathcal{M}^Y$;

b) for all $T \in \mathrm{Co}(Y)$, $V_eT * K$ exists and $V_eT * K = V_eT$;

c) for all $T \in \mathrm{Co}(Y)$, the map $x \mapsto \langle T, \tau(x)u\rangle_{\mathcal{S}}\langle \pi(x)u, u\rangle_{\mathcal{H}} = V_eT(x)\overline{K(x)}$ is in $L^1(G)$ and

$$\int_G \langle T, \tau(x)u\rangle_{\mathcal{S}}\langle \pi(x)u, u\rangle_{\mathcal{H}} dx = \langle T, u\rangle_{\mathcal{S}}; \tag{34}$$

d) for all $T \in \mathrm{Co}(Y)$, the map $x \mapsto V_eT\,(x)\,{}^t\!i(\pi(x)u) \in \mathcal{S}'_s$ is scalarly integrable and its scalar integral is T, that is

$$T = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}}\, {}^t\!i(\pi(x)u)\, dx. \tag{35}$$

Proof. By definition of coorbit space, $V_eT \in Y$ whenever $T \in \mathrm{Co}(Y)$. Hence a) is equivalent to b). Since $\mathrm{Co}(Y)$ is ${}^t\tau$-invariant, c) implies that the map $y \mapsto \langle\, {}^t\tau(x^{-1})T, \tau(y)u\rangle_{\mathcal{S}}\langle \pi(y)u, u\rangle_{\mathcal{H}}$ is integrable for all $x \in G$ and

$$\begin{aligned} V_eT\,(x) = \langle T, \tau(x)u\rangle_{\mathcal{S}} &= \int \langle\, {}^t\tau(x^{-1})T, \tau(y)u\rangle_{\mathcal{S}}\langle \pi(y)u, u\rangle_{\mathcal{H}}\, dy \\ &= \int \langle T, \tau(y)u\rangle_{\mathcal{S}}\langle \pi(y)u, \pi(x)u\rangle_{\mathcal{H}}\, dy. \end{aligned}$$

Hence c) implies item b) of Lemma 2.5. The converse is also true by evaluation at the identity. Therefore c) is equivalent to item b) of Lemma 2.5, which provides the equivalence between b) and c) and shows that d) implies c).

Assume now that $V_eT \in \mathcal{M}^Y$. Proposition 2.4 with $f = V_eT$ gives that V_eT satisfies (13), that $\pi(V_eT)u \in \mathcal{S}'$ exists and $V_e\pi(V_eT)u = V_eT$. Finally, since V_e is injective by Theorem 3.2, we know from item d) of Lemma 2.5 that a) implies d). □

3.3. **Dependency on the admissible vector.** We now examine the dependence of space $\mathcal{S}$ on the choice of the admisible vector u. For this reason, in this section, we write $\mathcal{S}_u$ instead of $\mathcal{S}$, and accordingly for other choices of admissible vectors.

Proposition 3.8. *Suppose that $j(\mathcal{T}) * j(\check{\mathcal{T}}) \subset j(\mathcal{T})$ and that for all $g \in \mathcal{T}$ the map*

$$f \mapsto f * \check{g} \tag{36}$$

is continuous from $\mathcal{T}$ into itself, where $j(f * \check{g}) = j(f) * \check{j}(g)$. If $\tilde{u} \in \mathcal{S}_u \subset \mathcal{H}$ is another admissible vector satisfying Assumptions 1, 2 and 3, then the test function spaces $\mathcal{S}_{\tilde{u}}$ and $\mathcal{S}_u$ coincide as Fréchet spaces. Furthermore, $\breve{\mathcal{S}}_u = \mathcal{S}_u$ for any admissible u and

$$V_{\tilde{u}}v = V_u v * \overline{\breve{V_u\tilde{u}}}.$$

for all $v \in \mathcal{S}_{\tilde{u}} = \mathcal{S}_u$.

Proof. Let $v \in \mathcal{S}_u$ and $x \in G$. Since π is reproducing and u is admissible,

$$\begin{aligned} V_{2,\tilde{u}}v\,(x) &= \langle v, \pi(x)\tilde{u}\rangle_{\mathcal{H}} \\ &= \int_G \langle v, \pi(y)u\rangle_{\mathcal{H}} \overline{\langle \pi(x)\tilde{u}, \pi(y)u\rangle_{\mathcal{H}}}\, dy \\ &= \int_G \langle v, \pi(y)u\rangle_{\mathcal{H}} \overline{\langle \tilde{u}, \pi(x^{-1}y)u\rangle_{\mathcal{H}}}\, dy \\ &= V_{2,u}v * \overline{\breve{V_{2,u}\tilde{u}}}\,(x), \end{aligned}$$

where $V_{2,u}v, V_{2,u}\tilde{u} \in j(\mathcal{T})$ since $v, \tilde{u} \in \mathcal{S}_u$. The hypothsis on $\mathcal{T}$ implies that $V_{2,\tilde{u}}v \in j(\mathcal{T})$, so that $\mathcal{S}_u \subset \mathcal{S}_{\tilde{u}}$. We now prove that the embedding of $\mathcal{S}_u$ into $\mathcal{S}_{\tilde{u}}$ is continuous. Fix a semi-norm $\|\cdot\|_{i,\mathcal{S}_{\tilde{u}}}$ of $\mathcal{S}_{\tilde{u}}$, *i.e*, fix a semi-norm $\|\cdot\|_{i,\mathcal{T}}$ of $\mathcal{T}$ such that $\|v\|_{i,\mathcal{S}_{\tilde{u}}} = \|V_{2,\tilde{u}}v\|_{i,\mathcal{T}}$ for all $v \in \mathcal{S}_{\tilde{u}}$. By (36) with $f = V_2 v$ and $g = \overline{V_0\tilde{u}}$, there exist a constant $C > 0$ and a semi-norm $\|\cdot\|_{j,\mathcal{T}}$ of $\mathcal{T}$ such that

$$\|v\|_{i,\mathcal{S}_{\tilde{u}}} = \|V_{2,\tilde{u}}v\|_{i,\mathcal{T}} \leqslant C\|V_2 v\|_{j,\mathcal{T}} = C\|v\|_{j,\mathcal{S}_u}$$

where $\|\cdot\|_{j,\mathcal{S}_u}$ is a semi-norm of $\mathcal{S}_u$. Hence, the embedding is continuous. Interchanging the roles of u and $\tilde{u}$, we obtain that $\mathcal{S}_{\tilde{u}} \subset \mathcal{S}_u$ with a continuous embedding. Finally by (77a) in the appendix and (23), for all $v \in \mathcal{S}_u$,

$$\breve{V_0 v} = \breve{V_0 v * K} = \overline{K} * \breve{V_0 v} \in \mathcal{T}$$

by assumption, so that $\breve{\mathcal{S}}_u \subset \mathcal{S}_u$ and, hence, $\breve{\mathcal{S}}_u = \mathcal{S}_u$. □

In the classical framework, π is irreducible and $\mathcal{T} = L^1(G, w\beta)$, where w is a continuous density satisfying (60a) and (60b) in the appendix and

$$w(x) = w(x^{-1})\Delta(x^{-1}). \tag{37}$$

This last condition implies that $\mathcal{T} = \check{\mathcal{T}}$ so that the hypotheses of the above proposition are satisfied. However, a stronger result holds true, namely

$$\{u \in \mathcal{H} \mid K_u \in \mathcal{T}\} = \mathcal{S},$$

which is the content of Lemma 4.2 in [1]. Note that the irreducibility ensures that, if $K_u \in \mathcal{T}$, then u is an admissible vector. However, if π is not irreducible, the above equality does not hold as shown by a counter-example in [22].

4. A model for the target space

In this section, we illustrate some examples. They include band-limited functions (Section 4.2), Shannon wavelets (Section 4.3) and *Schrödingerlets* (Section 4.4) that have inspired our theory.

4.1. **Intersection of all $L^p_w(G)$ with $1 < p < +\infty$.** In this section, $w : G \to (0, +\infty)$ will denote a continuous function, to be called *weight*, satisfying

$$w(xy) \leqslant w(x)w(y) \tag{38a}$$

$$w(x) = w(x^{-1}) \tag{38b}$$

for all $x, y \in G$. As a consequence, it also holds that

$$\inf_{x \in G} w(x) \geqslant 1. \tag{38c}$$

The notion of weight in [4] is based on the submultiplicative property (38a). The symmetry (38b) can always be satisfied by replacing w with $w + \check{w}$. This requirement is necessary for our development (see item g) of Theorem 4.4 below). Condition (38c) is explicitly stated in [4] and, in the classical $L^1(G)$ setting, it is necessary to ensure that the test space is a Banach space (see Theorem 5.5 below). In the usual irreducible L^1 setting, it is also assumed that the weight satisfies (37), which is actually incompatible with (38b). However, (37) is only necessary in order to see that the space of admissible vectors coincides with the test space (see Lemma 4.2 in [1]). In the non irreducible case, though, this set-theoretic equality is lost anyhow, as mentioned in the introduction [22].

For all $p \in [1, \infty)$ define the separable Banach space

$$L^p_w(G) = \{f \in L^0(G) \mid \int_G |w(x)f(x)|^p dx < +\infty\}$$

with norm

$$\|f\|^p_{p,w} = \int_G |w(x)f(x)|^p dx,$$

and the obvious modifications for $p = \infty$. Clearly, the map $J_p : L^p_w(G) \to L^p(G)$ defined by $J_p(f) = wf$ is a unitary operator. The following characterization of the Köthe dual holds true.

Lemma 4.1. *Fix $p \in [1, +\infty)$ and denote by $q = \frac{p}{p-1} \in (1, +\infty]$ the dual exponent. Then*

$$L^p_w(G)^\# = L^q_{w^{-1}}(G).$$

For all $g \in L^q_{w^{-1}}(G)$ and $f \in L^p_w(G)$, set

$$\langle g, f\rangle_{p,w} = \int_G g(x)\overline{f(x)}dx.$$

Then the map $g \mapsto \langle g, \cdot\rangle_{p,w}$ is an isomorphism from $L^q_{w^{-1}}(G)$ onto $L^p_w(G)'$. Under this identification, the transpose ${}^tJ_p : L^q(G) \to L^q_{w^{-1}}(G)$ is given by

$${}^tJ_p h = wh.$$

Proof. For $g \in L^0(G)$ we have $g \in L^p_w(G)^\#$ if and only if $gf \in L^1(G)$ for every $f \in L^p_w(G)$, which is equivalent to $(w^{-1}g)(wf) \in L^1(G)$ for every $wf \in L^p(G)$. This, in turn, happens if and only if $w^{-1}g \in L^q(G)$, which means that $g \in L^q_{w^{-1}}(G)$. Hence $L^q_{w^{-1}}(G) = L^p_w(G)^\# \subset L^p_w(G)'$, the pairing $\langle \cdot, \cdot\rangle_{p,w}$ is the pairing between $L^p_w(G)^\#$ and $L^p_w(G)$ given in Lemma 2.1, and

$$\|g\|_{L^p_w(G)'} = \sup_{\|f\|_{p,w} \leqslant 1} |\langle g, f\rangle_{p,w}| = \sup_{\|wf\|_p \leqslant 1} |\langle w^{-1}g, wf\rangle_p| = \|w^{-1}g\|_q = \|g\|_{q,w^{-1}}.$$

Thus the map $g \mapsto \langle g, \cdot\rangle_{p,w}$ is an isometry from $L^q_{w^{-1}}(G)$ into $L^p_w(G)'$ and it allows to identify $L^q_{w^{-1}}(G)$ with a closed subspace of $L^p_w(G)'$.

We now compute the transpose of J_p, taking into account that $L^p(G)' = L^q(G)$. For a fixed $h \in L^q(G)$ and all $f \in L^p_w(G)$ we have

$$\langle {}^tJ_p h, f\rangle_{p,w} = \int_G h(x)\overline{w(x)f(x)}dx,$$

so that, since w is positive, ${}^tJ_p h = wh \in L^p_w(G)^\# = L^q_{w^{-1}}(G)$. Since J_p is unitary, so is ${}^tJ_p$ and $L^p_w(G)' = L^q_{w^{-1}}(G)$. □

Lemma 4.2. *For all $p \in [1, +\infty)$, the left regular representation leaves $L^p_w(G)$ invariant. The restriction ℓ of λ to $L^p_w(G)$ is a continuous representation with $\|\ell(x)\| \leqslant w(x)$ for all $x \in G$.*

Proof. Fix $x \in G$. By (38a), for all $f \in L^p_w(G)$

$$\int_G |w(y)f(x^{-1}y)|^p dy = \int_G |w(xy)f(y)|^p dy \leqslant w(x)^p \int_G |w(y)f(y)|^p dy,$$

so that $\lambda(x)$ leaves $L^p_w(G)$ invariant and the norm of the restriction $\ell(x)$ is bounded by $w(x)$. We now prove that ℓ is continuous by applying the concluding remark of Section 6.2 in the appendix. For any compact subset $\mathcal{K}$ of G, since w is continuous, $w(\mathcal{K})$ is bounded and, hence, $\ell(\mathcal{K})$ is equicontinuous. Furthermore, if $f \in C_c(G)$, the map $x \mapsto \ell(x)f$ is clearly continuous from G into $L^q_w(G)$ by the dominated convergence theorem. The proof is completed by observing that $C_c(G)$ is a dense subset of $L^p_w(G)$. □

Let $I = (1, +\infty)$. We define the target space as the set

$$\mathcal{T}_w = \bigcap_{p \in I} L^p_w(G)$$

with the initial topology, which makes each inclusion $i_p : \mathcal{T}_w \hookrightarrow L^p_w(G)$ continuous, and endow

$$\mathcal{U}_w = \operatorname{span} \bigcup_{q \in I} L^q_{w^{-1}}(G)$$

with the final topology, which makes each inclusion $\tilde{\imath}_q : L^q_{w^{-1}}(G) \hookrightarrow \mathcal{U}_w$ continuous.

Recall that by definition of initial and of final topology, for any topological space X, a map $A : X \to \mathcal{T}_w$ is continuous if for all $p \in I$ there exists a continuous map $A_p : X \to L^p_w(G)$ such that $i_p A = A_p$, and a map $B : \mathcal{U}_w \to X$ is continuous if all $q \in I$ there exists a continuous map $B_q : L^q_{w^{-1}}(G) \to X$ such that $B\tilde{\imath}_q = B_q$.

As for notation, given the nature of $\mathcal{T}_w$, the inclusion $j : \mathcal{T}_w \to L^0(G)$ is set-theoretically tautological because the elements of $\mathcal{T}_w$ are (equivalence classes of) measurable functions. However, we keep it to emphasize that the two spaces, $\mathcal{T}_w$ and $L^0(G)$, have different topologies.

The following theorem states the main properties of $\mathcal{T}_w$ and $\mathcal{U}_w$.

Theorem 4.3. *The space $\mathcal{T}_w$ is a reflexive Fréchet space, whose topology is given by the fundamental family of semi-norms $\{\|\cdot\|_{p,w}\}_{p \in I}$. It is closed under complex conjugation and $f \mapsto \overline{f}$ is continuous. The canonical inclusion $j : \mathcal{T}_w \to L^0(G)$ is continuous, the left regular representation λ leaves $\mathcal{T}_w$ invariant and the restriction ℓ of λ to $\mathcal{T}_w$ is a continuous representation of G on $\mathcal{T}_w$.*

The space $\mathcal{U}_w$ is a complete reflexive locally convex topological vector space. For each $g \in \mathcal{U}_w$, the anti-linear map from $\mathcal{T}_w$ into $\mathbb{C}$ given by

$$f \mapsto \int_G g(x)\overline{f(x)}\, dx = \langle g, f\rangle_{\mathcal{T}_w}$$

is continuous and $g \mapsto \langle g, \cdot \rangle_{\mathcal{T}_w}$ identifies, as topological vector spaces, the dual of $\mathcal{T}_w$ with $\mathcal{U}_w$. Furthermore the Köthe dual of $\mathcal{T}_w$ is $\mathcal{U}_w$, so that

$$\mathcal{T}_w' = \mathcal{T}_w^{\#} = \mathcal{U}_w. \tag{39}$$

For each $f \in \mathcal{T}_w$, the anti-linear map from $\mathcal{U}_w$ to $\mathbb{C}$

$$g \mapsto \int_G f(x)\overline{g(x)}\, dx = \langle f, g \rangle_{\mathcal{U}_w}$$

is continuous and $f \mapsto \langle f, \cdot \rangle_{\mathcal{U}_w}$ identifies, as topological vector spaces, the dual of $\mathcal{U}_w$ with $\mathcal{T}_w$.

Proof. The proof is based on the content of the article [31], whose main results are summarized by Theorem 6.5 in the appendix, where $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{U} = \mathcal{U}_1$ (in [31] it is assumed that $w = 1$).

By definition of initial topology, $\mathcal{T}_w$ is a locally convex topological space and $\{\|\cdot\|_{p,w}\}_{p\in I}$ is a fundamental family of semi-norms.

Clearly, $\mathcal{T}_w$ is closed under complex conjugation and is left invariant by λ. We show that ℓ is a continuos representation. Given $x \in G$, $\ell(x) : \mathcal{T}_w \to \mathcal{T}_w$ is continuous because $i_p\ell(x) = \lambda(x)i_p$. Given $f \in \mathcal{T}_w$, the map $x \mapsto \ell(x)f$ is continuous from G to $\mathcal{T}_w$ since such are the maps $x \mapsto i_p\ell(x)f = \ell(x)i_pf$ for all $p \in I$. The proof that complex conjugation is continuous is similar.

Define the linear map $J : \mathcal{T}_w \to \mathcal{T}$, $Jf = wf$. Since $w > 0$, J is a bijection whose inverse is given by $J^{-1}g = w^{-1}g$. Both maps are continuous by definition of initial topology since

$$i_pJ = J_pi_p \qquad i_pJ^{-1} = J_p^{-1}i_p,$$

for all p. Hence J is a topological isomorphism. By Theorem 6.5, we infer that $\mathcal{T}$ is a reflexive Fréchet space and, hence, $\mathcal{T}_w$ is a reflexive Fréchet space, too.

Define $\tilde{J} : \mathcal{U} \to \mathcal{U}_w$, $\tilde{J}h = wh$, which is clearly bijective and whose inverse is $\tilde{J}^{-1}g = w^{-1}g$. By definition of final topology, both are continuos since for all $q \in I$

$$\tilde{J}\,\tilde{\imath}_q = \tilde{\imath}_q\, {}^tJ_{\frac{q}{q-1}} \qquad \tilde{J}^{-1}\tilde{\imath}_q = \tilde{\imath}_q\, J_q^{-1}$$

(with slight abuse, here $\tilde{\imath}_q$ denotes the inclusion of $L^q(G)$ into $\mathcal{U}$). Hence $\tilde{J}$ is an isomorphism from $\mathcal{U}$ onto $\mathcal{U}_w$. Therefore, by Theorem 6.5, $\mathcal{U}_w$ is a complete barelled locally convex topological vector space since such is $\mathcal{U}$.

Since J is an isomorphism between two Fréchet spaces, by Corollary 5 of Chapter IV.4.2 of [25], ${}^tJ$ is an isomorphism from $\mathcal{U}$ onto $\mathcal{T}_w'$ explicitly given by

$$\langle {}^tJh, f \rangle_{\mathcal{T}_w} = \sum_{i=1}^n c_i \int_G h_i(x)w(x)\overline{f(x)}dx = \int_G (\tilde{J}h)(x)\overline{f(x)}dx,$$

where $h = \sum c_ih_i$ with $c_1, \ldots, c_n \in \mathbb{C}$ and $h_1 \in L^{q_1}(G)$, $\ldots$, $h_n \in L^{q_n}(G)$. Hence, we can identify $\mathcal{T}_w'$ and $\mathcal{U}_w$ as topological vector spaces by means of the map $\tilde{J}\,{}^tJ^{-1}$, and the pairing between $\mathcal{U}_w$ and $\mathcal{T}_w$ is

$$\langle g, f \rangle_{\mathcal{T}_w} = \int_G g(x)\overline{f(x)}dx. \qquad \square$$

Observe that (38c) implies $w(x)^{-1} \leqslant w(x)$ for all $x \in G$, so that $\mathcal{T}_w \subset \mathcal{U}_w = \mathcal{T}_w'$. Furthermore, (38b) ensures that $\check{f} \in \mathcal{T}_w$ if and only if $\widecheck{wf} \in \mathcal{T}$.

We are now ready to state the main result of this section.

Theorem 4.4. *Take a reproducing representation π of G acting on the Hilbert space $\mathcal{H}$ and a weight w satisfying* (38a), (38b) *and* (38c). *Choose an admissible vector $u \in \mathcal{H}$ such that*

$$K(\cdot) = \langle u, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^p_w(G) \text{ for all } p \in I \tag{40}$$

and set

$$\mathcal{S}_w = \{v \in \mathcal{H} \mid \langle v, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^p_w(G) \text{ for all } p \in I\},$$

$$\|v\|_{p,\mathcal{S}_w} = \left(\int_G |\langle v, \pi(x)u\rangle_{\mathcal{H}}|^p w(x)^p dx\right)^{\frac{1}{p}}.$$

Then:

a) the space $\mathcal{S}_w$ is a reflexive Fréchet space with respect to the topology induced by the family of semi-norms $\{\|v\|_{p,\mathcal{S}_w}\}_{p\in I}$, the canonical inclusion $i : \mathcal{S}_w \to \mathcal{H}$ is continuous and with dense range;

b) the representation π leaves $\mathcal{S}_w$ invariant, its restriction τ is a continuous representation acting on $\mathcal{S}_w$, and

$$i(\tau(x)v) = \pi(x)i(v) \qquad x \in G,\ v \in \mathcal{S}_w;$$

c) if $\mathcal{H}$ and $\mathcal{S}'_w$ are endowed with the weak topology, the transpose ${}^t i : \mathcal{H} \to \mathcal{S}'_w$ is continuous, injective, with dense range and satisfies the intertwining

$${}^t\tau(x)\, {}^t i(v) = {}^t i(\pi(x)v) \qquad x \in G,\ v \in \mathcal{H};$$

d) the restricted voice transform $V_0 : \mathcal{S}_w \to \mathcal{T}_w$, given by

$$V_0 v(x) = \langle i(v), \pi(x)u\rangle_{\mathcal{H}} \qquad x \in G,\ v \in \mathcal{S}_w,$$

is an injective strict morphism onto the closed subspace

$$\mathcal{M}^{\mathcal{T}_w} = \{f \in \mathcal{T}_w \mid j(f) * K = j(f)\},$$

and it intertwines τ and ℓ;

e) every $f \in \mathcal{T}_w$, has at u a Fourier transform in $\mathcal{S}_w$ and

$$j(V_0\pi(f)u) = j(f) * K;$$

furthermore, the map

$$\mathcal{T}_w \ni f \mapsto \pi(f)u \in \mathcal{S}_w$$

is continuous and its restriction to $\mathcal{M}^{\mathcal{T}_w}$ is the inverse of V_0;

f) every $\Phi \in \mathcal{U}_w$ has at u a Fourier transform in $\mathcal{S}'_w$ and

$$V_e\pi(\Phi)u = \Phi * K;$$

g) the extended voice transform given by (30) *takes values in $\mathcal{U}_w$, it is injective, continuous (when both spaces are endowed with the strong topology) and intertwines ${}^t\tau$ and λ; the range of V_e is the closed subspace*

$$\mathcal{M}^{\mathcal{U}_w} = \{\Phi \in \mathcal{U}_w \mid \Phi * K = \Phi\} = \operatorname{span} \bigcup_{p\in I} \mathcal{M}^{L^p_w(G)} \subset L^\infty_{w^{-1}}(G) \tag{41}$$

and for all $T \in \mathcal{S}'_w$ and $v \in \mathcal{S}_w$

$$\langle T, v\rangle_{\mathcal{S}_w} = \langle V_e T, V_0 v\rangle_{\mathcal{T}_w}; \tag{42}$$

h) the map

$$\mathcal{M}^{\mathcal{U}_w} \ni \Phi \mapsto \pi(\Phi)u \in \mathcal{S}'_w$$

is the inverse of V_e *and coincides with the restriction of the map* ${}^tV_0$ *to* $\mathcal{M}^{\mathcal{U}_w}$*, namely*

$$V_e({}^tV_0\Phi) = V_e\pi(\Phi)u = \Phi \qquad \Phi \in \mathcal{M}^{\mathcal{U}_w}. \tag{43}$$

i) ${}^t\!ii(\mathcal{S}_w) = \{T \in \mathcal{S}'_w \mid V_eT \in \mathcal{T}_w\} = \{\pi(f)u \mid f \in \mathcal{M}^{\mathcal{T}_w}\}$.

The fundamental requirement (40) states that $K \in \mathcal{T}_w \subset \mathcal{T}'_w = \mathcal{T}^\#_w = \mathcal{U}_w$, whereas (42) is the reconstruction formula for the distributions in $\mathcal{S}'_w$, namely the fact that for any $T \in \mathcal{S}'_w$ the formula

$$\langle T, v\rangle_{\mathcal{S}_w} = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}_w} \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\, dx \tag{44}$$

holds for all $v \in \mathcal{S}_w$, where the integral converges since $\langle \pi(\cdot)u, i(v)\rangle_{\mathcal{H}}$ is in $\mathcal{T}_w$ by definition of $\mathcal{S}_w$, and $\langle T, \tau(\cdot)u\rangle_{\mathcal{S}_w}$ is in $\mathcal{U}_w = \mathcal{T}^\#_w$ since the range of V_e is contained in $\mathcal{U}_w$ (for arbitrary target spaces this last property could fail).

Proof. Since $K \in j(\mathcal{T}_w) \subset \mathcal{U}_w = \mathcal{T}^\#_w$, Assumption 1 is satisfied. Furthermore, since $j(\mathcal{T}_w) \subset L^2_w(G) \subset L^2(G)$ by (38c), also Assumption 2 is satisfied. The topology induced by the family of semi-norms $\{\|v\|_{p,\mathcal{S}_w}\}_{p\in I}$ is the initial topology on $\mathcal{S}_w$ induced by the map V_0, as in the proof of Corollary 2.7.

a) By Theorem 3.1 which does not depend on Assumption 3, the space $\mathcal{S}_w$ is a Fréchet space isomorphic to $\mathcal{M}^{\mathcal{T}_w}$ and the canonical inclusion $i : \mathcal{S}_w \to \mathcal{H}$ is continuous and with dense range. Furthermore, since $\mathcal{M}^{\mathcal{T}_w}$ is a closed subspace of a reflexive Fréchet space, both $\mathcal{M}^{\mathcal{T}_w}$ and $\mathcal{S}_w$ are reflexive.
b) Apply Theorem 3.1.
c) Apply Theorem 3.1 for the main statement; the intertwining property is easily checked.
d) Apply Theorem 3.2.
e) Fix $f \in \mathcal{T}_w$. By (38c) $j(f) \in L^2(G)$ and by Proposition 2.8 there exists $\pi(f)u \in \mathcal{H}$ such that $V_2\pi(f)u = j(f) * K$. We claim that $j(f)$ and K are convolvable (see the appendix for the definition) and $j(f) * K \in j(\mathcal{T}_w)$. It is enough to show that for all $r \in I$, $|wf| * |wK| \in L^r(G)$. Indeed, given $x \in G$

$$\begin{aligned} w(x)\int_G |f(y)||K(y^{-1}x)|\, dy &= \int_G |w(y)f(y)||w(y^{-1}x)K(y^{-1}x)|\frac{w(x)}{w(y)w(y^{-1}x)}dy \\ &\leqslant |wj(f)| * |wK|(x), \end{aligned}$$

by (38a). Define $p = q = \frac{2r}{1+r} > 1$, so that $\frac{1}{p} + \frac{1}{p} = \frac{1}{r} + 1$. Then by assumption $wj(f) \in L^p(G)$, $wK \in L^q(G)$ and $\widecheck{wK} = w\check{K} \in L^q(G)$. Hence (76b) applies, showing that $|wj(f)|$ and $|wK|$ are convolvable, $|wj(f)| * |wK| \in L^r(G)$ and $\||wj(f)| * |wK|\|_r \leqslant C\|wf\|_p$, where C is a constant depending on q and K. Hence $j(f) * K \in L^r_w(G)$ and

$$\|j(f) * K\|_{r,w} \leqslant C\|j(f)\|_{p,w}. \tag{45}$$

Therefore $j(f) * K \in j(\mathcal{T}_w)$. By definition of $\mathcal{S}_w$, $\pi(f)u \in \mathcal{S}_w$ and $j(V_0\pi(f)u) = j(f) * K$. The map $f \mapsto \pi(f)u$ is continuous by (45). By Theorem 3.2, the map $\mathcal{M}^{\mathcal{T}_w} \ni f \mapsto \pi(f)u \in \mathcal{S}_w$ is the inverse of V_0.
f) Fix $\Phi \in \mathcal{U}_w$. By linearity we can assume that Φ is in some $L^p_{w^{-1}}(G)$, so that $\Phi j(f)$ is in $L^1(G)$ for all $f \in \mathcal{T}_w$. In particular for all $v \in \mathcal{S}_w$ we have $\Phi j(V_0v) \in L^1(G)$ because

$V_0 v \in \mathcal{T}_w$, and by Proposition 2.4 there exists $\pi(\Phi)u \in \mathcal{S}'_w$. Furthermore, recalling that ${}^tV_0$ is a linear map from $\mathcal{T}'_w$ onto $\mathcal{S}'_w$, for all $v \in \mathcal{S}_w$ we have

$$\langle {}^tV_0\Phi, v\rangle_{\mathcal{S}_w} = \langle \Phi, V_0 v\rangle_{\mathcal{T}_w} = \int_G \Phi(x)\overline{\langle i(v), \pi(x)u\rangle_{\mathcal{H}}}dx = \int_G \Phi(x)\langle \pi(x)u, i(v)\rangle_{\mathcal{H}}dx.$$

Comparing this equation with the definition of $\pi(\Phi)u$ we get ${}^tV_0\Phi = \pi(\Phi)u$ and, by (31), $V_e\pi(\Phi)u = \Phi * K$.

g) Fix $T \in \mathcal{S}'_w$. Since ${}^tV_0$ is surjective, there exists $\Phi \in \mathcal{U}_w = \mathcal{T}'_w$ such that $T = {}^tV_0\Phi = \pi(\Phi)u$, so that $V_eT = \Phi * K$. To show that $V_eT \in \mathcal{U}_w$ we prove that Φ and K are convolvable whenever $\Phi \in \mathcal{U}_w$, their convolution $\Phi * K$ is in $\mathcal{U}_w$ and the map $\Phi \mapsto \Phi * K$ is continuous from $\mathcal{U}_w$ into $\mathcal{U}_w$. By definition of $\mathcal{U}_w$, it is enough to show that, given $p \in I$, for all $\Phi \in L^p_{w^{-1}}(G)$, Φ and K are convolvable, $\Phi * K \in L^{2p}_{w^{-1}}(G)$ and the map $\Phi \mapsto \Phi * K$ is continuous from $L^p_{w^{-1}}(G)$ into $L^{2p}_{w^{-1}}(G)$. As above,

$$\begin{aligned} w(x)^{-1}\int_G |\Phi(y)||K(y^{-1}x)|\,dy &= w(x)^{-1}\int_G |\Phi(xy)||K(y^{-1})|\,dy \\ &= \int_G |w^{-1}(xy)\Phi(xy)||K(y^{-1})|\frac{w(xy)}{w(x)}\,dy \\ &\leqslant \int_G |w^{-1}(xy)\Phi(xy)||w(y^{-1})K(y^{-1})|\,dy \\ &= |w^{-1}\Phi| * |wK|(x), \end{aligned}$$

where in the third line we used both (38a) and (38b). By assumption $w^{-1}\Phi \in L^p(G)$, $wK \in L^q(G)$ where we set $q = \frac{2p}{2p-1} > 1$, and $\widecheck{wK} = w\check{K} \in L^q(G)$. Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{2p} + 1$, (76b) gives that $|w^{-1}\Phi|$ and $|wK|$ are convolvable, $|w^{-1}\Phi| * |wK| \in L^{2p}(G)$ and

$$\| |w^{-1}\Phi| * |wK| \|_{2p} \leqslant C\|w^{-1}\Phi\|_p$$

where C is a constant depending on p and K. Hence $\Phi * K \in L^{2p}_{w^{-1}}(G)$ and

$$\|\Phi * K\|_{2p,w^{-1}} \leqslant C\|\Phi\|_{p,w^{-1}}.$$

This proves the claim. Note that, since $\Phi \mapsto \Phi * K$ is continuous, $\mathcal{M}^{\mathcal{U}_w}$ is closed. We observe *en passant* that $\mathcal{U}_w$ is not a Fréchet space, so that Proposition 2.2 does not apply.

We now prove that $V_eT = V_eT * K$. Since $|K| \in \mathcal{T}_w$, reasoning as in the proof of item e), $|K| * |K| \in \mathcal{T}_w$. Furthermore, if $g = |K| * |K|$, so that $\check{g} = g$, then as above Φ and g are convolvable and (77d) gives that

$$V_eT * K = (\Phi * K) * K = \Phi * (K * K) = \Phi * K = V_eT,$$

so that the range of V_e is contained in $\mathcal{M}^{\mathcal{U}_w}$. From f) we know that $\pi(\Phi)u \in \mathcal{S}'_w$ whenever $\Phi \in \mathcal{M}^{\mathcal{U}_w}$ and that $V_e\pi(\Phi)u = \Phi * K = \Phi$, showing that V_e is onto $\mathcal{M}^{\mathcal{U}_w}$ and that the map

$$\mathcal{M}^{\mathcal{U}_w} \ni \Phi \mapsto \pi(\Phi)u \in \mathcal{S}'_w$$

is the inverse of V_e, as claimed in item h).

Next, we prove (42). Fix $v \in \mathcal{S}_w$ and define the map $\Psi : G \to \mathcal{S}_w$ by

$$\Psi(x) = \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\tau(x)u = \overline{V_0v(x)}\tau(x)u.$$

For all $T \in \mathcal{S}'_w$, $V_eT \in \mathcal{U}_w$ and $V_0v \in \mathcal{T}_w$, so that $x \mapsto \langle T, \Psi(x)\rangle_{\mathcal{S}_w}$ is in $L^1(G)$.

Since $\mathcal{S}_w$ is a reflexive Fréchet space, we can regard $\mathcal{S}_w$ as the dual of $\mathcal{S}'_w$, which has the property (GDF) by Proposition 3 Chapter 6. Appendix No.2 of [30]. The fact that the map $x \mapsto \langle T, \Psi(x)\rangle_{\mathcal{S}_w}$ is in $L^1(G)$, means that Ψ is scalarly integrable. Theorem 6.4 shows that its (scalar) integral is in $\mathcal{S}_w$, *i.e.* that there exists $\psi \in \mathcal{S}_w$ such that

$$\langle T, \psi\rangle_{\mathcal{S}_w} = \int_G \langle T, \tau(x)u\rangle_{\mathcal{S}_w} \langle \pi(x)u, v\rangle_{\mathcal{H}} dx.$$

With the choice $T = {}^t i(z)$, $z \in \mathcal{H}$, (25) gives that $\langle z, i(\psi)\rangle_{\mathcal{H}} = \langle z, i(v)\rangle_{\mathcal{H}}$. Since this last equality holds true for all $z \in \mathcal{H}$ and i is injective, then $\psi = v$ and this proves (42). Furthermore, the reproducing formula (42) implies that V_e is injective.

Finally, we prove that V_e is continuous. Fix a bounded subset B in $\mathcal{T}_w$. By e) the map $f \mapsto \pi(f)u$ is continuous from $\mathcal{T}_w$ into $\mathcal{S}_w$. Then $B' = \pi(B)u$ is a bounded subset of $\mathcal{S}_w$. Furthermore, given $T \in \mathcal{S}'_w$

$$\sup_{f\in B} |\langle V_e T, f\rangle_{\mathcal{T}_w}| = \sup_{f\in B} |\int_G \langle T, \tau(x)u\rangle_{\mathcal{S}_w} \overline{f(x)} dx| = \sup_{f\in B} |\langle T, \pi(f)u\rangle_{\mathcal{S}_w}| = \sup_{v\in B'} |\langle T, v\rangle_{\mathcal{S}_w}|.$$

Since B' is a bounded subset of $\mathcal{S}_w$, the map $T \mapsto \sup_{v\in B'} |\langle T, v\rangle_{\mathcal{S}_w}|$ is continuous, hence V_e is such. The rightmost equality (41) is a consequence of the definition of $\mathcal{U}_w$ and the inclusion follows from the fact that $V_e \mathcal{S}' \subset L^\infty_{w^{-1}}(G)$. Indeed, for all $x \in G$

$$|V_e T(x)| = |\langle T, \tau(x)u\rangle|_{\mathcal{S}} \leqslant C_T \max_{i\leqslant n} \|\tau(x)u\|_{p_i,\mathcal{S}} \leqslant C \max_{i\leqslant n} \|\ell(x)K\|_{p_i} \leqslant C \max_{i\leqslant n} \|K\|_{p_i} \, w(x)$$

where C is a constant depending on T, $p_1, \dots, p_n$ are suitable numbers in I also depending on T, and the last bound is a consequence of Lemma 4.2.

h) See the proof of the above item.

i) Apply d) of Proposition 2.6 with $F = \mathcal{T}_w$ and $E = \mathcal{S}_w$, taking into account e). □

We summarize the findings in this section in the following theorem which is one of the main results of this paper since it shows that our analysis is indeed applicable.

Theorem 4.5. *If $K \in \mathcal{T}_w$, then Assumptions $1 \div 4$ are satisfied for $\mathcal{T}_w$.*

Proof. Under the hypothesis (40), Assumption 1 is satisfied, and $j(\mathcal{T}_w) \subset L^2_w(G) \subset L^2(G)$ implies Assumption 2. The reconstruction formula (44) clarifies that u is a cyclic vector for τ, which is equivalent to Assumption 3 since $V_0\tau(x)u = \ell(x)K$ and V_0 is an injective strict morphism from $\mathcal{S}_w$ onto $\mathcal{M}^{\mathcal{T}_w}$. Finally, (44) with $v = \tau(x)u$ implies that Assumption 4 holds true. □

Observe that Theorem 4.5 paves the way for a coorbit space theory with a specific choice of target space, namely $\mathcal{T}_w$. Indeed, if Y is a Banach space continuously embedded in $L^0(G)$ and λ-invariant, and we assume that $K \in Y^\#$ and that $\mathcal{M}xs^Y$ is a subspace of $\mathcal{U}_w$, then Assumptions 5 and 6 are satisfied. Hence Proposition 3.5 holds true, giving rise to a coorbit theory for Y.

We summarize the general scheme in the following picture.

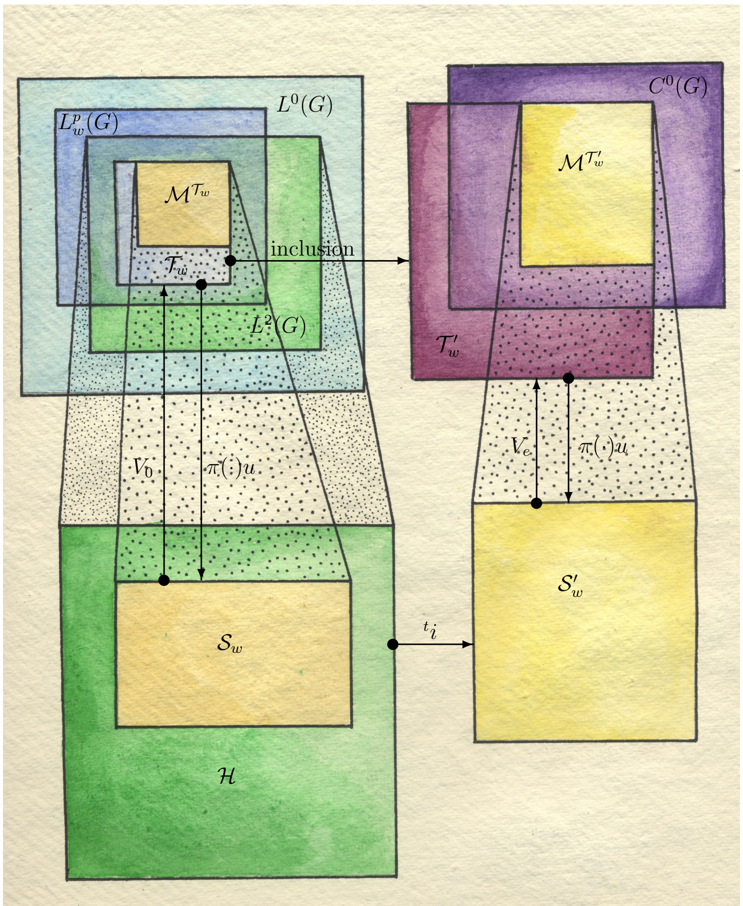

FIGURE 1. Objects on the group on top row, signals on bottom row.

4.2. **Band-limited functions.** As a toy example, we consider the space of band-limited signals on the real line. Although elementary, and certainly very natural, this case can not be handled by the classical coorbit machinery. This is somewhat unsatisfactory, because the sinc function is one of the first examples of reproducing kernels which comes to mind. Our theory does handle it, and the natural coorbit spaces that arise are the Paley–Wiener p-spaces.

In this section, G is the additive group $\mathbb{R}$ and the Haar measure is the Lebegue measure dx. We denote by $\mathbb{S}(\mathbb{R})$ the Fréchet space of rapidly decreasing functions and by $\mathbb{S}(\mathbb{R})'$ the space of tempered distributions. The Fourier transform on $\mathbb{S}(\mathbb{R})$ and $\mathbb{S}(\mathbb{R})'$ is denoted by $\mathcal{F}$. Regarding $L^2(\mathbb{R})$ as a subspace of $\mathbb{S}(\mathbb{R})'$, we set $\widehat{v} = \mathcal{F}v$ for any $v \in L^2(\mathbb{R})$.

The representation π is the regular representation restricted to the Paley–Wiener space of functions with band in the fixed compact interval $\Omega \subset \mathbb{R}$, namely

$$\mathcal{H} = B^2_\Omega = \{v \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{v}) \subseteq \Omega\}.$$

Strictly speaking, the elements of B^2_Ω are not functions, but equivalence classes of functions. However, in view of the Paley–Wiener–Schwartz theorem [32], each equivalence class in B^2_Ω has a unique representative which is continuous (in fact smooth). We are thus allowed to identify each class with its smooth representative, and we shall do so.

Since the group $\mathbb{R}$ acts on B^2_Ω by translations:

$$\pi(b)v(x) = v(x-b), \qquad v \in B^2_\Omega,$$

on the frequency side, $\widehat{\pi} = \mathcal{F}\pi\mathcal{F}^{-1}$ acts on $\mathcal{F}\mathcal{H} = L^2(\Omega)$ by modulations:

$$\widehat{\pi}(b)\widehat{v}(\xi) = \mathrm{e}^{-2\pi ib\xi}\,\widehat{v}(\xi), \qquad v \in B^2_\Omega.$$

This representation is not irreducible: any subset $\Xi \subseteq \Omega$ gives a subrepresentation on B^2_Ξ. For the reader's convenience, we summarize in the next proposition the main facts that are relevant to our discussion.

Proposition 4.6. *The representation π is reproducing and the following facts hold true.*

a) A vector $u \in B^2_\Omega$ is admissible if and only if $|\widehat{u}| = 1$ almost everywhere on Ω. In this case, the kernel K is

$$K = \langle u, \pi(\cdot)u\rangle_{\mathcal{H}} = \mathcal{F}^{-1}\chi_\Omega,$$

where χ_Ω is the characteristic function on Ω.

b) Let u be an admissible vector. Then $u(x) = K(x)$ for every $x \in \mathbb{R}$ if and only if the corresponding voice transform V_2 is the inclusion

$$V_2 : B^2_\Omega \hookrightarrow L^2(\mathbb{R}).$$

c) If $\Omega = [-\omega, \omega]$ is a symmetric interval, then the kernel is the sinc function

$$K(b) = 2\omega\,\mathrm{sinc}(2\omega\pi b),$$

where $\mathrm{sinc}\, x = \sin x/x$.

Proof. The fact that π is reproducing follows from item a).

a) Applying the Plancherel identity, we can compute the voice transform as

$$V_2v(b) = \langle v, \pi(b)u\rangle = \langle \widehat{v}, \widehat{\pi}(b)\widehat{u}\rangle = \int_{\widehat{\mathbb{R}}} \widehat{v}(\xi)\overline{\widehat{u}}(\xi)\,\mathrm{e}^{2\pi ib\xi}\,d\xi = \mathcal{F}^{-1}(\widehat{v}\overline{\widehat{u}})(b), \tag{46}$$

whose squared norm, again by Plancherel, is

$$\|V_2v\|^2 = \int_{\widehat{\mathbb{R}}} |\widehat{v}(\xi)\overline{\widehat{u}}(\xi)|^2 d\xi.$$

On the other hand, Plancherel also entails

$$\|v\|^2 = \int_{\mathbb{R}} |v(x)|^2 dx = \int_{\widehat{\mathbb{R}}} |\widehat{v}(\xi)|^2 d\xi.$$

Therefore, u is admissible if and only if $|\widehat{u}(\xi)| = 1$ for almost every $\xi \in \mathrm{supp}(\widehat{v}) \subseteq \Omega$. Since Ω is compact, vectors $u \in B^2_\Omega$ satisfying the above condition clearly exist and, hence, π is reproducing. If u is admissible, using (46) we obtain $K = V_2u = \mathcal{F}^{-1}(|\widehat{u}|^2) = \mathcal{F}^{-1}(\chi_\Omega)$.

b) Suppose that $u = K$. Then, in view of item (a) we have $\widehat{u} = \chi_\Omega$, and equality (46) gives $V_2v = \mathcal{F}^{-1}(\widehat{v}) = v$ for every $v \in B^2_\Omega$, so that V_2 is the natural inclusion. Conversely, if this is the case, then $K = V_2u = u$.

c) If $\Omega = [-\omega, \omega]$, from (46) it follows

$$K(b) = V_2u(b) = \int_{\widehat{\mathbb{R}}} |\widehat{u}(\xi)|^2\,\mathrm{e}^{2\pi ib\xi}\,d\xi = \int_{-\omega}^{\omega} \mathrm{e}^{2\pi ib\xi}\,d\xi = 2\omega\,\mathrm{sinc}(2\omega\pi b). \qquad \square$$

From now on we set $\Omega = [-\omega, \omega]$ and

$$u = K = \mathcal{F}^{-1}\chi_\Omega = 2\omega\,\mathrm{sinc}(2\omega\pi\cdot).$$

Clearly, K is not in $L^1(\mathbb{R})$, but it belongs to $L^p(\mathbb{R})$ for every $p > 1$. We thus choose the weight $w = 1$ and take

$$\mathcal{T} = \bigcap_{p \in I} L^p(\mathbb{R})$$

as target space to construct coorbits (recall that $I = (1, +\infty)$).

For $p \in [1, +\infty)$, we define the Paley–Wiener p-spaces

$$B^p_\Omega = \{f \in L^p(\mathbb{R}) : \operatorname{supp}(\mathcal{F}f) \subseteq \Omega\}.$$

Recall that the Fourier transform maps L^p to $L^{p'}$ for $p \leqslant 2$, while for $p > 2$ we get distributions that in general are not functions [32].

The spaces B^p_Ω are usually defined in the literature as the spaces of the entire functions of fixed exponential type whose restriction to the real line is p-integrable [33]. This definition is equivalent to ours since a Paley–Wiener theorem holds for all $p \in [1, +\infty)$. In particular, all these functions are indefinitely differentiable on $\mathbb{R}$. Moreover, if $f \in B^p_\Omega$ with $p < +\infty$, then $f(x) \to 0$ as $x \to \pm\infty$, hence

$$B^p_\Omega \subset C^\infty_0(\mathbb{R}), \qquad 1 \leqslant p < +\infty.$$

Consequently, the Paley–Wiener spaces are nested and increase with p:

$$B^p_\Omega \subseteq B^q_\Omega \qquad 1 \leqslant p \leqslant q < +\infty.$$

We are going to identify our coorbit spaces as Paley–Wiener spaces. To show this, we shall repeatedly make use of the following fact.

Lemma 4.7. *There exists a family of functions* $\{\widehat{g}_\varepsilon\} \subset C^\infty_c(\widehat{\mathbb{R}})$ *satisfying*

i) $\lim_{\varepsilon \to 0} \widehat{g}_\varepsilon = \chi_\Omega$ *in* $L^q(\widehat{\mathbb{R}})$ *for every* $q \geqslant 1$*;*
ii) $\chi_{[-\omega+\varepsilon, \omega-\varepsilon]} \leqslant \widehat{g}_\varepsilon \leqslant \chi_{[-\omega-\varepsilon, \omega+\varepsilon]}$*;*
iii) $\|\partial \widehat{g}_\varepsilon\|_\infty \lesssim \varepsilon^{-1}$.

such that for all $f \in L^p(\mathbb{R})$*, with* $p \geqslant 1$*, in* $\mathcal{S}'(\widehat{\mathbb{R}})$

$$\mathcal{F}(f * K) = \lim_{\varepsilon \to 0} \mathcal{F}(f)\widehat{g}_\varepsilon. \tag{47}$$

Take $f \in L^p(\mathbb{R})$. By Young's inequality (76b) we know that $f * K \in L^r$ for some $r > 1$, so that $f * K \in \mathcal{S}'(\mathbb{R})$ and the left hand side is the Fourier transform of a tempered distribution. Similarly, $\mathcal{F}(f)\widehat{g}_\varepsilon \in \mathcal{S}'(\mathbb{R})$ because $\widehat{g}_\varepsilon \in C^\infty_c(\mathbb{R})$, and on the right hand side we also have Fourier transforms of tempered distributions.

Proof. Take

$$\widehat{h} \in C^\infty_c(\mathbb{R}), \quad \operatorname{supp}(\widehat{h}) \subseteq [-1, 1], \quad \widehat{h} \geqslant 0, \quad \int_{\widehat{\mathbb{R}}} \widehat{h}(\xi) d\xi = 1,$$

and then consider the corresponding approximate identity $\{\widehat{h}_\varepsilon\}$ defined by the dilations of $\hat{h}$

$$\widehat{h}_\varepsilon(\xi) = \varepsilon^{-1}\widehat{h}(\xi/\varepsilon), \quad \varepsilon > 0,$$

so that $\widehat{h}_\varepsilon \in C^\infty_c(\widehat{\mathbb{R}})$, and define $\widehat{g}_\varepsilon = \widehat{h}_\varepsilon * \chi_\Omega$. Since both factors are L^1-functions, the convolution theorem gives that $g_\varepsilon = h_\varepsilon K$. A classical result, see Corollary 3.4 of [34], shows that $\widehat{g}_\varepsilon \to \chi_\Omega$ in $L^q(\mathbb{R})$ for every $q \geqslant 1$ and $\widehat{g}_\varepsilon \in C^\infty_c(\mathbb{R})$. Moreover,

$$\operatorname{supp}(\widehat{g}_\varepsilon) \subseteq \operatorname{supp}(\widehat{h}_\varepsilon) + \operatorname{supp}(\chi_\Omega) \subseteq [-\varepsilon, \varepsilon] + \Omega = [-\omega - \varepsilon, \omega + \varepsilon].$$

Expanding the convolution, we get

$$\widehat{g_\varepsilon}(\xi) = \varepsilon^{-1} \int_\Omega \widehat{h}((\xi - \tau)/\varepsilon) d\tau = \int_{(\Omega+\xi)/\varepsilon} \widehat{h}(\tau) d\tau$$

by a change of variable. Notice here that if $|\xi| \leqslant \omega - \varepsilon$, then $|\xi + \varepsilon\tau| \leqslant \omega$ whenever $|\tau| \leqslant 1$, which means that $(\Omega + \xi)/\varepsilon \supseteq [-1, 1] \supseteq \operatorname{supp}(\widehat{h})$. It follows that

$$\widehat{g_\varepsilon}(\xi) = \int_{\widehat{\mathbb{R}}} \widehat{h}(\tau) d\tau = 1$$

for every $\xi \in [-\omega + \varepsilon, \omega - \varepsilon]$. The derivative of $\widehat{g_\varepsilon}$ is $\partial(\widehat{h}_\varepsilon * \chi_\Omega) = \partial\widehat{h}_\varepsilon * \chi_\Omega$, and $\partial\widehat{h}_\varepsilon(\xi) = \varepsilon^{-2}\partial\widehat{h}(\xi/\varepsilon)$. A change of variable then yields

$$|\partial\widehat{g_\varepsilon}(\xi)| \leqslant \varepsilon^{-1} \int_{[-1,1]} |\partial\widehat{h}(\tau)| d\tau \leqslant 2 \sup(|\partial\widehat{h}|)\varepsilon^{-1}.$$

Finally, let $h_\varepsilon = \mathcal{F}^{-1}\widehat{h}_\varepsilon$. By dominated convergence, for all $x \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} h_\varepsilon(x) = \int_{\mathbb{R}} \hat{h}(\xi)\, d\xi = 1,$$

so that

$$\lim_{\varepsilon\to 0} g_\varepsilon(x) = \lim_{\varepsilon\to 0} h_\varepsilon(x)K(x) = K(x),$$

and at the same time

$$|h_\varepsilon(x)K(x)|^q \leqslant \|h\|_\infty^q |K(x)|^q$$

for any $q > 1$. Therefore $h_\varepsilon K \to K$ in $L^q(\mathbb{R})$, by dominated convergence. Young's inequality implies now that $f * h_\varepsilon K \to f * K$ in some $L^r(\mathbb{R})$, hence as tempered distributions. Therefore

$$\mathcal{F}(f * K) = \lim_{\varepsilon\to 0} \mathcal{F}(f * g_\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{F}(f)\mathcal{F}(g_\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{F}(f)\widehat{g_\varepsilon}$$

by the continuity of the Fourier transform and an application of the convolution theorem, because $f \in \mathbb{S}'(\mathbb{R})$ and $g_\varepsilon \in \mathbb{S}(\mathbb{R})$ (see Theorem XV, Ch. VII in [35]). □

We are now ready to state the characterization of the natural coorbit spaces relative to band limited functions.

Proposition 4.8. *Let $\Omega = [-\omega, \omega]$ and take $u = K = \mathcal{F}^{-1}\chi_\Omega$. The space of test functions is*

$$\mathcal{S} = \bigcap_{p \in I} B^p_\Omega$$

and the space of distributions is

$$\mathcal{S}' = \bigcup_{p \in I} B^p_\Omega.$$

The extended voice transform is the inclusion

$$V_e : \mathcal{S}' \hookrightarrow \mathcal{U}$$

and the coorbits of the L^p spaces are

$$\operatorname{Co}(L^p(\mathbb{R})) = \mathcal{M}^p = B^p_\Omega.$$

Proof. Since V_2 is the inclusion by item b) of Proposition 4.6, we have

$$\mathcal{S} = \{v \in B^2_\Omega : v \in L^p(\mathbb{R})\ \forall p > 1\} = \bigcap_{p\in I} B^p_\Omega.$$

In order to describe $\mathcal{S}'$, we first observe that

$$\bigcup_{p\in I} \mathcal{M}^p = \operatorname{span} \bigcup_{p\in I} \mathcal{M}^p = \mathcal{M}^{\mathcal{U}}.$$

Indeed, since $w = 1$, by item (41) $\mathcal{M}^p \subset L^\infty(G)$, so that $\mathcal{M}^p \subset \mathcal{M}^q$ whenever $p \leqslant q$. Furthermore, item h) of Theorem 4.4 implies that $V_e = {}^tV_0$ establishes a linear isomorphism from $\mathcal{M}^{\mathcal{U}} = \bigcup_{p\in I} \mathcal{M}^p$ to $\mathcal{S}'$. In particular, $\mathcal{S}'$ is the range ${}^tV_0\mathcal{M}^{\mathcal{U}}$.

Since $V_0 : \mathcal{S} \hookrightarrow \mathcal{T}$ is the inclusion, the transpose map ${}^tV_0 : \mathcal{T}' \to \mathcal{S}'$ is simply the restriction on the subspace $\mathcal{S}$. Therefore, we can explicitly represent $\mathcal{S}'$ as the space of anti-linear functionals

$$v \in \mathcal{S} \longmapsto \int_{\mathbb{R}} \Phi(x)\overline{v(x)}dx, \qquad \Phi \in \bigcup_{p\in I} \mathcal{M}^p.$$

The two spaces are thus canonically identified and, with this identification, V_e is the inclusion $\mathcal{S}' \hookrightarrow \mathcal{U}$.

We next prove that $\mathcal{M}^p = B^p_\Omega$ for all $p \in I$. Let $f \in \mathcal{M}^p$ and φ be a smooth function with compact support contained in Ω^c. Then, by Lemma 4.7, we have

$$\langle \widehat{f}, \varphi\rangle = \lim_{\varepsilon\to 0} \int \widehat{f}(\xi)\widehat{g_\varepsilon}(\xi)\varphi(\xi)d\xi,$$

for some $\widehat{g_\varepsilon}$ with $\operatorname{supp}(\widehat{g_\varepsilon}) \subseteq \Omega + [-\varepsilon, \varepsilon]$. Since $\Omega + [-\varepsilon, \varepsilon] \cap \operatorname{supp}(\varphi) = \varnothing$ for ε small enough, the limit is zero. This means that $\operatorname{supp}(\widehat{f}) \subseteq \Omega$, that is $f \in B^p_\Omega$.

Conversely, let $f \in B^p_\Omega$. We shall prove that $\mathcal{F}f = \mathcal{F}(f * K)$, whence $f = f * K$ and $f \in \mathcal{M}^p$. Thanks to formula IV' at page 111 in [33], there exists a continuous function ψ, periodic on Ω such that

$$f(x) = \int_\Omega [(1-x)\psi(\omega) + x\psi(\xi)]\, e^{2\pi i x\xi}\, d\xi.$$

Hence

$$f(x) = (1-x)\psi(\omega)\mathcal{F}^{-1}\chi_\Omega(x) + x\mathcal{F}^{-1}(\psi\chi_\Omega)(x),$$

so that, in $\mathbb{S}(\mathbb{R})$,

$$\mathcal{F}f = \psi(\omega)(1 - \frac{i}{2\pi}\partial)\chi_\Omega + \frac{i}{2\pi}\partial(\psi\chi_\Omega).$$

This tempered distribution acts on any function $\varphi \in \mathbb{S}(\mathbb{R})$ by

$$\begin{aligned}\langle \mathcal{F}f, \varphi\rangle &= \psi(\omega)\langle \chi_\Omega, (1 + \frac{i}{2\pi}\partial)(\varphi)\rangle - \langle \psi\chi_\Omega, \frac{i}{2\pi}\partial\varphi\rangle \\ &= \int_\Omega \left[\psi(\omega)(1 + \frac{i}{2\pi}\partial)\varphi(\xi) - \psi(\xi)\frac{i}{2\pi}\partial\varphi(\xi)\right] d\xi. \end{aligned} \tag{48}$$

On the other hand, we know from Lemma 4.7 that

$$\mathcal{F}(f * K) = \lim_{\varepsilon\to 0} \mathcal{F}(f)\widehat{g_\varepsilon},$$

where the limit is taken in $\mathbb{S}'(\mathbb{R})$. Compute now

$$\begin{aligned}\langle \mathcal{F}(f)\widehat{g}_\varepsilon, \varphi\rangle &= \langle \mathcal{F}f, \widehat{g}_\varepsilon\varphi\rangle \\ &= \psi(\omega)\langle \chi_\Omega, (1+\frac{i}{2\pi}\partial)(\widehat{g}_\varepsilon\varphi)\rangle - \langle \psi\chi_\Omega, \frac{i}{2\pi}\partial(\widehat{g}_\varepsilon\varphi)\rangle \\ &= \int_\Omega \left[\psi(\omega)(1+\frac{i}{2\pi}\partial)\varphi(\xi) - \psi(\xi)\frac{i}{2\pi}\partial\varphi(\xi)\right]\widehat{g}_\varepsilon d\xi \\ &+ \frac{i}{2\pi}\int_\Omega (\psi(\omega)-\psi(\xi))\varphi(\xi)\partial\widehat{g}_\varepsilon(\xi)d\xi.\end{aligned}$$

By ii) of Lemma 4.7, $\widehat{g}_\varepsilon \to \chi_\Omega$ pointwise, then the limit of the first integral is precisely (48). It remains to verify that the last integral tends to zero. Notice that it vanishes on $[-\omega+\varepsilon, \omega-\varepsilon]$ because, by Lemma 4.7, $\widehat{g}_\varepsilon = 1$, hence $\partial\widehat{g}_\varepsilon = 0$. The rest of the integral is dominated by

$$\begin{aligned}&\left[\sup_{-\omega\leqslant\xi\leqslant-\omega+\varepsilon}|\psi(\omega)-\psi(\xi)| + \sup_{\omega-\varepsilon\leqslant\xi\leqslant\omega}|\psi(\omega)-\psi(\xi)|\right]\|\varphi\|_\infty\|\partial\widehat{g}_\varepsilon\|_\infty\,\varepsilon \\ \lesssim &\sup_{-\omega\leqslant\xi\leqslant-\omega+\varepsilon}|\psi(\omega)-\psi(\xi)| + \sup_{\omega-\varepsilon\leqslant\xi\leqslant\omega}|\psi(\omega)-\psi(\xi)|,\end{aligned}$$

thanks to iii) of Lemma 4.7. But this tends to zero as $\varepsilon \to 0$, because $\psi(\omega) = \psi(-\omega)$. We have proven that $\mathcal{M}^p = B^p_\Omega$ for all p. Thus we finally obtain

$$\mathcal{S}' = \bigcup_{p\in I} B^p_\Omega \quad \text{and} \quad \mathrm{Co}(L^p(\mathbb{R})) = B^p_\Omega. \qquad \square$$

4.3. **Shannon wavelet.** We consider now the special case of a non integrable kernel for the wavelet representation of the affine group on $L^2(\mathbb{R})$. It is well known that this representation is reproducing and admits admissible vectors whose kernel is integrable [36, 37, 1]. The resulting coorbit spaces are completely understood as homogeneous Besov spaces [7, 1]. However, there are admissible vectors whose kernel is not integrable, as for example the Shannon wavelet, which provides another example for which our theory applies.

At first sight, this result might look surprising. Indeed, in [1], H. Feichtinger and K. Gröchenig claim that any band limited function whose Fourier transform has compact support bounded away from zero leads to an integrable kernel. However, a careful inspection of the proof reveals that the additional assumption that the Fourier transform is continuous is implicitly needed.

Let $G = \mathbb{R} \rtimes \mathbb{R}_+$ be the connected component of the affine group with left Haar measure $db\,da/a^2$. The wavelet representation π acts on $\mathcal{H} = L^2(\mathbb{R})$ by dilations and translations:

$$\pi(b,a)v(x) = a^{-1/2}v((x-b)/a).$$

The (real) Shannon wavelet is defined as

$$\widehat{u}(\xi) = \chi_{[1/4,1/2]}(|\xi|) = \chi_{[-1/2,1/2]}(\xi) - \chi_{[-1/4,1/4]}(\xi),$$

that is

$$u(x) = \mathrm{sinc}(\pi x) - \frac{1}{2}\mathrm{sinc}(\frac{\pi}{2}x) = \frac{1}{2}\mathrm{sinc}(\frac{\pi}{4}x)\cos(\frac{3}{4}\pi x).$$

It is easily seen that $u \notin L^1(\mathbb{R})$, but $u \in L^p(\mathbb{R})$ for all $p > 1$. We now prove that the corresponding kernel has the same behavior.

Lemma 4.9. *The kernel $K = V_2 u$ associated with the Shannon wavelet*

$$u(x) = \frac{1}{2}\,\mathrm{sinc}(\frac{\pi}{4}x)\cos(\frac{3}{4}\pi x)$$

is in $L^p(G)$ for all $p > 1$, but it is not in $L^1(G)$.

Proof. Since u is real and even, the voice transform V_2 is

$$V_2 v(b,a) = \int v(x) a^{-1/2} u((x-b)/a)dx = \left(v * \pi(0,a)u\right)(b).$$

The Shannon kernel is thus

$$K(b,a) = V_2 u(b,a) = (u * \pi(0,a)u)(b).$$

Since u is admissible, $K \in L^2(G)$ and, by Fubini theorem, $u * \pi(0,a)u \in L^2(\mathbb{R})$ for almost every $a > 0$. Then, by the convolution theorem for L^2-functions

$$\mathcal{F}(u * \pi(0,a)u)(\beta) = \hat{u}(\beta)\widehat{\pi(0,a)u}(\beta) = a^{1/2}\chi_{[1/4,1/2]\cap[1/4a,1/2a]}(|\beta|).$$

It follows that:

a) $K(\cdot,a) \neq 0$ only if $a \in (1/2, 2)$;
b) if $a \in (1/2,2)$ the Fourier transform of $K(\cdot,a)$ is a non-zero characteristic function.

By b), for almost all $a \in (1/2,2)$ the function $u * \pi(0,a)u$ cannot be in $L^1(\mathbb{R})$, otherwise its Fourier transform would be continuous. Hence $K \notin L^1(G)$. Let us show that $K \in L^r(G)$ for all $r > 1$. From a) we have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} |K(b,a)|^r db \frac{da}{a^2} = \int_{1/2}^{2}\int_{\mathbb{R}} |u * \pi(0,a)u(b)|^r db \frac{da}{a^2} = \int_{1/2}^{2} \|u * \pi(0,a)u\|_r^r \frac{da}{a^2}.$$

Recall that $u \in L^p(\mathbb{R})$ for all $p > 1$, so that the same holds for $\pi(0,a)u$. By Young's inequality (76b) for the unimodular group $G = \mathbb{R}$, we can estimate the inner norm and obtain

$$\|K\|_r^r \leqslant \|u\|_p^r \int_{1/2}^{2} \frac{\|\pi(0,a)u\|_q^r}{a^2} da,$$

where p and q are such that $1/p + 1/q = 1/r + 1$. This integral is finite, because the function $a \mapsto \|\pi(0,a)u\|_q^r/a^2$ is continuous and the interval $[1/2, 2]$ is compact. ☐

A Shannon wavelet coorbit theory can thus be implemented taking voices in the target space $\mathcal{T} = \bigcap_{p\in I} L^p(G)$, but not in $L^1(G)$.

4.4. **Schrödingerlets.** In this section, we illustrate the example that has motivated the search for a full coorbit theory in which one encounters reproducing kernels that do enjoy nice integrability properties but are not necessarily in $L^1(G)$. The main feature of this example is that once the admissibility conditions are worked out, it is relatively easy to exhibit kernels in $\bigcap_{p\in I} L^p(G)$ but hard to find a kernel in $L^1(G)$. This example has shown up in the classification of reproducing triangular subgroups of $Sp(2,\mathbb{R})$, which was recently achieved in [38, 18].

We shall be concerned with the three-dimensional group generated by rotations, dilations and flows of two-dimensional signals, in a sense to be made precise below. The group acts on functions via radial affine transformations, and the associated voice transform can thus be seen as a Fourier series of one-dimensional wavelets. This representation is highly reducible, and reproducing.

The group G is the direct product of the (connected component of) affine group of the line with the unit circle

$$G = (\mathbb{R} \rtimes \mathbb{R}_+) \times S^1$$

and its elements are parametrized by (b, a, φ) with $b \in \mathbb{R}$, $a > 0$ and $\varphi \in [0, 2\pi)$. A left Haar measure is

$$dx = \frac{db\, da}{a^2} \frac{d\varphi}{2\pi}.$$

Notice that G is not unimodular and has modular function $\Delta(b, a, \varphi) = a^{-1}$.

The representation π that we are going to define acts on $L^2(\mathbb{R} \times S^1)$, endowed with the tensor product of the Lebesgue measure and the normalized Haar measure on S^1. The action is

$$\pi(b, a, \varphi)v(x, \vartheta) = a^{-1/2}\ v((x - b)/a, \vartheta - \varphi), \qquad v \in L^2(\mathbb{R} \times S^1). \tag{49}$$

Since $L^2(\mathbb{R} \times S^1) = L^2(\mathbb{R}) \otimes L^2(S^1)$, π is simply the tensor product $\pi = w \otimes \lambda$ where w is the wavelet representation of the affine group

$$w(b, a)g(x) = a^{-1/2}\ g((x - b)/a), \qquad g \in L^2(\mathbb{R}), \tag{50}$$

and λ is the left regular representation of S^1 on $L^2(S^1)$, namely

$$\lambda(\varphi)h(\vartheta) = h(\vartheta - \varphi), \qquad h \in L^2(S^1). \tag{51}$$

In what follows, we denote by $\mathcal{F}_x$ the unitary Fourier transform from $L^2(\mathbb{R})$ onto $L^2(\widehat{\mathbb{R}})$, which is also regarded as a unitary map from $L^2(\mathbb{R}\times S^1)$ onto $L^2(\widehat{\mathbb{R}})\otimes L^2(S^1)$. Furthermore, we denote by $\mathcal{F}_\vartheta$ the unitary Fourier transform from $L^2(S^1)$ onto $\ell^2(\mathbb{Z})$, which is also regarded as a unitary map from $L^2(\mathbb{R} \times S^1)$ onto $L^2(\mathbb{R}) \otimes \ell^2(\mathbb{Z})$. Explicitly, if $v \in C_c^\infty(\mathbb{R} \times S^1)$, then

$$\mathcal{F}_x v(\rho, \vartheta) = \int_{\mathbb{R}} v(x, \vartheta)\, \mathrm{e}^{-2\pi i \rho x}\, dx, \tag{52a}$$

$$\mathcal{F}_\vartheta v(x, n) = \int_{\mathbb{R}} v(x, \vartheta)\, \mathrm{e}^{-in\vartheta}\, \frac{d\vartheta}{2\pi} = \int_{\mathbb{R}} v(x, \vartheta)\overline{e_n(\vartheta)} \frac{d\vartheta}{2\pi}, \tag{52b}$$

where $e_n(\vartheta) = \mathrm{e}^{in\vartheta}$. The partial Fourier transform $\mathcal{F}_\vartheta v$ of any $v \in L^2(\mathbb{R} \times S^1)$ can be identified with the sequence of functions $(v_n)_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, where $v_n = \mathcal{F}_\vartheta v(\cdot, n)$. Hence

$$v = \sum_{n\in\mathbb{Z}} v_n \otimes e_n, \qquad \|v\|^2_{L^2(\mathbb{R}\times S^1)} = \sum_{n\in\mathbb{Z}} \|v_n\|^2_{L^2(\mathbb{R})}. \tag{53}$$

To simplify the computations, we restrict the representation π to the closed subspace $\mathcal{H} = \mathcal{F}_x^{-1} L^2(\widehat{\mathbb{R}}_+) \otimes L^2(S^1)$, so that the wavelet representation w acts irreducibly on $\mathcal{F}_x^{-1} L^2(\widehat{\mathbb{R}}_+)$. Given a vector $u \in \mathcal{H}$, we denote by V the voice transform corresponding to the representation π of G and the analyzing vector u, namely

$$Vv(b, a, \vartheta) = \langle v, \pi(b, a, \vartheta)u\rangle_{\mathcal{H}} \qquad v \in \mathcal{H},$$

and by V_n^{w} the voice transform corresponding to the representation w of the affine group and the analyzing vector u_n, *i.e.*

$$V_n^{\mathrm{w}} g(b, a) = \langle g, w(b, a)u_n\rangle_{L^2(\mathbb{R})}.$$

We use the unitary operator $\mathcal{F}_x : \mathcal{H} \to L^2(\widehat{\mathbb{R}}_+ \times S^1)$ to obtain an intermediate equivalent version of π, denoted $\mathcal{F}_x(\pi)$, acting on $L^2(\widehat{\mathbb{R}}_+ \times S^1)$. This is defined via the intertwining

$\mathcal{F}_x \circ \pi(g) = \mathcal{F}_x(\pi)(g) \circ \mathcal{F}_x$ for every $g \in G$. The analytic expression of $\mathcal{F}_x(\pi)$ is immediately computed to be

$$\mathcal{F}_x(\pi)(b,a,\varphi)v(\xi,\vartheta) = a^{1/2}\, \mathrm{e}^{-2\pi i b\xi}\, v(a\xi, \vartheta - \varphi),$$

whereas from the structural point of view it may be written as

$$\mathcal{F}_x(\pi) = \widehat{w} \otimes \lambda,$$

where $\widehat{w}(b,a) = \mathcal{F}_x \circ w(b,a) \circ \mathcal{F}_x^{-1}$.

The group G can be realized as the triangular subgroup of $Sp(2,\mathbb{R})$ consisting of the matrices

$$\begin{bmatrix} a^{-1/2}R & 0 \\ ba^{-1/2}R & a^{1/2}R \end{bmatrix}, \qquad b \in \mathbb{R}, a > 0, R \in SO(2).$$

Thus, G may also be seen as the semidirect product $\mathbb{R} \rtimes (\mathbb{R}_+ \times SO(2))$, where the homogeneous factor $\mathbb{R}_+ \times SO(2)$ acts on the normal subgroup $\mathbb{R}$ by isotropic dilations. We shall not distinguish between S^1 and $SO(2)$ and write rotations as

$$R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \qquad \varphi \in [0, 2\pi).$$

We show below that π is equivalent to the metaplectic representation μ as restricted to the above group, defined in the frequency domain by

$$\mu(b,a,\varphi)v(\xi) = a^{1/2} e^{-2\pi i b|\xi|^2} v(a^{1/2} R_{-\varphi}\xi), \qquad v \in L^2(\widehat{\mathbb{R}^2}).$$

The space-domain version of this representation explains the reason of the name *Schrödingerlets*. Denote by $\widehat{\mu}$ the representation obtained by conjugating μ with the Fourier transform, namely

$$\widehat{\mu}(g)f = \mathcal{F}^{-1} \circ \mu(g) \circ \mathcal{F}.$$

We now interpret $b \in \mathbb{R}$ as a time parameter and look at the evolution *flow* of $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$

$$(b,x) \mapsto \widehat{\mu}_b f(x) = \widehat{\mu}(b,1,0)f(x) = \int_{\widehat{\mathbb{R}^2}} \hat{f}(\xi) e^{-2\pi i b|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi.$$

It is then straightforward to verify that the flow $\widehat{\mu}_b f$ satisfies the Schrödinger equation

$$\Big(2\pi i \frac{\partial}{\partial b} + \Delta\Big) \widehat{\mu}_b f(x) = 0,$$

where Δ is the spacial Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

It is in this sense that the group is generated by (dilations, rotations and) flows.

We now prove the equivalence. The unitary map $\Psi : L^2(\widehat{\mathbb{R}^2}) \to L^2(\widehat{\mathbb{R}}_+ \times S^1)$, defined by

$$\Psi v(\xi,\vartheta) = \pi^{1/2}\; v(\sqrt{\xi}\cos\vartheta, \sqrt{\xi}\sin\vartheta),$$

intertwines μ with $\mathcal{F}_x(\pi)$ because, for $v \in L^2(\widehat{\mathbb{R}^2})$, we have on the one hand

$$\begin{aligned}
\Psi\left(\mu(b,a,\varphi)v\right)(\xi,\vartheta) &= \Psi\left(a^{1/2} e^{-2\pi i b|\cdot|^2} v(a^{1/2}R_{-\varphi}(\cdot))\right)(\xi,\vartheta) \\
&= \pi^{1/2} a^{1/2} e^{-2\pi i b|(\sqrt{\xi}\cos\vartheta, \sqrt{\xi}\sin\vartheta)|^2} v\left(a^{1/2}R_{-\varphi}(\sqrt{\xi}\cos\vartheta, \sqrt{\xi}\sin\vartheta)\right) \\
&= \pi^{1/2} a^{1/2} e^{-2\pi i b\xi} v\left(\sqrt{a}(\sqrt{\xi}\cos(\vartheta-\varphi), \sqrt{\xi}\sin(\vartheta-\varphi))\right)
\end{aligned}$$

and on the other hand

$$\begin{aligned}\mathcal{F}_x(\pi)(b,a,\varphi)\,(\Psi v)\,(\xi,\vartheta) &= a^{1/2}\,\mathrm{e}^{-2\pi ib\xi}(\Psi v)(a\xi,\vartheta-\varphi)\\ &= a^{1/2}\,\mathrm{e}^{-2\pi ib\xi}\,\pi^{1/2}v(\sqrt{a\xi}\cos(\vartheta-\varphi),\sqrt{a\xi}\sin(\vartheta-\varphi)),\end{aligned}$$

as claimed. In conclusion, since π and $\mathcal{F}_x(\pi)$ are equivalent, so are π and μ.

We point out that Ψ is simply the change from rectangular to polar-like coordinates $(\sqrt{\xi},\vartheta)$, together with the appropriate L^2-normalization.

Since the wavelet representation is irreducible, while λ completely reduces to $\oplus_{n\in\mathbb{Z}}e_{-n}$, where each function e_n is regarded as a character of S^1, then

$$\pi = \bigoplus_{n\in\mathbb{Z}} w\otimes e_{-n},$$

which expresses π as a sum of irreducibles. This allows us to view the voice transform of π as a Fourier series of one-dimensional wavelet transforms, as clarified in the next proposition.

Proposition 4.10. *Let $u=\sum_{n\in\mathbb{Z}}u_n\otimes e_n\in\mathcal{H}$. The voice transform V associated with π and u admits the trigonometric expansion*

$$Vv(b,a,\varphi)=\sum_{n\in\mathbb{Z}}V_n^{\mathrm{w}}v_n(b,a)\,\mathrm{e}^{in\varphi}, \tag{54}$$

where the series converges pointwise for all $(b,a,\varphi)\in G$ and where

$$V_n^{\mathrm{w}}v_n(b,a)=\int_{S^1}Vv(b,a,\vartheta)e^{-in\vartheta}\frac{d\vartheta}{2\pi}. \tag{55}$$

Proof. Since $\mathcal{F}_\vartheta$ is a unitary map, for all $(b,a,\varphi)\in G$

$$\begin{aligned}\langle v,\pi(b,a,\varphi)u\rangle_{\mathcal{H}} &= \langle\mathcal{F}_\vartheta v,\mathcal{F}_\vartheta\,(w(b,a)\otimes\lambda(\vartheta))\,u\rangle_{L^2(\mathbb{R})\otimes\ell^2(\mathbb{Z})}\\ &= \langle\mathcal{F}_\vartheta v,\left(w(b,a)\otimes\mathcal{F}_\vartheta\lambda(\varphi)\mathcal{F}_\vartheta^{-1}\right)\mathcal{F}_\vartheta u\rangle_{L^2(\mathbb{R})\otimes\ell^2(\mathbb{Z})}\\ &= \sum_{n\in\mathbb{Z}}\langle v_n,w(b,a)u_n\rangle_{L^2(\mathbb{R})}\overline{e_{-n}(\varphi)}\\ &= \sum_{n\in\mathbb{Z}}V_n^{\mathrm{w}}v_n(b,a)e^{in\varphi},\end{aligned}$$

where the third line is due to the fact that the action of $\mathcal{F}_\vartheta\lambda(\vartheta)\mathcal{F}_\vartheta^{-1}$ on $\ell^2(\mathbb{Z})$ is the multiplication operator by the sequence $(e_{-n}(\varphi))_n$. For fixed $(b,a)\in\mathbb{R}\rtimes\mathbb{R}_+$, the function $\vartheta\mapsto Vv(b,a,\vartheta)$ is continuous and, hence, integrable on S^1. Therefore, by de la Vallée–Poussin theorem, (see iii) of Theorem 11.3 in [39]), we obtain (55). □

The Fourier expansion (54) shows how to construct admissible vectors as series of wavelets. This result has been originally obtained in [18] and in general setting in [40]. For the reader's convenience, we give here a more direct proof.

Proposition 4.11. *The representation π is reproducing and a vector $u=\sum_{n\in\mathbb{Z}}u_n\otimes e_n\in\mathcal{H}$ is admissible for π if and only if for all $n\in\mathbb{Z}$*

$$\int_{\mathbb{R}_+}|\mathcal{F}_x u_n(\xi)|^2\frac{d\xi}{\xi}=1. \tag{56}$$

Furthermore, given a sequence $(u_n)_{n\in\mathbb{Z}}$ *with* $u_n \in \mathcal{F}^{-1}L^2(\widehat{\mathbb{R}}_+)$, *we have*

$$\sum_{n\in\mathbb{Z}} u_n \otimes e_n \in \mathcal{H} \qquad \Longleftrightarrow \qquad \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}_+} |\mathcal{F}_x u_n(\xi)|^2 d\xi < +\infty. \tag{57}$$

If u *is an admissible vector, the voice transform from* $\mathcal{H}$ *into* $L^2(G) = L^2(\mathbb{R} \rtimes \mathbb{R}_+) \otimes L^2(S^1)$ *is*

$$V_2 v = \sum_{n\in\mathbb{Z}} V_n^w v_n \otimes e_n \qquad v = \sum_{n\in Z} v_n \otimes e_n \in \mathcal{H} \tag{58}$$

and the series (54) *converges also in* $L^2(G)$.

Proof. Admissibility of u means that $\|Vv\|_{L^2(G)} = \|v\|_{\mathcal{H}}$ must hold for all $v \in \mathcal{H}$. Fix $n \in \mathbb{Z}$ and choose $v = v_n \otimes e_n$ with $v_n \in \mathcal{F}^{-1}L^2(\widehat{\mathbb{R}}_+)$. By (54), when computing the norm, the integral on the circle is equal to 1, whereas the integration on $\mathbb{R} \rtimes \mathbb{R}_+$ provides the classical admissibility condition, namely Calderón's equations (56).

Conversely, suppose (56) true for every $n \in \mathbb{Z}$. Fix $v \in \mathcal{H}$. Given $(b, a) \in \mathbb{R} \rtimes \mathbb{R}_+$, (55) implies that the function $Vv(b, a, \cdot)$ is in $L^2(S^1)$ if and only if the sequence $(Vv_n(a, b))_{n\in\mathbb{Z}}$ is in $\ell^2(\mathbb{Z})$ and, under this assumption, Fubini theorem yields

$$\|Vv\|^2_{L^2(G)} = \int_{\mathbb{R}\rtimes\mathbb{R}_+} \sum_{n\in\mathbb{Z}} |V_n^{\mathrm{w}} v_n(b, a)|^2 \frac{dbda}{a^2} = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |v_n(x)|^2 dx = \|v\|^2_{\mathcal{H}} \tag{59}$$

because by (56) for each n the voice V_n^{w} is an isometry from $\mathcal{F}_x^{-1}L^2(\widehat{\mathbb{R}_+})$ into $L^2(\mathbb{R} \rtimes \mathbb{R}_+, dbda/a^2)$. The last equality is due to (53). Equation (57) is a consequence of (53) and the fact that $\mathcal{F}_x$ is unitary.

To prove that π is reproducing, it is enough to show there exists a sequence $(u_n)_n$ in $\mathcal{F}^{-1}L^2(\widehat{\mathbb{R}}_+)$ satisfying both (56) and (57). Fix $u_0 \in \mathcal{F}^{-1}L^2(\widehat{\mathbb{R}}_+)$ satisfying (56), *i.e.*, an admissible vector for the wavelet representation w. For all $n \in \mathbb{Z}$ define the functions $u_n \in \mathcal{F}^{-1}L^2(\widehat{\mathbb{R}}_+)$ as

$$u_n(x) = a_n u_0(a_n x), \qquad \mathcal{F}_x u_n(\xi) = \mathcal{F}_x u_0(a_n^{-1}\xi),$$

where $a_n > 0$ and $\sum_{n\in\mathbb{Z}} a_n < +\infty$. Since u_0 satisfies (56), so do all the functions u_n. Further,

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}_+} |\mathcal{F}_x u_0(\xi)|^2 d\xi = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}_+} |\mathcal{F}_x u_0(a_n^{-1}\xi)|^2 d\xi = \|u_0\|^2 \sum_{n\in\mathbb{Z}} a_n < +\infty,$$

so that by (57) the vector $u = \sum u_n \otimes e_n$ is in $\mathcal{H}$ and is admissible for π.

Finally, we prove (58). By (59) the series $\sum_{n\in\mathbb{Z}} V_n^{\mathrm{w}} v_n \otimes e_n$ converges in $L^2(G)$ to $V_2 v$. □

Now we come to the integrability question. The idea is based on the very simple observation that Calderón's equation (56) is invariant under dilations.

Proposition 4.12. *There exist admissible vectors* $u \in \mathcal{H}$ *whose kernel* $K = V_2 u$ *belongs to* $\bigcap_{p\in I} L^p(G)$ *but not to* $L^1(G)$.

Proof. Define u as in the second part of the proof of the above proposition. Using (58) we write $K = \sum K_n \otimes e_n$ where $K_n = V_n^{\mathrm{w}} u_n$ and the series converges both in $L^2(G)$ and pointwise. By a simple change of variable, we get that $K_n(b, a) = a_n K_0(a_n b, a)$. Therefore for any $p \in I$

$$\|K\|_p \leqslant \sum_{n\in\mathbb{Z}} a_n \left(\int_{\mathbb{R}\rtimes\mathbb{R}_+} |K_0(a_n b, a)|^p \frac{dbda}{a^2} \right)^{1/p} = \|K_0\|_p \sum_{n\in\mathbb{Z}} a_n^{1-1/p}.$$

In order to construct u, it is therefore sufficient to take a positive sequence for which $\sum_{n\in\mathbb{Z}} a_n^\alpha$ converges for every $\alpha \in (0, 1]$.

We now prove that the kernel is not in $L^1(G)$. By contradiction, assume that $K \in L^1(G)$. Fubini's theorem implies that for almost all $a \in \mathbb{R}_+$ the function $K(\cdot, a, \cdot)$ is in $L^1(\mathbb{R} \times S^1)$. Hence, regarding $\mathbb{R} \times S^1$ as an abelian group, its Fourier transform

$$\begin{aligned}\mathcal{F}K(\xi, n) &= \int_{\mathbb{R}\times S^1} K(b, a, \varphi)e^{-in\varphi}e^{-2\pi ib\xi}db\frac{d\varphi}{2\pi}\\ &= \int_{\mathbb{R}}\left(\int_{S^1} K(b, a, \varphi)e^{-in\varphi}db\right)e^{-2\pi ib\xi}\frac{d\varphi}{2\pi}\end{aligned}$$

is in $C_0(\widehat{\mathbb{R}} \times \mathbb{Z})$. By (55), it holds that

$$\begin{aligned}\mathcal{F}K(\xi, n) &= \int_{\mathbb{R}} a_n K_0(a_n b, a)e^{-2\pi ib\xi}\,db\\ &= \int_{\mathbb{R}} K_0(b)e^{-2\pi ib\frac{\xi}{a_n}}\,db = \hat{g}(\frac{\xi}{a_n}),\end{aligned}$$

where $\hat{g}$ is the Fourier transform of the function $K_0(\cdot, a)$, which is in $L^1(\mathbb{R})$ by Fubini's theorem. Fix $\overline{\xi} \in \widehat{\mathbb{R}}$ and set $\xi = a_n\overline{\xi}$ in the above equality. Then

$$\hat{g}(\overline{\xi}) = \lim_{n\to\infty} \mathcal{F}K(a_n\overline{\xi}, n) = 0,$$

because $\mathcal{F}K \in C_0(\widehat{\mathbb{R}} \times \mathbb{Z})$. Hence, by injectivity of the Fourier transform, $K_0(b, a_0) = 0$ for almost all $b \in \mathbb{R}$. Since the above equality holds for almost all $a \in \mathbb{R}_+$, we get that $K_0 = 0$, which is a contradiction. □

5. L^1-KERNELS: THE NON IRREDUCIBLE COORBIT THEORY

In this section, we apply our machinery and show that the standard setup of coorbit theory makes sense without assuming that the representation π is irreducible, because it corresponds to the case arising from the classical choice $\mathcal{T} = L^1_w(G)$. The fact that irreducibility is a somewhat redundant assumption has been perhaps known to some extent, but it is not easy to pin down precise statements in the literature. Theorem 5.1 below contains a summary of the most relevant facts.

It is perhaps worthwhile observing that the present case is structurally different from the case discussed in Section 4.1 because $L^1_w(G)$ is not a reflexive space.

Throughout this section, we fix a continuous function $w : G \to (0, +\infty)$ satisfying the following assumptions (see [4]):

$$w(xy) \leqslant w(x)w(y) \tag{60a}$$

$$w(x) \geqslant 1 \tag{60b}$$

for all $x, y \in G$. We choose as target space $\mathcal{T}$ the Banach space $L^1_w(G)$ and denote by j the canonical inclusion into $L^1_{\mathrm{loc}}(G) \subset L^0(G)$. Since j is canonical, we do not write it explicitly, especially because it would conflict with the current literature, where no explicit mention of the embedding is ever made.

Since Lemma 4.1 and Lemma 4.2 do not depend on Assumption (38b),

$$L^1_w(G)^\# = \{\Phi \in L^0(G) \mid w^{-1}\Phi \in L^\infty(G)\}$$

the left regular representation λ leaves $L^1_w(G)$ invariant, and the restriction ℓ of λ to $L^1_w(G)$ is a continuous representation that satisflies

$$\|\lambda(x)\| \leqslant w(x), \qquad x \in G. \tag{61}$$

We assume that there exists an admissible vector $u \in \mathcal{H}$ whose voice transform $V_2 u$ is in $L^1_w(G)$ and construct the corresponding reservoir $\mathcal{S}_w$ of test functions. We are in a position of stating the main properties of the standard setup, without the assumption that the representation is irreducible.

Theorem 5.1. *Take a reproducing representation π of G acting on the Hilbert space $\mathcal{H}$ and a weight w satisfying* (60a) *and* (60b)*. Choose an admissible vector $u \in \mathcal{H}$ such that*

$$K(\cdot) = \langle u, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^1_w(G)$$

and set

$$\mathcal{S}_w = \{v \in \mathcal{H} \mid \langle v, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^1_w(G)\},$$
$$\|v\|_{\mathcal{S}_w} = \int_G |\langle v, \pi(x)u\rangle_{\mathcal{H}}| w(x) dx.$$

a) The space $\mathcal{S}_w$ is a Banach space and the canonical inclusion $i : \mathcal{S}_w \to \mathcal{H}$ is continuous and with dense range.

b) The representation π leaves $\mathcal{S}_w$ invariant, its restriction τ is a continuous representation acting on $\mathcal{S}_w$, the operator norms satisfy $\|\tau(x)\| \leqslant w(x)$ for all $x \in G$, and

$$i(\tau(x)v) = \pi(x)i(v) \qquad x \in G,\, v \in \mathcal{S}_w.$$

c) Endowing $\mathcal{H}$ with the weak topology and $\mathcal{S}'$ with the weak-$*$ topology, the transpose mapping ${}^t i : \mathcal{H} \to \mathcal{S}'_w$ is continuous, injective and with dense range, and satisfies the intertwining

$${}^t\tau(x)\, {}^t i(w) = {}^t i(\pi(x)w) \qquad x \in G,\, w \in \mathcal{H}.$$

d) The restricted voice transform $V_0 : \mathcal{S}_w \to L^1_w(G)$ is an isometry intertwining τ and λ and its range is the closed subspace

$$\mathcal{M}^1 = \{f \in L^1_w(G) \mid f * K = f\},$$

e) For all $f \in L^1_w(G)$, the Fourier transform of f at u exists in $\mathcal{S}_w$ and satisfies

$$V_0 \pi(f)u = f * K.$$

Furthermore, the map

$$L^1_w(G) \ni f \mapsto \pi(f)u \in \mathcal{S}_w$$

is continuous and its restriction to $\mathcal{M}^1$ is the inverse of V_0.

f) The extended voice transform $V_e : \mathcal{S}'_w \to L^\infty_{w^{-1}}(G)$ is injective, continuous (when both spaces are endowed with the topology of the compact convergence) and intertwines ${}^t\tau$ and λ. The range of V_e is the closed subspace

$$\mathcal{M}^\infty = \{\Phi \in L^\infty_{w^{-1}}(G) \mid \Phi * K = \Phi\} \subset C(G).$$

For all $T \in \mathcal{S}'_w$ and $v \in \mathcal{S}_w$

$$\langle T, v\rangle_{\mathcal{S}_w} = \langle V_e T, V_0 v\rangle_{L^1_w(G)}. \tag{62}$$

g) For all $\Phi \in L^\infty_{w^{-1}}(G)$ the Fourier transform of Φ exists at u in $\mathcal{S}'_w$ and satisfies

$$V_e \pi(\Phi)u = \Phi * K.$$

h) The map

$$\mathcal{M}^\infty \ni \Phi \mapsto \pi(\Phi)u \in \mathcal{S}'_w$$

is the inverse of V_e and coincides with the restriction of the map ${}^tV_0$ to $\mathcal{M}^\infty$, namely

$$V_e({}^tV_0\Phi) = V_e\pi(\Phi)u = \Phi, \qquad \Phi \in \mathcal{M}^\infty. \tag{63}$$

i) ${}^t\!ii(\mathcal{S}_w) = \{T \in \mathcal{S}'_w \mid V_eT \in L^1_w(G)\} = \{\pi(f)u \mid f \in \mathcal{M}^1\}$.

Proof. Since $L^1_w(G) \subset L^1(G)$ and $K, V_2v \in L^\infty(G)$ for all $v \in \mathcal{H}$, Assumptions 1 and 2 are satisfied.

a) By the first part of Theorem 3.1 which does not depend on Assumption 3. The space $\mathcal{S}$ is a Banach space because $\mathcal{M}^1$ is such.

b) Apply Theorem 3.1. Moreover, by (61)

$$\|\tau(x)v\|_{\mathcal{S}_w} = \|V_0\tau(x)v\|_{1,w} = \|\lambda(x)V_0v\|_{1,w} \leqslant w(x)\|V_0v\|_{1,w} = w(x)\|v\|_{\mathcal{S}_w}.$$

c) Apply Theorem 3.1.

d) Apply Theorem 3.2.

e) Fix $f \in L^1_w(G)$ and set $\Psi : G \to \mathcal{S}_w$, $\Psi(x) = f(x)\tau(x)u$. We show that Ψ is Bochner-integrable with respect to β. The map Ψ is β-measurable since $f \in L^0(G)$ and $x \mapsto \tau(x)u$ is continuous from G into $L^1_w(G)$, and, by item b),

$$\|\Psi(x)\|_{\mathcal{S}_w} = |f(x)|\|\tau(x)u\|_{\mathcal{S}_w} \leqslant w(x)\|u\|_{\mathcal{S}_w}|f(x)|,$$

which is in $L^1(G)$ since $f \in L^1_w(G)$. Set

$$\pi(f)u = \int_G f(x)\tau(x)u\,dx.$$

Clearly, for all $v \in \mathcal{S}_w$

$$\langle {}^t\!ii\pi(f)u, v\rangle_{\mathcal{S}_w} = \int_G f(x)\langle i(\tau(x)u), i(v)\rangle_{\mathcal{H}}dx = \int_G f(x)\langle \pi(x)u, i(v)\rangle_{\mathcal{H}}dx.$$

Hence ${}^t\!ii\pi(f)u$ satisfies (14) and we can identify $\pi(f)u$ with ${}^t\!ii\pi(f)u$. So $V_e\pi(f)u = V_0\pi(f)u$. The fact that $V_0\pi(f)u = \Phi * K$ follows from (20) with $F = L^1_w(G)$ and $E = \mathcal{S}_w$. The fact that $f \mapsto \pi(f)u$ is the inverse of V_0 follows from (21c) in Proposition 2.6.

f) We first prove that $V_e\mathcal{S}'_w \subset L^\infty_{w^{-1}}(G)$. Take $T \in \mathcal{S}'_w$. For all $x \in G$, by (61)

$$|\langle T, \tau(x)u\rangle_{\mathcal{S}_w}| \leqslant \|T\|_{\mathcal{S}'_w}\|\tau(x)u\|_{\mathcal{S}_w} = \|T\|_{\mathcal{S}'_w}\|\lambda(x)Vu\|_{1,w} \leqslant w(x)\|T\|_{\mathcal{S}'_w}\|K\|_{1,w},$$

so that $w^{-1}V_eT$ is bounded and continuous. We now prove the reconstruction formula (62). Fix $v \in \mathcal{S}_w$ and define the map $\Psi : G \to \mathcal{S}_w$ by $\Psi(x) = \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\tau(x)u = \overline{V_0v(x)}\tau(x)u$. We show that it is Bochner-integrable with respect to β. It is continuous since both V_0v and $\tau(\cdot)u$ are continuous, and

$$\|\Psi(x)\|_{\mathcal{S}_w} = |V_0v(x)|\|\tau(x)u\|_{\mathcal{S}_w} \leqslant w(x)|V_0v(x)|\|K\|_{1,w},$$

which is in $L^1(G)$ by definition of $\mathcal{S}_w$. Hence there exists $w_v \in \mathcal{S}_w$ such that

$$w_v = \int_G \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\tau(x)u\,dx.$$

For all $z \in \mathcal{H}$ we have ${}^t\!i(z) \in \mathcal{S}'_w$ and

$$\begin{aligned}\langle z, i(w_v)\rangle_{\mathcal{H}} = \langle {}^t\!i(z), w_v\rangle_{\mathcal{S}_w} &= \int_G \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\langle {}^t\!i(z), \tau(x)u\rangle_{\mathcal{S}_w}dx \\ &= \int_G \langle \pi(x)u, i(v)\rangle_{\mathcal{H}}\langle z, \pi(x)u\rangle_{\mathcal{H}}dx = \langle z, i(v)\rangle_{\mathcal{H}},\end{aligned}$$

that is, $w_v = v$. Hence

$$v = \int_G \langle \pi(x)u, i(v)\rangle_{\mathcal{H}} \tau(x)u \, dx,$$

where the integral is a Bochner integral in $\mathcal{S}_w$. Take $T \in \mathcal{S}_w'$. Then for all $v \in \mathcal{S}_w$

$$\langle T, v\rangle_{\mathcal{S}_w} = \int_G \langle \pi(x)u, i(v)\rangle_{\mathcal{H}} \langle T, \tau(x)u\rangle_{\mathcal{S}_w} \, dx,$$

which proves the reconstruction formula. This, in turn, implies that V_e is injective.

g) Apply item a) of Proposition 2.6 with $E = \mathcal{S}_w$ and $F = L^\infty_{w^{-1}}(G)$.

h) Items b) and d) of Proposition 2.6 with $F = L^\infty_{w^{-1}}(G)$ and $E = \mathcal{S}_w$ show that the range of V_e is the closed subspace $\mathcal{M}^\infty$ and that the inverse of V_e is $\Phi \mapsto \pi(\Phi)u$.

i) Since $V_0\mathcal{S}_w \subset L^1_w(G)$ and $V_0 v = V_e\, {}^t i i(v)$ for all $v \in \mathcal{S}_w$, it follows that ${}^t i i(\mathcal{S}_w) \subset \{T \in \mathcal{S}_w' \mid V_e T \in L^1_w(G)\}$. Furthermore, Proposition 2.6 with $F = L^1_w(G)$ and $E = \mathcal{S}_w$ gives that

$$\{T \in \mathcal{S}_w' \mid V_e T \in L^1_w(G)\} = \{\pi(f)u \mid f \in \mathcal{M}^1\}.$$

Item d) of this theorem finally implies that $\{\pi(f)u \mid f \in \mathcal{M}^1\} = \mathcal{S}_w$. □

As shown in the previous proof, Assumptions 1 and 2 are satisfied. The reconstruction formula (62) makes clear that u is a cyclic vector for τ, which is equivalent to Assumption 3 because $V_0\tau(x)u = \ell(x)K$ and V_0 is an isometry from $\mathcal{S}_w$ onto $\mathcal{M}^1$. Furthermore, (44) with $v = \tau(x)u$ implies that also Assumption 4 holds true.

From now until the end of this section we choose a Banach space Y with a continuous embedding $j : Y \to L^1_{\mathrm{loc}}(G)$, denoted $f \mapsto f(\cdot)$, and with a continuous involution $f \mapsto \overline{f}$. We further suppose that there are two continuous representations ℓ and r of G on Y satisfying

i) for all $x \in G$ and all $f \in Y$

$$j(\ell(x)f) = \lambda(x)j(f), \qquad j(r(x)f) = \rho(x)j(f); \tag{64}$$

ii) for all $f \in Y$ and almost every $x \in G$,

$$j(\overline{f})(x) = \overline{j(f)(x)}. \tag{65}$$

Proposition 5.2. *Assume that Y is a Banach space with a continuous representation r for which there exists a continuous embedding $j : Y \to L^1_{loc}(G)$, denoted $f \mapsto f(\cdot)$, such that, for all $x \in G$, all $f \in Y$ and almost every $y \in G$ it holds that $r(x)f\,(y) = f(yx)$. Suppose that $g \in L^1_{loc}(G)$ is such that for all $f \in Y$*

$$\int_G |g(x^{-1})| \|r(x)f\|_Y \, dx < +\infty. \tag{66}$$

Then $j(f)$ and g are convolvable, there exists $f * g \in Y$ satisfying $j(f) * g = j(f * g)$ and

$$\|f * g\|_Y \leqslant \int_G |g(x^{-1})| \|r(x)f\|_Y \, dx.$$

Proof. The proof is closely related to the proof of Proposition 6 Chapter VIII.4.2 of [29]. Fix $f \in Y$ and $g \in L^1_{\mathrm{loc}}(G)$ and set

$$\Psi : G \to Y, \qquad \Psi(x) = g(x^{-1})r(x)f.$$

We claim that Ψ is β-integrable in the Bochner sense. Since r is a continuous representation, the map $x \mapsto r(x)f$ is continuous from G to Y and hence it is β-measurable. Since $g \in L^1_{\mathrm{loc}}(G)$, so is $\check{g}$, and hence Ψ is β-measurable. Furthermore,

$$x \mapsto \|\Psi(x)\|_Y = |g(x^{-1})| \|r(x)f\|_Y$$

is β-integrable by assumption. Hence Ψ is β-integrable. Define

$$v = \int_G g(x^{-1}) r(x) f \, dx \in Y,$$

which clearly satisfies

$$\|v\|_Y \leqslant \int_G |g(x^{-1})| \|r(x)f\|_Y. \tag{67}$$

Recall that $C_c(G) \subset Y^{\#}$ and take $\varphi \in C_c(G)$. Then by (8) with $q_i(f) = \|f\|$ (Y is normed)

$$\int_G |v(y)||\varphi(y)| \, dy = \int_G |g(x^{-1})| \left(\int_G |f(yx)\varphi(y)| dy \right) dx \leqslant C \int_G |g(x^{-1})| \|r(x)f\|_Y \, dx,$$

which is finite by assumption. By Fubini theorem, the function

$$(x, y) \mapsto g(x^{-1}) f(yx) \overline{\varphi(y)}$$

is in $L^1(G \times G)$ and, hence, there exists a negligible set N_φ such that for all $y \notin N_\varphi$ the function

$$x \mapsto g(x^{-1}) f(yx) \overline{\varphi(y)}$$

is in $L^1(G)$. Put $E_\varphi = \{x \in G \mid \varphi(x) \neq 0\}$. By the change of variable $x \mapsto y^{-1}x$, for all $y \in E_\varphi \backslash N_\varphi$, the function

$$x \mapsto f(x) g(x^{-1}y)$$

is in $L^1(G)$. Take a countable family $\{\varphi_k\}_{k \in \mathbb{N}}$ such that $\bigcup_k E_{\varphi_k} = G$ and set $N = \bigcup_k N_{\varphi_k}$. Then N is negligible and for all $y \notin N$ the map $x \mapsto f(x)g(x^{-1}y)$ is integrable. Hence, for all $\varphi \in C_c(G)$, Fubini theorem gives

$$\begin{aligned}
\int_G v(y) \overline{\varphi(y)} dy &= \langle v, \varphi \rangle_Y \\
&= \int_G g(x^{-1}) \langle r(x) f, \varphi \rangle_Y \, dx \\
&= \int_G g(x^{-1}) \left(\int_G f(yx) \overline{\varphi(y)} \, dy \right) dx \\
&= \int_G \overline{\varphi(y)} \left(\int_G f(yx) g(x^{-1}) \, dx \right) dy
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_Y$ denotes the duality between Y and $Y^{\#} \subset C_c(G)$ introduced in (6). By the change of variable $x \mapsto y^{-1}x$ in the inner integral, we get

$$\int_G v(y) \overline{\varphi(y)} dy = \int_G \left(\int_G f(x) g(x^{-1}y) dx \right) \overline{\varphi(y)} dy. \tag{68}$$

This means that $j(f)$ and g are convolvable, $j(v) = j(f) * g$ and, by (67), the inequality $\|f * g\|_Y \leqslant \int_G |g(x^{-1})| \|r(x)f\|_Y$ holds true. □

Corollary 5.3. *Take a weight w such that $\|r(x)\| \leqslant w(x)$ for all $x \in G$. Then:*

a) For all $f \in Y$ and $\check{g} \in L^1_w(G)$, $j(f)$ and g are convolvable, there exists $f * g \in Y$ such that $j(f * g) = j(f) * g$, the map

$$Y \ni f \mapsto f * g \in Y$$

is continuous and $\|f * g\|_Y \leqslant \|f\|_Y \|g\|_{1,w}$.

b) The set

$$\mathcal{M}^Y = \{f \in Y \mid f * g = f\},$$

is a closed ℓ-invariant subspace of Y.

Proof. Item a) follows from Proposition 5.2, observing that (67) is satisfied and

$$\int_G |g(x^{-1})| \|r(x)f\|_Y \, dx \leqslant \|f\|_Y \|\check{g}\|_{1,w}.$$

As for b), since Y is a metrizable topological space, it is sufficient to prove that $\mathcal{M}^Y$ is sequentially closed. Take a sequence $(f_n)_n$ in $\mathcal{M}^Y$ converging to $f \in Y$. Possibly passing to a subsequence, we can assume that there exists a negligible set N such that for all $x \notin N$ the sequence $(f_n(x))_n$ converges to $f(x)$. Furthermore, possibly changing N, we can also assume that, for all n and $x \notin N$, $j(f_n) * g\,(x) = f_n(x)$.

Since $f \mapsto f * g$ and j are continuous, $j(f_n) * g$ converges to $j(f) * g$ in $L^1_{\mathrm{loc}}(G)$. Hence, by Lemma 6.1 in the appendix, possibly passing again to a subsequence and again redefining N, we can also assume that for all $x \notin N$ $\lim_n j(f_n) * g\,(x) = j(f) * g\,(x)$. Then

$$j(f) * g\,(x) = \lim_n j(f_n) * g\,(x) = \lim_n f_n(x) = f(x),$$

so that $j(f) * g = j(f)$ in $L^0(G)$, that is $f \in \mathcal{M}^Y$. Finally, given $x \in G$ and $f \in \mathcal{M}^Y$, by (77b) ,

$$j(\ell(x)f) = \lambda(x)j(f) = \lambda(x)(j(f) * g) = \lambda(x)j(f) * g = j(\ell(x)f) * g,$$

which means that $\ell(x)f \in \mathcal{M}^Y$. □

We apply the above corollary with the choice $g = K$, which is in $L^1_w(G)$ by assumption, together with $\check{K} = \overline{K}$. Notice that, although Assumption 5 is not satisfied, b) of Corollary 5.3 guarantees that $\mathcal{M}^Y$ is a closed subspace of Y. Furthermore we assume that

$$\mathcal{M}^Y \subset L^\infty_{w^{-1}}(G). \tag{69}$$

Since by construction $V_2 v \in L^1_w(G)$ for all $v \in \mathcal{S}$ and $L^\infty_{w^{-1}}(G) = L^1_w(G)^\#$, (69) implies Assumption 6. Hence we can define

$$\begin{aligned} \mathrm{Co}(Y) &= \{T \in \mathcal{S}'_w \mid V_e T \in Y\}, \\ \|T\|_{\mathrm{Co}(Y)} &= \|V_e T\|_Y. \end{aligned}$$

The inclusion (69) is satisfied by all the classical Banach spaces considered in [2]. This fact is shown in the proof of Proposition 4.3.

Theorem 5.4. *The space $\mathrm{Co}(Y)$ is a π-invariant Banach space and the restriction of V_e to $\mathrm{Co}(Y)$ is an isometry from $\mathrm{Co}(Y)$ onto $\mathcal{M}^Y$. For all $\Phi \in \mathcal{M}^Y$, $\pi(\Phi)u$ exists in $\mathcal{S}'_w$, it actually belongs to $\mathrm{Co}(Y)$ and satisfies*

$$\pi(V_e T)u = T, \qquad T \in \mathrm{Co}(Y) \tag{70a}$$

$$V_e \pi(\Phi)u = \Phi, \qquad \Phi \in \mathcal{M}^Y. \tag{70b}$$

Proof. Apply Proposition 2.6. □

5.1. **Completeness and weights bounded from below.**

Take a continuous function $w : G \to (0,+\infty)$ satisfying only (60a). Take a square-integrable representation π acting on $\mathcal{H}$ and fix an admissible vector $u \in \mathcal{H}$ such that

$$K(\cdot) = \langle u, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^1_w(G),$$

and set

$$\mathcal{S}_w = \{v \in \mathcal{H} \mid \langle v, \pi(\cdot)u\rangle_{\mathcal{H}} \in L^1_w(G)\}$$

$$\|v\|_{\mathcal{S}_w} = \int_G |\langle v, \pi(x)u\rangle_{\mathcal{H}}| w(x)dx.$$

Theorem 5.5. *The space $\mathcal{S}_w$ is a Banach space if and only if*

$$\inf_{x\in G} w(x) > 0. \tag{71}$$

Proof. Assume that $\inf_{x\in G} w(x) \geqslant c > 0$. We can always suppose that $c = 1$ so that (60b) holds true. Indeed, if $c < 1$, we redefine w as w/c, so that

$$\frac{w(xy)}{c} \leqslant \frac{w(x)w(y)}{c^2}c \leqslant \frac{w(x)}{c}\frac{w(y)}{c}$$

and w/c satisfies (60a). Since $L^1_w(G) = L^1_{w/c}(G)$ with equivalent norms, clearly the fact that $\mathcal{S}_w$ is a Banach space does not depend on the choice of c. Item a) of Theorem 5.1 states that $\mathcal{S}_w$ is a Banach space.

Assume that $\mathcal{S}_w$ is a Banach space. Define $\mathcal{S}^*_w$ as the vector space $\mathcal{S}_w$ with the norm

$$\|v\|_* = \max\{\|v\|_{\mathcal{S}_w}, \|v\|_{\mathcal{H}}\}.$$

We claim that $\mathcal{S}^*_w$ is complete. Take a Cauchy sequence $(v_n)_n$ with respect to $\|\cdot\|_*$. By construction, it is a Cauchy sequence also with respect to both $\|\cdot\|_{\mathcal{S}_w}$ and $\|\cdot\|_{\mathcal{H}}$. Since $\mathcal{S}_w$ and $\mathcal{H}$ are complete, there exist $v' \in \mathcal{S}_w$ and $v \in \mathcal{H}$ such that

$$\lim_{n\to+\infty} \|v_n - v\|_{\mathcal{H}} = 0 \qquad \lim_{n\to+\infty} \|v_n - v'\|_{\mathcal{S}_w} = 0.$$

Since the voice trasform is an isometry both from $\mathcal{H}$ into $L^2(G)$ and from $\mathcal{S}_w$ into $L^1_w(G)$, the sequence $(Vv_n)_n$ converges to Vv in $L^2(G)$ and to Vv' in $L^1_w(G)$. Hence, possibly passing to a subsequence, $(Vv_n)_n$ converges almost everywhere to Vv and to Vv'. Since $w > 0$, $Vv = Vv'$ almost everywhere and, hence, $v = v'$ by the injectivity of V, so that $v \in \mathcal{S}^*_w$. Furthermore

$$\begin{aligned}\lim_{n\to+\infty} \|v_n - v\|_* &= \lim_{n\to+\infty} \max\{\|v_n - v\|_{\mathcal{S}_w}, \|v_n - v\|_{\mathcal{H}}\}\\ &= \max\{\lim_{n\to+\infty} \|v_n - v\|_{\mathcal{S}_w}, \lim_{n\to+\infty} \|v_n - v\|_{\mathcal{H}}\} = 0.\end{aligned}$$

Hence $\mathcal{S}^*_w$ is complete and the natural inclusion $i : \mathcal{S}^*_w \to \mathcal{S}_w$ is clearly continuous and bijective. Since $\mathcal{S}_w$ is a Banach space, the open mapping theorem implies that the inverse is also continuous, so that there is a constant $c > 0$ such that

$$c\|v\|_* \leqslant \|v\|_{\mathcal{S}_w} \leqslant \|v\|_*.$$

As usual, for all $x \in G$ and $v \in \mathcal{S}_w$

$$\|\pi(x)v\|_{\mathcal{S}_w} = \|\lambda(x)Vv\|_{L^1_w(G)} \leqslant w(x)\|v\|_{\mathcal{S}_w}, \qquad \|\pi(x)v\|_{\mathcal{H}} = \|v\|_{\mathcal{H}}.$$

Then, if $v \neq 0$, for all $x \in G$

$$c\|v\|_{\mathcal{H}} = c\|\pi(x)v\|_{\mathcal{H}} \leqslant c\|\pi(x)v\|_* \leqslant \|\pi(x)v\|_{\mathcal{S}_w} \leqslant w(x)\|v\|_{\mathcal{S}_w}$$

taking the infimum over G, we get

$$0 < c \leqslant \inf_{x \in G} w(x).$$

$\square$

6. Appendix: some functional analysis

6.1. **Notation.** The set $L^0(G)$ is a metrizable complete topological vector space, and $C_0(G)$ is dense in $L^0(G)$ (Propositions 19 and 20 Chapter IV.5.11 of [30]). Since G is second countable, $C_0(G)$ is separable (with respect to the convergence on compact subsets, hence with respect to the convergence in measure) and $L^0(G)$ is separable, too. Furthermore, if (f_n) is a sequence converging to f in $L^0(G)$, then there exist a subsequence $(f_{n_k})_k$ and a negligible set $N \subset G$ such that

$$\lim_{k\to+\infty} f_{n_k}(x) = f(x) \qquad \text{for all } x \in G\backslash N. \tag{72}$$

(Corollary of Proposition 19 Chapter IV.5.11 of [30]).

If G is compact $L^1_{\text{loc}}(G) = L^1(G)$ is a separable Banach space. Otherwise, a saturated fundamental system of semi-norms is given as follows (recall that a family is saturated if the maximum of any finite set of seminorms is in the family). Since G is second countable, take a countable increasing family $(\mathcal{K}_i)_{i\in\mathbb{N}}$ of compact subsets of G such that $\mathcal{K}_i \subset \mathring{\mathcal{K}}_{i+1}$ and $\bigcup_i \mathcal{K}_i = G$. For all $i \in \mathbb{N}$ put

$$q_i(f) = \int_{\mathcal{K}_i} |f(x)|dx. \tag{73}$$

Then, for each compact set $\mathcal{K}$ there exists $i \in \mathbb{N}$ such that $\mathcal{K} \subset \mathcal{K}_i$ and

$$\int_{\mathcal{K}} |f(x)|dx \leqslant q_i(f).$$

With the induced topology, $L^1_{\text{loc}}(G)$ is complete (see Ex. 31 Chapter V.5 of [30]), hence it is a Fréchet space.

Lemma 6.1. *If (f_n) is a sequence in $L^1_{loc}(G)$ converging to f in $L^1_{loc}(G)$, then there exists a subsequence $(f_{n_k})_k$ that converges to f almost everywhere.*

Proof. If G is compact, the claim is clear. If not, take the increasing sequence of compact subsets $(K_i)_{i\in\mathbb{N}}$ defining the fundamental family of semi-norms (73). The topology of $L^1_{\text{loc}}(G)$ is such that $(f_n)_n$ converges to f in $L^1(K_i)$ for all $i \in \mathbb{N}$. We procede by induction on $\mathbb{N}$. Suppose that we have found a subsequence $(f_{n_k^{(i)}})_k$ and a negligible subset $N_i \subset K_i$ such that

$$\lim_{k\to+\infty} f_{n_k^{(i)}}(x) = f(x) \qquad \text{for all } x \in K_i\backslash N_i.$$

Clearly $(f_{n_k^{(i)}})_k$ converges to f in $L^1(K_{i+1})$ and we can further extract a subsequence $(f_{n_k^{(i+1)}})_k$ for which there exists a negligible subset $N_{i+1} \subset K_{i+1}$ such that

$$\lim_{k\to+\infty} f_{n_k^{(i+1)}}(x) = f(x) \qquad \text{for all } x \in K_{i+1}\backslash N_{i+1}.$$

Set $N = \bigcup_{i\in\mathbb{N}} N_i$ and $f_{n_k} = f_{n_k^{(k)}}$. Given $x \notin N$, fix h such that $x \in K_h$, so that $x \in K_i\backslash N_i$ for all $i \geqslant h$. Then $(f_{n_k^{(k)}}(x))_{k\geqslant h}$ is a subsequence of $(f_{n_k^{(h)}}(x))_{k\geqslant h}$ which converges to $f(x)$. $\square$

Given $f \in L^0(G)$, $\check{f} \in L^0(G)$ since a subset $E \subset G$ is negligible if and only if E^{-1} is negligible. Notice that

$$\check{f} \in L^p(G) \quad \Longleftrightarrow \quad \Delta^{-1/p} f \in L^p(G) \quad \Longleftrightarrow \quad f \in L^p(G, \Delta^{-1} \cdot \beta) \tag{74}$$

$$\|\check{f}\|_p = \|\Delta^{-1/p} f\|_p. \tag{75}$$

Since Δ is continuous, $L^1_{\mathrm{loc}}(G)$ is invariant under the map $f \mapsto \check{f}$.

6.2. **Representations.** Let E be a locally convex space with a saturated fundamental system $\{q_i\}_i$ of semi-norms and τ a (linear) representation of G on E.

i) The representation τ is separately continuous if
   a) for all $x \in G$, $\tau(x)$ is continuous from E to E;
   b) for all $v \in E$, $x \mapsto \tau(x)v$ is continuous from G into E.
ii) The representation τ is continuous if
   a) if $(x, v) \mapsto \tau(x)v$ is continuous from $G \times E$ into E.
iii) The representation τ is equicontinuous if
   a) if $(x, v) \mapsto \tau(x)v$ is continuous from $G \times E$ into E;
   b) $\tau(G)$ is equicontinuous, *i.e.* for every semi-norm q_i there exists a semi-norm q_j and a constant C such that $q_i(\tau(x)) \leqslant C q_j(\tau(x))$ for all $x \in G$.

If E is a Fréchet space, then i) implies ii) (Proposition 1 Chapter VIII.2.1 of [30]). Furthermore, τ is continuous if and only if for any compact set Q of G, $\tau(Q)$ is equicontinuous and the map $x \mapsto \tau(x)v$ is continuous for all $v \in D$, where D is a total set in E (Remark 2 of Definition 1 Chapter VIII.2.1 of [30]).

6.3. **Convolutions.** The basic property of convolution is given by the following lemma.

Lemma 6.2. *If $f * g$ exists, it is a β-measurable function whose equivalence class in $L^0(G)$ depends only on the equivalence classes of f and g.*

Proof. Without loss of generality, we can suppose that both f and g are positive. The topological isomorphism $\psi : G \times G \to G \times G$, $\psi(x, y) = (x, y^{-1}x)$ has the property that a set $E \subset G \times G$ is $\beta \otimes \beta$-negligible if and only if $\psi^{-1}(E)$ is $\beta \otimes \beta$-negligible. Indeed, take E a Borel measurable subset of $G \times G$, then

$$\beta \otimes \beta(\psi^{-1}(E)) = \int_G \beta(\psi^{-1}(E)_x) dx = \int_G \beta(x E_x^{-1}) dx = \int_G \beta(E_x^{-1}) dx$$

where $E_x = \{y \in G \mid (x, y) \in E\}$ and $\psi^{-1}(E)_x = \{y \in G \mid (x, y^{-1}x) \in E\} = x E_x^{-1}$. Hence $\beta \otimes \beta(\psi^{-1}(E)) = 0$ if and only if $\beta(E_x^{-1}) = 0$ for almost all $x \in G$, which is equivalent to the fact that $\beta(E_x) = 0$ for almost all $x \in G$, *i.e.* $\beta \otimes \beta(E) = 0$. As a consequence, the map $\varphi = (f \otimes g)\psi$ is $\beta \otimes \beta$-measurable, and if we change $f \otimes g$ on a negligible set, φ will change on a negligible set, too. Since G is second countable, the measure β is moderated and Proposition 7.b) Chapter V.8.3 of [30] shows that the map $x \mapsto \int_G \varphi(x, y)\, dy$ is β-measurable, where the integral is finite by assumption. Therefore, $\int_G \varphi(x, y) dy$ depends only on the equivalence class of f and g. □

If $f, g \in L^1_{\mathrm{loc}}(G)$, $f * g$ exists and $|f| * |g|$ is in $L^1_{\mathrm{loc}}(G)$, then we say that f and g are *convolvable.* Since $f, g \in L^1_{\mathrm{loc}}(G)$, then $\mu = f \cdot \beta$ and $\nu = g \cdot \beta$ are (Radon) measures on G. The fact that f and g are convolvable is equivalent to the fact that μ and ν admit a convolution, *i.e.* for all $\varphi \in C_c(G)$ the function $(x, y) \mapsto \varphi(xy)$ is $\mu \otimes \nu$-integrable, namely

$$\int_{G \times G} |\varphi(xy)||f(x)||g(x)| dx dy < +\infty, \qquad \varphi \in C_c(G).$$

The two definitions agree, since:

i) if μ and ν admit a convolution, the map $\varphi \mapsto \int_G \varphi(xy) d\mu(x) d\nu(y)$ defines a measure on G whose density is precisely $f \ast g$ (Proposition 10 Chapter VIII.3.2 of [29] and Proposition 10 Chapter VIII.4.5 of [29]);
ii) if $|f| \ast |g|$ exists and is in $L^1_{\mathrm{loc}}(G)$, then μ and ν admit a convolution (Proposition 9 Chapter VIII.4.5 of [29]).

We recall the following sufficient conditions.

a) Corollary 20.14 of [41]: if $f \in L^1(G)$ and $g \in L^p(G)$ with $p \in [1, +\infty]$, then f and g are convolvable, $f \ast g$ belongs to $L^p(G)$ and

$$\|f \ast g\|_p \leqslant \|f\|_1 \|g\|_p. \tag{76a}$$

b) Theorem 20.18 of [41]: if $f \in L^p(G)$, $g \in L^q(G)$ and $\check{g} \in L^q(G)$ with $1 < p < +\infty$, $1 < q < +\infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $r > 1$, then f and g are convolvable and $f \ast g$ belongs to $L^r(G)$. Furthermore, if $\|\check{g}\|_q = \|g\|_q$, then

$$\|f \ast g\|_r \leqslant \|f\|_p \|g\|_q. \tag{76b}$$

c) Theorem 20.16 of [41]: under the same assumptions on f and g as in the previous item, if $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < +\infty$, then f and g are convolvable and $f \ast g$ belongs to $C_0(G)$ and

$$\|f \ast g\|_\infty \leqslant \|f\|_p \|\check{g}\|_q. \tag{76c}$$

d) Theorem 20.16 of [41]: if $f \in L^1(G)$ and $g \in L^\infty(G)$ (which is equivalent to $\check{g} \in L^\infty(G)$) or if $f \in L^\infty(G)$ and $\check{g} \in L^1(G)$, then f and g are convolvable, $f \ast g$ is a bounded continuous function, and

$$\|f \ast g\|_\infty \leqslant \|f\|_1 \|g\|_\infty \quad \text{or } \|f \ast g\|_\infty \leqslant \|f\|_\infty \|\check{g}\|_1 \tag{76d}$$

In general, the convolution is not associative. We recall a sufficient condition as well as some other useful relations.

Lemma 6.3. *Given $f, g \in L^0(G)$,*

$$\widecheck{f \ast g} = \check{g} \ast \check{f} \tag{77a}$$

and, for all $x \in G$,

$$\lambda(x) f \ast g = \lambda(x)(f \ast g) \qquad \rho(x) f \ast g = \Delta(x^{-1})(f \ast \lambda(x^{-1}) g) \tag{77b}$$

$$f \ast \lambda(x)\, g = \Delta(x^{-1})(\rho(x^{-1}) f \ast g) \qquad f \ast \rho(x) g = \rho(x)(f \ast g), \tag{77c}$$

provided that one of the two sides of each equality exists.

If $f, g, h \in L^0(G)$ are such that either $|f| \ast |g|$ and $(|f| \ast |g|) \ast |h|$ exist or $|g| \ast |h|$ and $|f| \ast (|g| \ast |h|)$ exist, then

$$f \ast (g \ast h) = (f \ast g) \ast h \tag{77d}$$

and all the convolutions exist.

Proof. To prove (77a) just compute

$$\check{g} \ast \check{f}(x) = \int_G \check{f}(y^{-1}x) \check{g}(y) dy = \int_G f(x^{-1}y) g(y^{-1}) dy = \int_G f(y) g(y^{-1}x^{-1}) dy = f \ast g(x^{-1}).$$

Next we prove (77d). Fubini theorem gives that, for all $x \in X$

$$\begin{aligned}(|f| * |g|) * |h|(x) &= \int_{G\times G} |f(y)||g(y^{-1}z)||h(z^{-1}x)|dydz \\ &= \int_{G\times G} |f(y)||g(z)||h(z^{-1}y^{-1}x)|dydz = |f| * (|g| * |h|)(x)\end{aligned}$$

by the change of variable $z \mapsto yz$. Hence the two assumptions implies that the map $(y, z) \mapsto f(y)g(y^{-1}z)h(z^{-1}x)$ is in $L^1(G\times G)$. Since $|f * g| \leqslant |f| * |g|$, all the convolutions in (77d) exist and Fubini theorem implies the claimed equality. The remaining assertions are standard. □

6.4. **Scalar integration.** Let E be a locally convex topological vector space, and let X be a Hausdorff locally compact second countable topological space with a positive measure dx, which is finite on all compact subsets. A function $\Psi : X \to E$ is called scalarly integrable if the scalar function $\langle T, \Psi(\cdot)\rangle_E$ is integrable for every $T \in E'$. If Ψ is scalarly integrable, the map

$$T \mapsto \int_X \langle T, \Psi(x)\rangle_E dx$$

defines a linear functional on E', possibly not continuous; that is, there exists an element in the algebraic dual E'^*, called the scalar integral of Ψ and denoted

$$\int_X \Psi(x)dx \in E'^*,$$

such that

$$\langle T, \int_X \Psi(x)dx\rangle_E = \int_X \langle \Psi(x), T\rangle_E dx.$$

Usually one is interested to understand under which conditions the scalar integral lies in E. In our paper we often look ar the case in which the argument takes its values in a dual space (or in a space which embeds into a dual space), namely

$$\Psi : X \longrightarrow E'_s,$$

where E'_s is the space E' endowed twith the weak*-topology, namely the topology of simple convergence, so that $(E'_s)' = E$.

A locally convex space E is said to have the property (GDF)[7] if every linear map from E to a Banach space which has sequentially closed graph is actually continuous (that is, the closed graph theorem holds true for Banach space-valued linear maps defined on E). All the Fréchet spaces enjoy the property (GDF) ([25], Chapter I.3.3, Corollary 5). Also, the dual space of any reflexive Fréchet space satisfies the property with respect to the strong topology, namely the topology of the convergence on bounded sets ([30], Chapter 6, Appendix, n° 2, Proposition 3).

The key theorem for the convergence of scalar integrals with values in a dual vector space is the following.

Theorem 6.4 (Gelfand–Dunford, [30], Theorem 1, Chapter VI.1.4)**.** *Let E be a Hausdorff locally convex topological vector space with the property (GDF). Then, for any scalarly integrable function $\Psi : X \to E'_s$, we have*

$$\int_X \Psi(x)dx \in E'.$$

[7]The acronym GDF stands for "graphe dénombrablement fermé", namely "countably closed graph".

6.5. **Intersections of L^p spaces.** In this final section we recall, for the reader's convenience, the main results obtained in [31], specialized to our setting. Set $I = (1, +\infty)$ and define

$$\mathcal{T} = \bigcap_{p \in I} L^p(\mu)$$

with the initial topology, which makes each inclusion $\mathcal{T} \hookrightarrow L^p(G)$ continuous, and

$$\mathcal{U} = \operatorname{span} \bigcup_{q \in I} L^q(G)$$

with the final topology, which makes each inclusion $L^q(G) \hookrightarrow \mathcal{U}$ continuous.

Theorem 6.5 ([31]). *The space $\mathcal{T}$ is a reflexive Fréchet space and $\mathcal{U}$ is a complete reflexive locally convex topological vector space. For each $g \in \mathcal{U}$, the linear map*

$$f \mapsto \int_G g(x) f(x)\, dx = g(f)$$

is continuous and $g \mapsto g(\cdot)$ identifies, as topological vector spaces, the dual of $\mathcal{T}$ with $\mathcal{U}$. For each $f \in \mathcal{T}$, the linear map

$$g \mapsto \int_G f(x) g(x)\, dx = f(g)$$

is continuous and $f \mapsto f(\cdot)$ identifies, as topological vector spaces, the dual of $\mathcal{U}$ with $\mathcal{T}$.

Proof. Here we refer to [31]. Observe that the Haar measure β is σ-finite since G is locally compact and second countable and β is finite on compact subsets. Furthermore, denoted by $I' = \{\frac{p}{p-1} \mid p \in I\}$, clearly $I' = I$. Proposition 2.1 and the following remark show that the map $f \mapsto \langle f, \cdot \rangle_{\mathcal{U}}$ is a topological isomorphism from $\mathcal{T}$ onto the strong dual of $\mathcal{U}$.

Theorem 2.1 and Corollary 3.2 show that the map $g \mapsto \langle g, \cdot \rangle_{\mathcal{T}}$ is a topological isomorphism from $\mathcal{U}$ onto the strong dual of $\mathcal{T}$.

Hence we can identify, as topological vector spaces, the dual of $\mathcal{T}$ with $\mathcal{U}$ and the dual of $\mathcal{U}$ with $\mathcal{T}$. So that both $\mathcal{T}$ and $\mathcal{U}$ are reflexive locally convex vector spaces. Theorem 3.1 proves that $\mathcal{T}$ is a Frechét space and Corollary 3.3 shows that $\mathcal{U}$ is complete. $\square$

Note that, since $\mathcal{T}$ and $\mathcal{U}$ are reflexive spaces, they are barrelled (Theorem 2 IV.2.3 of [25]).

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