FRAMES FROM IMPRIMITIVITY SYSTEMS

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Abstract. Let (P, V) be an irreducible imprimitivity system for a group H based on a dual group $\hat{A}$ of an abelian group A and acting on a Hilbert space $\mathcal{H}$. Given $\psi \in \mathcal{H}$, we find necessary and sufficient conditions in order that the set of vectors

$$\{\int_{\hat{A}} \overline{\langle \hat{a}, a\rangle} V_h^{-1} dP(\hat{a})\psi \ : \ a \in A,\ h \in H\}$$

be a frame in $\mathcal{H}$. Moreover, we apply these results to some examples that are considered in the literature in the context of square-integrability modulo a coset space.

1. Introduction

It is well known that a large class of continuous frames is defined in terms of square-integrable representations U of locally compact groups G by means of the equation

$$G \ni g \mapsto U_g\psi \in \mathcal{H}, \tag{1}$$

where $\mathcal{H}$ is the Hilbert space carrying the representation U and ψ is any nonzero vector belonging to the domain of the formal degree $K_U^{-1/2}$ of U. The space labelling the frame is the group G endowed with a left invariant Haar measure μ_G^l. As a consequence of the properties of square-integrable representations, [5], the above frames are tight and the frame bound can be computed using the orthogonality relation

$$\int_G \langle \phi, U_g\psi\rangle \overline{\langle \phi', U_g\psi'\rangle}\, d\mu_G^l(g) = \langle \phi, \phi'\rangle \left\langle K_U^{-\frac{1}{2}}\psi', K_U^{-\frac{1}{2}}\psi \right\rangle. \tag{2}$$

The groups that are semidirect products with an abelian normal factor, *i.e.* $G = A \times' H$, are of particular interest in the applications. For these groups, there is a one-to-one correspondence between the representations U of G and the imprimitivity systems for H based on the dual group $\hat{A}$ of A, explicitly given by

$$\begin{aligned} U_a &= \int_{\hat{A}} \langle \hat{a}, a\rangle \ dP(\hat{a}) \\ V_h &= U_h, \end{aligned}$$

where $a \in A$ and $h \in H$. Moreover, a right invariant Haar measure μ_G^r for G is simply given by $\mu_G^r = \mu_A^r \otimes \mu_H^r$ where μ_A^r and μ_H^r are right invariant

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Haar measures on A and H, respectively; hence it is convenient to perform in Eq. (1) the change of variable $g \mapsto g^{-1}$ in order to use the measure μ_G^r instead of μ_G^l. Taking into account these facts Eq. (1) becomes

$$A \times H \ni (a,h) \mapsto \left(\int_{\hat{A}} \overline{\langle \hat{a}, a \rangle} V_h^{-1} dP(\hat{a}) \psi \right) \in \mathcal{H}, \tag{3}$$

and the corresponding frame is labelled by the points of the topological direct product $A \times H$ with the product measure $\mu_A^r \otimes \mu_H^r$.

The structure of the above relation suggests the possibility of defining frames by means of Eq. 3 also in the case that H acts on $\hat{A}$ without preserving the composition law of $\hat{A}$.

The need to consider the above classes of frames is due to the fact that there are groups G of interest in the applications, as the euclidean group, the Galilei and Poincaré groups, that are semidirect products, but do not admit square integrable representations.

In this paper, we consider two locally compact second countable (lcsc) groups A, H such that A is abelian and H acts on $\hat{A}$, the dual group of A, and an irreducible imprimitivity system (P, V) for H based on $\hat{A}$ and acting on a Hilbert space $\mathcal{H}$. We assume that the orbits of H in $\hat{A}$ are locally closed, so that, due to the theorem of Mackey about imprimitivity systems, see, for example [6], there is a one-to-one correspondence between the equivalence classes of irreducible imprimitivity systems (P, V) and the couples $(X, [m])$, where X is an orbit in $\hat{A}$ and $[m]$ is an equivalence class of irreducible representations of the the stability subgroup at a fixed $x_0 \in X$. Hence, we can (and we do) associate the maps defined by Eq. (3) with the couples $(x_0, [m])$ instead of (P, V).

In Section 3, we prove that the set of vectors given by Eq. (3) is a frame for some $\psi \in \mathcal{H}$ if and only if the Haar measure of $\hat{A}$ restricted to X is not singular with respect to the H-quasi-invariant measures on X and m is a square-integrable representation. Moreover we show that these frames are tight, that their frame bound can be computed by a *generalised orthogonality relation* and we characterise the set of admissible vectors. We stress that, with our assumptions, we have to show these last properties since orthogonality relations are not proved to exist for imprimitivity systems, but only for square-integrable representations. In the particular case that H preserves the group law of $\hat{A}$ (so that, by duality, it is defined the semidirect product $A \times' H$) we obtain the results of paper [3].

In Section 4, we restrict ourself to the case of imprimitivity systems such that A and H are Lie groups and the stability subgroup H_0 is compact and we consider the sets of vectors of the form

$$A \times (H/H_0) \times J \ni (a, x, i) \mapsto \left(\int_{\hat{A}} \langle \hat{a}, a \rangle V_{q(x)} dP(\hat{a}) \psi_i \right) \in \mathcal{H},$$

where J is the set of the first n numbers with the counting measure μ_c (n is the dimension of the Hilbert space where m acts) and q is any measurable section from the quotient space H/H_0 to H. We prove that, with a suitable choice of the vectors ψ_i, the above set is a tight frame labelled by the points of the space $(A \times H \times J)$ endowed with the measure $\mu^r_A \otimes \mu^r_H \otimes \mu_c$.

In the literature there are attempts to define frames by the equation

$$G/G_0 \ni x \mapsto U_{q(x)}\psi \in \mathcal{H} \tag{4}$$

where G_0 is a closed subgroup of G, q is a suitable section from the quotient space G/G_0 to G and the frame is labelled by the points of G/G_0 endowed with a G-quasi invariant measure (in this case U is called *square-integrable modulo a coset space*), see [1] and references therein. Nevertheless, there are no general results on the square-integrable representations modulo a coset space and one has to prove for each representation both the fact that the set given by Eq. (4) is a frame and that the frame is tight with an equation for computing the frame bound. In the examples we give in the final section we show that the frames defined by Eq. (4) are particular cases of our procedure, so that our results provide a rigorous and general framework to this class of frames. An alternative approach to this problem can be found in [2].

In the first section we introduce the mathematical notations and we give some preliminary results, as standard reference we use [6].

2. Mathematical preliminaries

Let X be a locally compact second countable Hausdorff topological space. We denote by $\mathcal{K}(X)$ the vector space of continuous functions with compact support on X, by $\mathcal{B}(X)$ the Borel σ-algebra of X and by $M(X)$ the Banach space of complex measures on $\mathcal{B}(X)$.

By *lcsc group* G we mean a locally compact second countable Hausdorff topological group. We denote by μ_G a right invariant Haar measure and by Δ_G its modular function.

Let A be an abelian lcsc group. We denote by $\hat{A}$ the dual group of A. Let $\mathcal{F}$ be the Fourier transform on $M(\hat{A})$ defined as

$$(\mathcal{F}\nu)(a) = \int_{\hat{A}} \overline{\langle x, a\rangle} d\nu(x) \qquad a \in A,\ \nu \in M(\hat{A})$$

The same symbol $\mathcal{F}$ denotes the Fourier-Plancherel operator on $L^2(\hat{A}, \mu_{\hat{A}})$. We choose the Haar measure on $\hat{A}$ in such a way that $\mathcal{F}$ is unitary from $L^2(\hat{A}, \mu_{\hat{A}})$ onto $L^2(A, \mu_A)$.

Let G be a lcsc group and X be a lcsc continuous G-space. We denote the action of $g \in G$ on $x \in X$ by $g[x]$, the G-orbit of $x \in X$ by $G[x]$ and the stability subgroup of G at x by G_x.

We recall the following standard results about transitive G-spaces.

Lemma 1. *Let G be a lcsc topological group and $X = G[x_0]$ a transitive G-space.*

1. *There exists a regular section based on* x_0*,* i.e. *a measurable map* q *from* X *to* G *such that*

$$\begin{aligned} q(x_0) &= e \\ q(x)[x_0] &= x \qquad \forall x \in X \end{aligned}$$

   and, for any compact set K *in* G*, the set* $\{q(g[x_0]) \ : \ g \in K\}$ *has compact closure in* G*.*
2. *There exists a strongly quasi-invariant measure* ν *on* X*,* i.e. *a Radon measure* ν *such that*

$$\nu(g[E]) = \int_E \lambda_\nu(g,y)d\nu(y) \qquad g \in G, \ E \in \mathcal{B}(X),$$

   where λ_ν *is the cocycle of* ν*,* i.e. *it is a continuous function from* $G \times X$ *to* $]0,\infty[$ *satisfying*

$$\begin{aligned} \lambda_\nu(g_1g_2,y) &= \lambda_\nu(g_1,g_2[y])\lambda_\nu(g_2,y) \qquad g_1,g_2 \in G, y \in X \\ \lambda_\nu(h,x) &= \frac{\Delta_{G_x}(h)}{\Delta_G(h)} \qquad h \in G_x. \end{aligned}$$

3. *Two strongly quasi-invariant measures on* X *are mutually absolutely continuous and the corresponding density is a continuous positive function.*

The following lemma is a particular case of the Mackey-Bruhat formula, which allows us to compute the integrals on G as integrals on $G/H \times H$ where H is a closed subgroup of G, see [3] for an elementary proof based on the fact that G is a second countable space.

Lemma 2. *Let* G *be a lcsc group,* $X = G[x_0]$ *be a transitive* G*-space and* H *be the stability subgroup at* x_0*. Given a a strongly quasi invariant measure* ν *and a regular section* q *based on* x_0*, define* $\beta : X \times H \to G$ *as the map*

$$\beta(x,h) = q(x)h \qquad x \in X, \ h \in H.$$

Then the map β *is an isomorphism of measurable spaces and the image measure of*

$$\frac{1}{\Delta_G(q(x))\lambda_\nu(q(x),x_0)}\nu(x) \otimes \mu_H(h),$$

under the map β*, is a Haar measure of* G*.*

By *Hilbert space* we mean a complex separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, linear in the first argument. We use the word *representation* to mean a continuous unitary representation of G acting in a Hilbert space $\mathcal{H}$. If a representation U is square integrable, we denote by K_U the corresponding *formal degree.*

Finally, we recall the notion of *square-integrability modulo a coset space* (see [1] and references therein).

Let G be a lcsc group, X a transitive G-space and U an irreducible representation of G acting in a Hilbert space $\mathcal{H}$. The representation U is said to be *square integrable modulo* X if there exist n nonzero vectors

$$\psi_1, \dots, \psi_n \in \mathcal{H}$$

and a section q from X to G such that

$$\Gamma : X \times \{1, \dots, n\} \ni (x, i) \mapsto U_{q(x)}\psi_i \in \mathcal{H} \tag{5}$$

is a frame over the space $(X \times \{1, \dots, n\}, \mu \otimes \mu_c)$, where μ is a strongly quasi-invariant measure on X and μ_c the counting measure on $\{1, \dots, n\}$.

3. Frames on a direct product

Let H be a lcsc group and A an abelian lcsc group such that H acts continuously on the dual group $\hat{A}$ of A. Let M be the direct topological product $M = A \times H$, endowed with the measure $\mu^M = \mu_A \otimes \mu_H$.

Let $x_0 \in \hat{A}$ and $X = H[x_0]$ be the corresponding orbit. We assume that X is locally closed in $\hat{A}$. Fix a strongly quasi-invariant measure ν on X with cocycle λ and a regular section q from X to H based on x_0.

Let m be an irreducible representation of the stability subgroup H_0 at x_0. Define μ_X as the restriction of $\mu_{\hat{A}}$ to X (since X is locally closed μ_X is a Radon measure) and (P, V) be the imprimitivity system based on $\hat{A}$ for the group H acting in $\mathcal{H} = L^2(X, \nu, \mathcal{K})$ as

$$(P_E\phi)(x) = \chi_{E\cap X}(x)\phi(x) \tag{6}$$

$$(U_a\phi)(x) = \langle x, a\rangle\, \phi(x) \tag{7}$$

$$(V_h\phi)(x) = (\lambda(h^{-1}, x))^{1/2} m(q(x)^{-1}hq(h^{-1}[x]))\phi(h^{-1}[x]) \tag{8}$$

where $x \in X$, $a \in A$, $h \in H$, $E \in \mathcal{B}(\hat{A})$ and $\phi \in L^2(X, \nu, \mathcal{K})$.

Let $\psi \in \mathcal{H}$ and define $\mathbb{F}_\psi$ as the map

$$M \ni (a, h) \mapsto V_{h^{-1}}U_{-a}\psi \in \mathcal{H},$$

and, for all $\phi \in \mathcal{H}$, the map $c_{\phi,\psi}$

$$M \ni (a, h) \mapsto \langle \phi, V_{h^{-1}}U_{-a}\psi\rangle_{\mathcal{H}} \in \mathbb{C}. \tag{9}$$

As in the case of square integrable representations, we say that ψ is an *admissible vector* for (P, V) if the map $\mathbb{F}_\psi$ is a frame over (M, μ^M), *i.e* if the following conditions are satisfied:

1. for all $\phi \in \mathcal{H}$, $c_{\phi,\psi} \in L^2(M, \mu^M)$;
2. there exist two positive numbers α and β such that

$$\alpha \|\phi\|^2_{\mathcal{H}} \leq \|c_{\phi,\psi}\|^2_{L^2(M,\mu^M)} \leq \beta \|\phi\|^2_{\mathcal{H}}, \qquad \forall\phi \in \mathcal{H}.$$

We are now in position to state the main result of the paper. We give necessary and sufficient conditions on the imprimitivity system in order that the set of admissible vectors for (P, V) is not void. Moreover, we show that the corresponding frame $\mathbb{F}_\psi$ is tight, *i.e.* $\alpha = \beta$, and its frame bound α can be computed by means of a sort of *orthogonality relation*, as Eq. (2).

Theorem 1. *There is $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame in $\mathcal{H}$ over (M, μ^M) if and only if ν and μ_X are not disjoint and m is a square-integrable representation of H_0.*

This theorem is a consequence of the following proposition. Let

$$\nu = \nu_s + f\mu_X \tag{10}$$

be the decomposition of ν with respect to μ_X given by the Lebesgue- Radon-Nikodym theorem where f is non-negative and measurable. Defined $\gamma : X \to]0, \infty[$ as

$$\gamma(x) = \frac{f(x)}{\Delta_H(q(x))\lambda(q(x), x_0)},$$

the following result holds.

Proposition 1. *Let $\psi, \phi \in \mathcal{H}$, $\psi, \phi \neq 0$. The function $c_{\phi,\psi}$ is in $L^2(M, \mu^M)$ if and only if the following conditions hold:*

a) *m is a square-integrable representation of H_0 (we denote by K_m its formal degree);*
b) *$\psi(x) = 0$ for ν_s-almost all $x \in X$;*
c) *$\psi(x) \in \mathrm{Dom}\, K_m^{-1/2}$ for ν-almost all $x \in X$;*
d) *the function*

$$X \ni x \mapsto \gamma(x) \left\| K_m^{-1/2}\psi(x) \right\|_{\mathcal{K}}^2$$

is in $L^1(X, \nu)$.

In this case, we have that

$$\|c_{\phi,\psi}\|^2_{L^2(G,\mu_G)} = \|\phi\|^2_{\mathcal{H}} \int_X \gamma(x) \left\| K_m^{-1/2}\psi(x) \right\|_{\mathcal{K}}^2 d\nu(x). \tag{11}$$

Remark 1. We observe that

1. the measure ν defines uniquely the density f, hence also γ, up to a ν-negligible set;
2. the function γ is invariant, up to a positive constant factor and a ν-negligible set, if we change ν with another quasi-invariant measure;
3. the condition that ν and μ_X are not disjoint implies that $\mu_X(X) > 0$.

Proof. Let $\phi, \psi \in \mathcal{H}$, $\psi, \phi \neq 0$. Then, for all $(a, h) \in M$,

$$\begin{aligned} c_{\phi,\psi}(a,h) &= \int_X \langle x, a \rangle \, \langle (V_h\phi)(x), \psi(x) \rangle_{\mathcal{K}} \, d\nu(x) \\ &= (\mathcal{F}\nu_h)(-a), \end{aligned}$$

where ν_h is the canonical extension to $\hat{A}$ of the complex measure on X having $\langle (V_h\phi)(x), \psi(x) \rangle_{\mathcal{K}}$ as density with respect to ν. Applying Fubini theorem and taking into account that A is unimodular, the condition

$$c_{\phi,\psi} \in L^2(M, \mu^M)$$

is equivalent to the following two conditions

– For μ_H-almost all $h \in H$ the function
$$A \ni a \mapsto (\mathcal{F}\nu_h)(a)$$
is in $L^2(A, \mu_A)$;

– the function
$$H \ni h \mapsto \int_A |(\mathcal{F}\nu_h)(a)|^2 \, d\mu_A(a)$$
is in $L^1(H, \mu_H)$.

Using a standard result on Fourier transform (see, for example, Theorem 31.33 of [7]) and the fact that the Fourier-Plancherel operator is unitary, the two conditions above turn out to be equivalent to

– For μ_H-almost all $h \in H$, there is $l_h \in L^2(\hat{A}, \mu_{\hat{A}})$ such that
$$l_h \mu_{\hat{A}} = \nu_h$$

– the function
$$H \ni h \mapsto \int_{\hat{A}} |l_h(x)|^2 \, d\mu_{\hat{A}}(x)$$
is in $L^1(H, \mu_H)$.

Since $\nu_h(\hat{A} \setminus X) = 0$, $l_h = 0$ for $\mu_{\hat{A}}$-almost any $x \notin X$ and
$$l_h(x)\mu_X = \langle (V_h\phi)(x), \psi(x) \rangle \, (\nu_s + f(x)\mu_X).$$
This last equation shows that the above conditions are in fact equivalent to

a) For μ_H-almost all $h \in H$ and ν_s-almost all $x \in X$,
$$\langle (V_h\phi)(x), \psi(x) \rangle = 0,$$

b) For μ_H-almost all $h \in H$ the function
$$X \ni x \mapsto \langle (V_h\phi)(x), \psi(x) \rangle \, f(x)$$
is in $L^2(X, \mu_X)$;

c) the function
$$H \ni h \mapsto \int_X |\langle (V_h\phi)(x), \psi(x) \rangle_{\mathcal{K}} \, f(x)|^2 \, d\mu_X(x)$$
is in $L^1(H, \mu_H)$.

Denoted by $\mathcal{P}$ the orthogonal projection on the singular part ν_s, we claim that the first of the above conditions is equivalent to the fact that $\mathcal{P}\psi = 0$. Indeed, the first condition implies that for μ_H-almost all $h \in H$, for all $E \in \mathcal{B}(\hat{A})$ and for ν-almost all $x \in X$
$$\langle (P_E V_h \phi)(x), (\mathcal{P}\psi)(x) \rangle_{\mathcal{K}} = 0,$$
hence, by integration over X and taking into account the fact that the map $h \mapsto V_h$ is continuous with respect to the strong operator topology, one has that, for all $h \in H$ and for all $E \in \mathcal{B}(\hat{A})$
$$\langle P_E V_h \phi, \mathcal{P}\psi \rangle_{\mathcal{H}} = 0.$$

Since $\phi \neq 0$ and (P, V) is irreducible, it follows that $\mathcal{P}\psi = 0$. The converse implication is evident. Taking into account Eq. (10), one obtains the following conditions

– $\mathcal{P}\psi = 0$,
– for μ_H-almost all $h \in H$ the function
$$X \ni x \mapsto \langle (V_h\phi)(x), \psi(x)\rangle \sqrt{f(x)}$$
is in $L^2(X, \nu)$;
– the function
$$H \ni h \mapsto \int_X |\langle (V_h\phi)(x), \psi(x)\rangle_{\mathcal{K}}|^2 \, d\nu(x) f(x)$$
is in $L^1(H, \mu_H)$.

Fixed $h \in H$, performing the change of variables $x \to h[x]$, these conditions are equivalent to

– $\mathcal{P}\psi = 0$,
– for μ_H-almost all $h \in H$ the function
$$X \ni x \mapsto \left|\left\langle \phi(x), m(q(x)^{-1}h^{-1}q(h[x]))\psi(h[x])\right\rangle_{\mathcal{K}}\right|^2 f(h[x])$$
is in $L^1(X, \nu)$;
– the function
$$H \ni h \mapsto \int_X \left|\left\langle \phi(x), m(q(x)^{-1}h^{-1}q(h[x]))\psi(h[x])\right\rangle_{\mathcal{K}}\right|^2 f(h[x]) \, d\nu(x)$$
is in $L^1(H, \mu_H)$.

Again by Fubini theorem and, fixed $x \in X$, performing the change of variables $h \to hq(x)^{-1}$, the above conditions turn out to be equivalent to

– $\mathcal{P}\psi = 0$,
– for ν-almost all $x \in X$ the function
$$H \ni h \mapsto \left|\left\langle \phi(x), m(h^{-1}q(h[x_0]))\psi(h[x_0])\right\rangle_{\mathcal{K}}\right|^2 f(h[x_0])$$
is in $L^1(H, \mu_H)$;
– the function
$$X \ni x \mapsto \int_H \left|\left\langle \phi(x), m(h^{-1}q(h[x_0]))\psi(h[x_0])\right\rangle_{\mathcal{K}}\right|^2 f(h[x_0]) \, d\mu_H(h)$$
is in $L^1(X, \nu)$.

Finally, using Lemma 2 to compute the integral in the variable h and Fubini theorem, the last conditions are equivalent to

– $\mathcal{P}\psi = 0$,
– for ν-almost all $x, y \in X$ the function
$$H_0 \ni s \mapsto \gamma(y) \left|\left\langle \phi(x), m(s^{-1})\psi(y)\right\rangle_{\mathcal{K}}\right|^2$$
is in $L^1(H_0, \mu_{H_0})$;

– For ν-almost all $x \in X$ the function

$$X \ni y \mapsto \gamma(y) \int_{H_0} \left| \langle \phi(x), m(s^{-1})\psi(y) \rangle_{\mathcal{K}} \right|^2 d\mu_{H_0}(s)$$

is in $L^1(X,\nu)$.

– the function

$$X \ni x \mapsto \int_X \gamma(y) \int_{H_0} \left| \langle \phi(x), m(s^{-1})\psi(y) \rangle_{\mathcal{K}} \right|^2 d\mu_{H_0}(s)\, d\nu(y)$$

is in $L^1(X,\nu)$.

The equivalence of these last four conditions to the ones contained in the statement of the proposition are now consequence of the properties of the formal degree operator, see Th. 3 of [5]. Equation (11) follows from Fubini theorem. □

Proof of theorem. "only if": By definition of frame, there are $\psi, \phi \in \mathcal{H}$, $\psi, \phi \neq 0$, such that $c_{\phi,\psi} \in L^2(M,\mu^M)$, applying Prop. 1, one has that $\psi(x) = 0$ ν_s-almost all $x \in X$ and m is square-integrable. The first condition and the fact that $\psi \neq 0$ imply that the measures ν and μ_X are not disjoint.

"if": Since the measures ν and μ_X are not disjoint, there is a set $C \in \mathcal{B}(X)$ such that

$$\begin{aligned} \nu(C) &> 0 \\ \nu_s(C) &= 0 \\ f(x) &> 0 \quad \forall x \in C \end{aligned}$$

Since ν is a regular measure, we can always assume that C is compact. Moreover, due to the fact that m is square-integrable, there is $v \in \mathcal{K}$, $v \neq 0$, such that $v \in \operatorname{Dom} K_m^{-\frac{1}{2}}$. Define $\psi : X \to \mathcal{K}$ as

$$\psi(x) = \begin{cases} \frac{v}{\sqrt{f(x)}} & x \in C \\ 0 & x \notin C \end{cases}$$

Obviously $\psi \in \mathcal{H}$, $\psi \neq 0$. Taking into account that $\frac{1}{\Delta_G(q(x))\lambda_\nu(q(x),x_0)}$ is bounded on the compact sets and the definition of γ, one has that the four conditions of Prop. 1 are satisfied for all $\phi \in \mathcal{H}$, $\phi \neq 0$. Hence, $c_{\phi,\psi}$ is in $L^2(M,\mu^M)$ for all $\phi \in \mathcal{H}$. □

Corollary 1. *With the notations of Th. 1, there is $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame in $\mathcal{H}$ over (M,μ^M) if and only if there are $\psi, \phi \in \mathcal{H}$, $\psi, \phi \neq 0$, such that $c_{\phi,\psi}$ is in $L^2(M,\mu^M)$.*

The admissible vectors ψ are the ones satisfying

$$\begin{aligned} \psi &\neq 0 \\ \psi(x) &= 0 \quad \nu_s\text{-almost all } x \in X \\ \int_X \gamma(x) \left\| K_m^{-1/2}\psi(x) \right\|_{\mathcal{K}}^2 d\nu(x) &< \infty, \end{aligned}$$

and the corresponding frame $\mathbb{F}_\psi$ is tight with frame bound

$$\alpha = \int_X \gamma(x) \left\| K_m^{-1/2} \psi(x) \right\|_{\mathcal{K}}^2 d\nu(x),$$

where K_m is the formal degree of m .

Proof. It follows easily from Proposition 1. □

We are able neither to prove that, for the frames $\mathbb{F}_\psi$ given by Th. 1, the singular part ν_s of the strongly quasi-invariant measure ν is zero nor to find an example of such frames such that $\nu_s \neq 0$. Nevertheless, there are two partial results in this direction.

The first one applies in the case that H and A are Lie groups.

Corollary 2. *Let H and A be Lie groups such that the action of H on $\hat{A}$ is smooth and the orbits are locally closed. There is $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame in $\mathcal{H}$ over (M, μ^M) if and only if the orbit X is open in $\hat{A}$ and m is a square-integrable representation of H_0.*

In particular, the restriction μ_X of the Haar measure $\mu_{\hat{A}}$ to X is a strongly quasi-invariant measure, so in Eq. (10) *$\nu_s = 0$ and f can be chosen continuous and strictly positive.*

The proof of the corollary is based on the following standard results on the Lebesgue measures. We recall that, if M is a manifold of dimension n, a Radon measure μ on M is said to be a *Lebesgue measure* if for each chart (U, ϕ) the image under ϕ of the induced measure μ_U is equivalent to the restriction of the Lebesgue measure of $\mathbb{R}^n$ to $\phi(U)$ with a C^∞ density. Then

1. There exist Lebesgue measures and two of them are equivalent (see 16.22.2 of [4]).
2. If N is a submanifold with dimension strictly lower than n and it is locally closed in M, then N is negligible with respect to any Lebesgue measure (see 16.22 of [4]).
3. If G is a Lie group, then the Haar measure of G is a Lebesgue measure (see 19.6 of [4]).
4. If G is a Lie group acting on M and the action is transitive and smooth, then a Radon measure μ is G-quasi-invariant if and only if it is equivalent to a Lebesgue measure (see, for example, [11]).

Proof. As a consequence of Th. 1, there is $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame in $\mathcal{H}$ over (M, μ^M) if and only if m is a square-integrable representation of H_0 and there is a measurable subset C of X such that $\nu(C) > 0$ and $\mu_{\hat{A}}(C) > 0$. So, we can assume, without loss of generality, that $\mu_{\hat{A}}(X) > 0$. Under this assumption, we show that the restriction μ_X of $\mu_{\hat{A}}$ to X is quasi-invariant. Indeed, by means of a standard result of Lie groups, the orbit X is a (locally closed) submanifold and, since $\mu_{\hat{A}}$ is a Lebesgue measure and $\mu(X) > 0$, then $\dim X = \dim \hat{A}$, i.e. X is open in $\hat{A}$, and μ_X is a Lebesgue measure (with respect to the manifold X). By the results referred above, the claim is now evident.

The statement of the corollary is now clear. □

The same conclusion can be proved if we assume that the action of H on $\hat{A}$ preserves the composition law of $\hat{A}$.

Corollary 3. *Let H and A be lcsc groups such that the action of H on $\hat{A}$ satisfies*

$$h[x_1 + x_2] = h[x_1] + h[x_2] \qquad h \in H,\ x_1, x_2 \in \hat{A}, \tag{12}$$

and the corresponding orbits are locally closed. There is $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame in $\mathcal{H}$ over (M, μ^M) if and only if $\mu_{\hat{A}}(X) > 0$ and m is a square-integrable representation of H_0.

In particular, the restriction μ_X of the Haar measure $\mu_{\hat{A}}$ to X is a strongly quasi-invariant measure, so in Eq. (10) $\nu_s = 0$ and f can be chosen continuous and strictly positive.

The proof is based on the following Lemma, see, for example, [3], for the proof.

Lemma 3. *For any $h \in H$ and $E \in \mathcal{B}(\hat{A})$ we have*

$$\mu_A(h[E]) = \rho(h^{-1})\mu_A(E)$$

where $\rho : H \to (0, \infty)$ is a continuous group homomorphism.

Proof of the corollary. Arguing as in the proof of the previous corollary, we can assume, without loss of generality, that $\mu_{\hat{A}}(X) > 0$. Hence, due to the above Lemma, the measure μ_X is a strongly quasi-invariant measure on X. The thesis in now evident. □

We observe that, if Eq. (12) holds, the frame (9) is associated in a natural way with a square-integrable representation. Indeed, since the action of H on $\hat{A}$ satisfies (12), by duality, H acts also on A in such a way that

$$h[a_1 + a_2] = h[a_1] + h[a_2] \qquad h \in H,\ a_1, a_2 \in A,$$

and the set M acquires a structure of a lcsc group with respect to the group law

$$(a_1, h_1)(a_2, h_2) = (a_1 + h_1[a_2], h_1 h_2).$$

To stress the structure of group, we denote M by G in the following.

By definition, the group G is the semidirect product of A and H, and the measure $\mu_A \otimes \mu_H$ is a right invariant Haar measure. Moreover there is a one-to-one correspondence between the irreducible imprimitivity systems (P, V) and the irreducible representations U of G, explicitly given by

$$(P, V) \mapsto U_{(a,h)} := U_a V_h,$$

where $U_a = \int_{\hat{A}} \langle x, a \rangle \; dP(x) \quad a \in A$. As a consequence, the fact that there is $\psi \in \mathcal{H}$ such that the map $\mathbb{F}_\psi$ given by Eq. (9) is a frame is precisely the fact that U is a square-integrable representation of G and, hence, Corollary 3 characterises completely the square-integrable representations of groups that are semidirect products with an abelian normal factor, [9], [3].

4. Frames on a quotient space

In this section, we assume that A be an abelian Lie group and H a Lie group acting on $\hat{A}$, the dual group of A.

Fix $x_o \in \hat{A}$ and an irreducible representation m of the stability subgroup H_0 at x_0. Let $X = H[x_o]$ be the orbit of x_0 (as usual we assume that X is locally closed) and $\mathcal{K}$ the Hilbert space where m acts. We suppose that

1. the stability subgroup H_0 is compact;
2. the orbit $X := H[x_0]$ of x_0 has positive measure with respect to the Haar measure $\mu_{\hat{A}}$ of $\hat{A}$.

From these assumptions one has the following properties.

1. The stability subgroup H_0 is unimodular and we can normalise the Haar measure μ_{H_0} in such a way that $\mu_{H_0}(H_0) = 1$.
2. The orbit X admits a H-invariant measure ν, which is equivalent to the restriction μ_X of $\mu_{\hat{A}}$ to X. Let f be the continuous and positive function from X to $\mathbb{R}$ such that
$$\nu = f\mu_X,$$
so that the cocycle λ of μ_X is
$$\lambda(h,x) = \frac{f(x)}{f(h[x])} \qquad x \in X,\ h \in H,$$
(since ν is invariant the cocycle of ν is the identity.)
3. The Hilbert space $\mathcal{K}$, in which the representation m acts, is finite-dimensional, so we can assume that $\mathcal{K} = \mathbb{C}^n$. Moreover, the representation m is square-integrable and the corresponding operator of formal degree is proportional to the identity.

Let (P,V) be the imprimitivity system for H based on $\hat{A}$ and acting on $\mathcal{H} = L^2(X,\nu,\mathbb{C}^n)$ whose equivalence class corresponds to $(X,[m])$. If q is a regular section from X to H based on x_0, then the system (P,V) is explicitly given by Eqs. (6-8).

Choose $\Psi \in \mathcal{K}(X)$, $\Psi \neq 0$, and define ψ from X to $\mathbb{C}$ as

$$\psi(x) = \int_{H_0} \sqrt{\frac{f(s[x])}{f(x)}}\Psi(s[x])\,d\mu_{H_0}(s), \tag{13}$$

the integral being finite since Ψ has compact support, f is continuous and H_o is compact. Taking into account that H_0 is unimodular, for all $s' \in H_0$ and $x \in X$

$$\psi(s'[x]) = \sqrt{\lambda(s',x)}\psi(x). \tag{14}$$

Indeed, we have:

$$\begin{aligned}
\psi(s'[x]) &= \int_{H_0} \sqrt{\frac{f(ss'[x])}{f(s'[x])}}\Psi(ss'[x])\, d\mu_{H_0}(s) \\
&= (s \mapsto ss'^{-1}) \\
&= \int_{H_0} \sqrt{\frac{f(s[x])}{f(s'[x])}}\Psi(s[x])\, d\mu_{H_0}(s) \\
&= \sqrt{\frac{f(x)}{f(s'[x])}} \int_{H_0} \sqrt{\frac{f(s[x])}{f(x)}}\Psi(s[x])\, d\mu_{H_0}(s) \\
&= \sqrt{\lambda(s',x)}\psi(x).
\end{aligned}$$

Moreover, since H_0 is compact and $\Psi \in \mathcal{K}(X)$, still $\psi \in \mathcal{K}(X)$ and, recalling that the cocycle of ν is equal to 1, one has that

$$\alpha := \int_X \frac{f(x)}{\Delta_H(q(x))}|\psi(x)|^2\, d\nu(x) < \infty. \tag{15}$$

Finally, let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, J be the set of the first n numbers and μ^c the corresponding counting measure. Then:

Corollary 4. *With the above notations, the map*

$$A \times X \times J \ni (a, x, i) \mapsto V_{q(x)} U_a(\psi e_i) \in \mathcal{H}$$

is a tight frame in $\mathcal{H}$, over $(A \times X \times J,\ \mu_{\hat{A}} \otimes \nu \otimes \mu^c)$, with frame bound α.

Proof. Let V^0 and U^0 be the representations of H and A corresponding to the choice of the trivial representation m^0 of H_0 ($\mathcal{K}^0 = \mathbb{C}$). Since ψ, defined as above, satisfies Eq. (15), then

$$(a, h) \mapsto V^0_{h^{-1}} U^0_{-a}\psi \tag{16}$$

is a tight frame in $L^2(X, \nu)$ over $(A \times H, \mu_{\hat{A}} \otimes \mu_H)$, with frame bound α. Now, let ϕ be any vector in $\mathcal{H}$. Then, by the fact that (16) is a frame, it follows that:

$$
\begin{aligned}
\alpha \|\phi\|^2_{\mathcal{H}} &= \int_{A\times H} \left\langle \|\phi(x)\|_{\mathbb{C}^n}, V^0_{h^{-1}} U^0_{-a}\psi \right\rangle_{L^2(X,\nu)} d\mu_A(a) \otimes d\mu_H(h) \\
&= \int_H \int_A \left| \int_X \|\phi(x)\|_{\mathbb{C}^n} \overline{(V^0_{h^{-1}} U^0_{-a}\psi)(x)}\, d\nu(x) \right|^2 d\mu_A(a)\, d\mu_H(h) \\
&= (x \mapsto h^{-1}[x]) \\
&= \int_H \int_A \left| \int_X \left\|\phi(h^{-1}[x])\right\|_{\mathbb{C}^n} \overline{(U^0_{-a}\psi)(x)}\, d\nu(x) \right|^2 d\mu_A(a)\, d\mu_H(h) \\
&= (\text{Fourier transform and } \nu = f\mu_X) \\
&= \int_H \int_X \left\|\phi(h^{-1}[x])\right\|^2_{\mathbb{C}^n} |\psi(x)|^2 f(x)\, d\nu(x)\, d\mu_H(h) \\
&= (h \mapsto h^{-1}) \\
&= \int_H \int_X \|\phi(h[x])\|^2_{\mathbb{C}^n} |\psi(x)|^2 f(x)\, d\nu(x)\, d\mu_H(h).
\end{aligned}
$$

Now, using the Mackey-Bruhat formula in order to compute the integral on H as an integral on $X \times H_0$, we have:

$$
\begin{aligned}
\alpha\|\phi\|^2_{\mathcal{H}} &= \int_X \int_{H_0} \int_X \|\phi((q(y)s)[x])\|^2_{\mathbb{C}^n} |\psi(x)|^2 f(x)\, d\nu(x)\, d\mu_{H_0}(s)\, d\nu(y) \\
&= (x \mapsto s^{-1}[x] \text{ and unitarity of } m) \\
&= \int_X \int_{H_0} \int_X \left\|(V_{q(y)^{-1}}\phi)(x)\right\|^2_{\mathbb{C}^n} \left|\psi(s^{-1}[x])\right|^2 f(s^{-1}[x])\, d\nu(x)\, d\mu_{H_0}(s)\, d\nu(y) \\
&= (\text{Eq. (14) }) \\
&= \int_X \int_{H_0} \int_X \left\|(V_{q(y)^{-1}}\phi)(x)\right\|^2_{\mathbb{C}^n} |\psi(x)|^2 f(x)\, d\nu(x)\, d\mu_{H_0}(s)\, d\nu(y).
\end{aligned}
$$

At this point, recalling that we have set $\mu_{H_0}(H_0) = 1$, we can perform the integral over H_0; thus, we obtain that

$$
\alpha\|\phi\|^2_{\mathcal{H}} = \int_X \int_X \|(V_{q(y)^{-1}}\phi)(x)\|^2_{\mathbb{C}^n} |\psi(s^{-1}[x])|^2 f(x)\, d\nu(x)\, d\nu(y).
$$

Then, since $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{C}^n$, we find:

$$
\begin{aligned}
\alpha\|\phi\|^2_{\mathcal{H}} &= \int_X \int_X \sum_i \left| \left\langle (V_{q(y)^{-1}}\phi)(x), \psi(x)e_i \right\rangle_{\mathbb{C}^n} \right|^2 f(x)\, d\nu(x)\, d\nu(y) \\
&= (\text{Fourier transform and } \nu = f\mu_X) \\
&= \int_X \int_A \sum_i \left| \left\langle (V_{q(y)^{-1}}\phi), U_{-a}(\psi e_i) \right\rangle_{\mathcal{H}} \right|^2 d\mu_A(a)\, d\nu(y). \\
&= (a \mapsto -a) \\
&= \int_X \int_A \sum_i \left| \left\langle \phi, V_{q(y)} U_a(\psi e_i) \right\rangle_{\mathcal{H}} \right|^2 d\mu_A(a)\, d\nu(y).
\end{aligned}
$$

The thesis is now evident. □

5. Examples

In this section we apply the previous results to some specific groups that are considered in the literature.

5.1. **The causal group.** Let G be the *causal group*, namely the semidirect product

$$G = \mathbb{R}^4 \times' (SO_0(3,1) \times \mathbb{R}_*^+),$$

with $S0_0(3,1)$ is the connected component with the identity of the Lorentz group and $\mathbb{R}_*^+$ the multiplicative group of strictly positive real numbers. This group has been considered in the context of wavelet electrodynamics, [8].

In order to take into account the projective representations of G, we will study the universal covering group of G, *i.e.*

$$G^* = \mathbb{R}^4 \times' (SL(2,\mathbb{C}) \times \mathbb{R}_*^+).$$

The group G^* is the semidirect product of the abelian group $A = \mathbb{R}^4$ and the group $H = SL(2,\mathbb{C}) \times \mathbb{R}_*^+$, so by duality H acts on the dual group $\mathbb{P}^4$ of $\mathbb{R}^4$, preserving the composition law of $\mathbb{P}^4$. Hence, we can consider unitary representations of G^* instead of imprimitivity systems for H based on $\mathbb{P}^4$.

In order to apply Corollary 3, we identify the dual group $\mathbb{P}^4$ of $\mathbb{R}^4$ with $\mathbb{R}^4$ by means of the pairing

$$\langle p, a\rangle = p_0a_0 - p_1a_1 - p_2a_2 - p_3a_3.$$

The Haar measure on $\mathbb{P}^4$ is the Lebesgue measure dp and the action of H on $\mathbb{P}^4$ is

$$(h,d)[p] = \frac{1}{d}L(h)a \qquad (h,d) \in SL(2,\mathbb{C}) \times \mathbb{R}_*^+,\ p \in \mathbb{P}^4,$$

where $L : SL(2,\mathbb{C}) \to SO_0(3,1)$ is the covering homomorphism. The corresponding orbits of $SL(2,\mathbb{C}) \times \mathbb{R}_+^*$ can be classified in the following way:

1. two "massive" orbits O^+, O^- defined by
$$O^\pm = G^*[p^\pm] = \{p \in \mathbb{R}^4 \ : \ \pm p_0 > 0,\ \|p\| > 0\},$$
where $p^\pm = (\pm 1, 0, 0, 0)$ and $\|p\| = \sqrt{p_0^2 - p_1^2 - p_2^2 - p_3^2}$;
2. the orbit O defined by
$$O = G^*[(0,0,0,1)] = \{p \in \mathbb{R}^4 \ : \ \|p\| < 0\};$$
3. the singleton orbit $\{(0,0,0,0)\}$.

Since the orbits are locally closed, we can apply Corollary 3 to select the square-integrable representations. The orbits having positive Lebesgue measure are $O^\pm, O$. The stability subgroup at $(0,0,0,1) \in O$ is the euclidean group $E(2)$, which does not admit square-integrable representations, whereas the one associated with $O^{\pm}$ is isomorphic to $SU(2)$. Since $SU(2)$ is compact, every irreducible representation is square-integrable and let

$$\{D^j \ : \ j = 0, \frac{1}{2}, 1, \ldots\} \tag{17}$$

be the canonical maximal set of inequivalent irreducible representations of $SU(2)$ where each D^j acts on $\mathbb{C}^{2j+1}$. Then, a maximal set of inequivalent square-integrable representations of G^* is

$$\{U^{\pm,j} := \mathrm{Ind}_{\mathbb{R}^4 \times' SU(2)}^{G^*}(p^{\pm} D^j) \ : \ j = 0, \frac{1}{2}, 1, \ldots\}.$$

Observing that the restriction ν of the measure $\|p\|^{-4} dp$ to the orbit $O^{\pm}$ is invariant with respect to the action of G^* , for each j the representation $U^{\pm,j}$ acts on the Hilbert space $\mathcal{H}^j := L^2(O^{\pm}, \nu; \mathbb{C}^{2j+1})$ as

$$\big(U^j_{(a,h,d)}\psi\big)(p) = e^{i\langle p,a\rangle} D^j\big(q(p)^{-1}(hd)\, q((hd)^{-1}[p])\big)\, \psi((hd)^{-1}[p])$$

where $\psi \in \mathcal{H}^j$, $(a, h, d) \in G^*$ and q is a regular section from $O^{\pm}$ into $SL(2, \mathbb{C}) \times \mathbb{R}^{\pm}_{*}$. The corresponding set $\mathcal{A}^j$ of admissible vectors is given by Corollary 1. To this aim, notice that ν has density $f(p) = \|p\|^{-4}$ with respect to dp (according to Corollary 3, the singular part ν_s of ν with respect to dp is zero), $SL(2, \mathbb{C}) \times \mathbb{R}^{\pm}_{*}$ is unimodular and the formal degree of the representation D^j is proportional to the identity, then

$$\mathcal{A}^j = \int_{O^{\pm}} \|\psi(p)\|^2_{\mathbb{C}^{2j+1}} \ \|p\|^{-4} \ d\mu(p) < \infty.$$

5.2. **Lorentz invariant frames.** It is well known that the Poincaré group does not admit square integrable representations, so that there are no frames given by Eq. (9), which, obviously, are invariant with respect to the full Poincaré group. Nevertheless we can define frames that are invariant with respect to the Lorentz group by means of the imprimitivity systems for the Lorentz group based on $\mathbb{R}^3$. The action of the Lorentz group on $\mathbb{R}^3$ is defined by identifying, as a manifold, $\mathbb{R}^3$ with with the the *massive* orbit in the momentum space

$$X_m = \{(p_0, \vec{p}) \in \mathbb{R}^4 \ : \ p_0^2 - \vec{p}^2 = m^2, \ p_0 > 0\}.$$

This kind of frames was introduced by means of suitable representations of the Poincaré group that are square-integrable modulo a coset space, see [1] and references therein.

As usual we consider the universal covering group $H = SL(2, \mathbb{C})$ of the connected component $SO_0(3, 1)$ of the Lorentz group. We denote by $\mu_{SL(2,\mathbb{C})}$ a Haar measure of $SL(2, \mathbb{C})$, which is a unimodular group, and by L the covering homomorphism from $SL(2, \mathbb{C})$ onto $SO_0(3, 1)$.

Let A be the abelian Lie group $\mathbb{R}^3$. As usual, we identify the dual group $\mathbb{P}^3$ of A with $\mathbb{R}^3$ by means of the euclidean scalar product. Define the diffeomorphism ϖ_m from $\mathbb{R}^3$ onto X_m as

$$\varpi_m(\vec{p}) = (\sqrt{m^2 + \vec{p}^2}, \vec{p})$$

with inverse

$$\varpi_m^{-1}(p_0, \vec{p}) = \vec{p}.$$

The space $\mathbb{P}^3$ becomes a transitive H-space with respect to the smooth action

$$h[\vec{p}] = \varpi_m^{-1}(L(h)\varpi_m(\vec{p}))$$

with $h \in SL(2,\mathbb{C})$ and $\vec{p} \in \mathbb{P}^3$.

We recall the following facts about $\mathbb{P}^3$ as an H-space.

1. The H-invariant measure on $\mathbb{P}^3$ is
$$\nu_m = \frac{d\vec{p}}{\sqrt{m^2 + \vec{p}^2}},$$
with density $f(\vec{p}) = \frac{1}{\sqrt{m^2+\vec{p}^2}}$ with respect to the Lebesgue measure $d\vec{p}$ of $\mathbb{P}^3$.
2. The stability subgroup at the origin $\vec{0}$ is $H_0 = SU(2)$, which is compact.
3. The set $\{D^j\}$ given by Eq. (17) is a set of inequivalent representations of $SU(2)$.

Fixed $j = 0, \frac{1}{2}, 1, \ldots$, each couple $(\mathbb{P}^3, [D^j])$ defines two classes of frames in the Hilbert space $\mathcal{H}^j = L^2(\mathbb{P}^3, \nu_m, \mathbb{C}^{2j+1})$. Indeed, let

$$\begin{aligned}(U^j_{\vec{a}}\phi)(\vec{p}) &= e^{i\vec{a}\cdot\vec{p}}\phi(\vec{p}) \\ (V^j_h\phi)(\vec{p}) &= D^j\left(q(\vec{p})^{-1}hq(h^{-1}\vec{p})\right)\phi(h^{-1}\vec{p}),\end{aligned}$$

where $\vec{p} \in \mathbb{P}^3$, $\phi \in \mathcal{H}^j$, $h \in H$, $\vec{a} \in A$ and q is any regular section from $\mathbb{P}^3$ to H.

We observe that the formal degree of the representations D^j is proportional to the identity and that

$$\gamma(\vec{p}) = f(\vec{p}) = \frac{1}{\sqrt{m^2 + \vec{p}^2}}$$

since H is unimodular and ν_m is invariant. Then, by means of Corollary 2 and Corollary1, if $\psi \in \mathcal{A}^j$ where

$$\mathcal{A}^j = \{\psi \in \mathcal{H} \;:\; \psi \neq 0,\; \frac{\psi(\vec{p})}{(m^2 + \vec{p}^2)^{\frac{1}{2}}} \in L^2(\mathbb{P}^3, d\vec{p}, \mathbb{C}^{2j+1})\},$$

then the map

$$\mathbb{R}^3 \times SL(2,\mathbb{C}) \ni (a, h,) \mapsto V_h^{-1}U_a^{-1}\psi \in \mathcal{H}^j \tag{18}$$

is a tight frame over $(\mathbb{R}^3 \times SL(2,\mathbb{C}), d\vec{a} \otimes \mu_{SL(2,\mathbb{C})})$ with frame bound

$$\alpha = \int_{\mathbb{P}^3} \frac{\|\psi(\vec{p})\|^2_{\mathbb{C}^{2j+1}}}{(m^2 + \vec{p}^2)}\, d\vec{p}.$$

Moreover, we notice that the measure ν_m is an H-invariant measure on $\mathbb{P}^3$ with density

$$f(\vec{p}) = \frac{1}{\sqrt{m^2 + \vec{p}^2}}$$

with respect to the Lebesgue measure $d\vec{p}$ and we define J as the set of the first n numbers and μ^c the corresponding counting measure.

Let now $(e_i)_{i=1}^{2j+1}$ be the standard basis of $\mathbb{C}^{2j+1}$ and $\Psi \in \mathcal{K}(\mathbb{P}^3)$. Define ψ by means of Eq. (13), then, as a consequence of Corollary 4, the map

$$\mathbb{R}^3 \times \mathbb{P}^3 \times J \ni (\vec{a}, \vec{p}, i) \mapsto V^j_{q_m(\vec{p})} U^j_{\vec{a}}(\psi e_i) \in \mathcal{H}^j \tag{19}$$

is a tight frame in $L^2(\mathbb{P}^3, \nu_m, \mathbb{C}^{2j+1})$ over $(A \times \mathbb{P}^3 \times J, d\vec{a} \otimes \nu_m \otimes \mu^c)$ with frame bound

$$\alpha = \int_{\mathbb{P}^3} \frac{\|\psi(\vec{p})\|^2_{\mathbb{C}^{2j+1}}}{(m^2 + \vec{p}^2)}\, d\vec{p}.$$

These classes of frames were introduced in the paper [1] by the use of an irreducible representation W of the universal covering group G of the connected component of the Poincaré group such that W be *square integrable modulo a section*. We now show the relationship between the two constructions.

Consider first the frame (18) and define $\mathbb{T}$ as the subgroup of the time translations, *i.e.*

$$\mathbb{T} = \{(p_0, \vec{0}, I) \in G \ : \ p_0 \in \mathbb{R}\}.$$

Then, the manifold $A \times H$ is diffeomorphic to the space of left cosets $G/\mathbb{T}$ and it becomes a transitive G-space. One has that the map $c : A \times H \to G$ is

$$(\vec{a}, h) \mapsto (h[(0, \vec{a})], h)$$

is a smooth section and the measure $d\vec{a} \otimes \mu_{SL(2,\mathbb{C})}$ is G-invariant.

Let $G_0 = \mathbb{R}^4 \times' SU(2)$ and χ_m be the character of $\mathbb{R}^4$

$$\chi_m(a_0, \vec{a}) = e^{ia_0 m}, \qquad (a, \vec{a})\mathbb{R}^4,$$

then $\chi \otimes D^j$ is an irreducible representation of G_0 and we can consider the representation $W^{m,j}$ unitarily induced from G_0 to G by $\chi_m \otimes D^j$, which is explicitly given by

$$(W^{m,j}_{(a,h)}\phi)(\vec{p}) = e^{i(a_0\sqrt{m^2+\vec{p}^2} - \vec{a}\cdot\vec{p})} D^j\left(q_m(\vec{p})^{-1} h q_m(h^{-1}\vec{p})\right)\phi(h^{-1}\vec{p})$$

where $a \in \mathbb{R}^4$, $h \in H$, $p \in \mathbb{P}^3$ and $\phi \in \mathcal{H}^j$. Comparing the form of $W^{m,j}$ with the one of U^j and V^j, it follows that

$$W^{m,j}_{c(-\vec{a},h)} = V^j_h U^j_{\vec{a}}.$$

Hence, we can conclude that, given any $\psi \in \mathcal{A}$, the representation $W^{m,j}$ is square integrable mod. $(\mathbb{R}^3 \times SL(2, \mathbb{C}), c, \psi)$ and that the frame (18) coincides with the one defined in [1].

Let now consider the frame (19). The space $A \times \mathbb{P}^3$ is a transitive G-space with respect to the action

$$(b, h)[(\vec{a}, \vec{p})] := \left(\mathrm{pr}_A(q_m(h[\vec{p}])^{-1}[b] + (q_m(\vec{p})^{-1} h q_m(h^{-1}[\vec{p}]))[\vec{a}]), q_m(\vec{p})\right),$$

where $b \in \mathbb{R}^4$, $h \in SL(2,\mathbb{C})$, $\vec{a} \in A$, $\vec{p} \in \mathbb{P}^3$ and pr_A is the canonical projection from $\mathbb{R}^4$ to A, *i.e.*

$$\mathrm{pr}_A(b_0, \vec{b}) = \vec{b}.$$

Observe that

1. the stability subgroup at $(\vec{0}, \vec{0})$ is $\mathbb{T} \times' SU(2)$, where $\mathbb{T}$ is the subgroup of the time translations and, since $SU(2)$ does not act on $\mathbb{T}$ the semidirect product is in fact a direct product;
2. a continuous section from $A \times \mathbb{P}^3$ to G is
$$\widetilde{q}_m(\vec{a}, \vec{p}) = (q_m(\vec{p})[(0, \vec{a})], q_m(\vec{p}));$$
3. the G-invariant measure is $d\vec{a} \otimes \nu_m$.

Hence one has that

$$W^{(m,j)}_{\widetilde{q}_m(-\vec{a},\vec{p})} = V^j_{q_m(\vec{p})} U^j_{\vec{a}}.$$

Since the map (19) is a frame, then the representation $W^{(m,j)}$ is square-integrable modulo $(A \times \mathbb{P}^3, \widetilde{q}_m, \psi e_1, \ldots, \psi e_{2j+1})$.

5.3. **The Galilei group.** We now consider a class of frames that are invariant under the homogeneous Galilei group.

Let $\mathcal{V} := \mathbb{R}^3$ and $d\vec{v}$ be the corresponding Lebesgue measure. Let $SU(2)$ be the universal covering of the rotation group in $\mathbb{R}^3$ and $\mu_{SU(2)}$ be the corresponding normalised Haar measure. The group $SU(2)$ acts on $\mathcal{V}$ by means of the covering homomorphism δ and we can consider the semidirect product,

$$H := \mathcal{V} \times' SU(2).$$

The group H is unimodular with Haar measure $d\vec{v} \otimes \mu_{SU(2)}$.

Moreover, let $A = \mathbb{R}^3$ and $\mathbb{P}^3$ be the corresponding dual group identified with $\mathbb{R}^3$ by means of the euclidean scalar product. The corresponding Lebesgue measures are denote by $d\vec{a}$ and $d\vec{p}$, respectively.

Fixed $m \in \mathbb{R}$, $m \neq 0$, the space $\mathbb{P}^3$ is a transitive G-space with respect to the nonlinear transitive smooth action of H

$$(\vec{v}, h)[\vec{p}] = \delta(h)\vec{p} + m\vec{v}. \tag{20}$$

The measure $d\vec{p}$ is H-invariant and the stability subgroup H_0 at $\vec{0}$ is $SU(2)$.

As in the previous case, for each $j = 0, \frac{1}{2}, 1, \ldots$, the couple $(X, [D^j])$ defines defines two classes of frames in the Hilbert space $\mathcal{H}^j = L^2(\mathbb{P}^3, d\vec{p}, \mathbb{C}^{2j+1})$. Indeed, let

$$\begin{aligned} (U^j_{\vec{a}}\phi)(\vec{p}) &= e^{i\vec{a}\cdot\vec{p}}\phi(\vec{p}), \\ (V^j_h\phi)(\vec{p}) &= D^j\left(q(\vec{p})^{-1} h q(h^{-1}\vec{p})\right)\phi(h^{-1}\vec{p}) \end{aligned}$$

where $\vec{a} \in A$, $h \in H$, $p \in \mathbb{P}^3$, $\phi \in \mathcal{H}^j$ and $q : \mathbb{P}^3 \to H$ is a section for the action of H on $\mathbb{P}^3$ (for example $q(\vec{p}) = (\frac{\vec{p}}{m}, I)$).

Indeed, by Corollary 2, for *any* nonzero $\psi \in \mathcal{H}^j$, the map

$$A \times H \ni (\vec{a}, h) \mapsto V^j_{h^{-1}} U^j_{-\vec{a}}\psi \in \mathcal{H}^j \tag{21}$$

is a tight frame in $\mathcal{H}$ over $(A \times H, d\vec{a} \otimes \mu_{SU(2)})$ with frame bound $\|\psi\|^2$. It is easy to prove that two frames of the type (21) corresponding to the same value of j and to the same analysing vector ψ, but differing for the value of the mass $m \neq 0$, are unitarily equivalent.

Moreover, let $(e_i)_{i=1}^{2j+1}$ be the standard basis of $\mathbb{C}^{2j+1}$, J the set of the first n numbers and μ^c the corresponding counting measure. Given $\Psi \in \mathcal{K}(\mathbb{P}^3)$, define ψ by means of Eq. (13), then, as a consequence of Corollary 4, the map

$$(\vec{a}, \vec{p}, i) \mapsto V^j_{q_m(\vec{p})} U^j_{\vec{a}}(\psi e_i) \tag{22}$$

is a tight frame in $L^2(\mathbb{P}^3, d\vec{p}, \mathbb{C}^{2j+1})$ over $(A \times \mathbb{P}^3 \times J, d\vec{a} \otimes d\vec{p} \otimes \mu^c)$.

In this particular case, one can check by direct computation that in Eq. (22) any vector $\psi \in L^2(\mathbb{P}^3, d\vec{p})$ can be used to define a frame.

As in the previous example, we can show that the above frames can be obtained using a (projective) representation of covering group G of the full Galilei group which is square integrability modulo a section. Let $W^{m,j}$ the (projective) representation of G associated with the Galilei invariant quantum particle of mass m and spin j (for an explicit description of this representations see, for example, [10]), then one can show, as in the case of the Poincaré group, that $W^{m,j}$ is square integrable modulo the left coset space $G/\mathbb{T}$ where $\mathbb{T}$ is the subgroup of the time translations since the map (21) is a frame and that $W^{m,j}$ is square integrable modulo the left coset space $G/(\mathbb{T} \times' SU(2)$ since the map (22) is a frame.

5.4. **The Weyl-Heisenberg group.** Let us consider the (2n+1)-dimensional Weyl-Heisenberg group H_n, which is the group associated with the canonical quantisation of a classical mechanical system.

Fixed $n \in \mathbb{N}$, $n > 0$, let $T = \mathbb{R}$ and $A_n = \mathcal{V}_n = \mathbb{R}^n$ the usual vectors groups, then the Lie group H_n is the manifold $T \times A_n \times \mathcal{V}_n$ with the composition law

$$(\tau_1, \vec{a}_1, \vec{u}_1)\,(\tau_2, \vec{a}_2, \vec{u}_2) = (\tau_1 + \tau_2 - \frac{1}{2}(\vec{a}_1 \cdot \vec{u}_2 - \vec{a}_2 \cdot \vec{u}_1), \vec{a}_1 + \vec{a}_2, \vec{u}_1 + \vec{u}_2).$$

The group H_n is the semidirect product of the abelian group $T \times A_n$ and the group $\mathcal{V}_n$, whose action is given by

$$\vec{u}[(\tau, \vec{a})] = (\tau + \vec{a} \cdot \vec{u}, \vec{a}), \quad (\tau, \vec{a}) \in T \times A_n,\ \vec{u} \in \mathcal{V}_n,$$

where the dot denotes the euclidean product. Identifying the dual group $\mathbb{P} \times \mathbb{P}^n$ of $T \times A_n$ with $T \times A_n$ itself by means of the euclidean product, the dual action of $\mathcal{V}_n$ on $\mathbb{P} \times \mathbb{P}^n$ is given by:

$$\vec{u}[(h, \vec{p})] = (h, \vec{p} - h\vec{u}), \quad (h, \vec{p}) \in \mathbb{P} \times \mathbb{P}^n,\ \vec{u} \in \mathcal{V}_n. \tag{23}$$

Then, the $\mathcal{V}_n$-orbits in $\mathbb{P} \times \mathbb{P}^n$ are

- for all $\vec{p} \in \mathbb{R}^n$, the singleton orbits

$$\mathcal{O}_{0,\vec{p}} = \{(0, \vec{p})\};$$

- for all $h \in \mathbb{R}$, $h \neq 0$, the orbits
$$\mathcal{O}_h = \{(h,\vec{p})|\ \vec{p} \in \mathbb{R}^n\}.$$

Since all the orbits are closed, according to Corollary 3, the group H_n has not square-integrable representations. Nevertheless, fixed $h \neq 0$, the orbit $\mathcal{O}_h$ is canonically identified with $\mathbb{P}^n$, which is the dual group of $\mathbb{R}^n$, so that we can use our procedure to define frames associated with irreducible imprimitivity systems for the group $\mathcal{V}_n$ based on the transitive $\mathcal{V}_n$-space $\mathbb{P}^n$.

Since the stability subgroup at $\vec{0} \in \mathbb{P}_n$ is the identity, we have only the couple $(\mathbb{P}_n, I)$ and it defines a class of tight frames in $\mathcal{H} = L^2(\mathbb{P}^n, d\vec{p})$. Indeed, let

$$\begin{aligned} (U_{\vec{a}}\phi)(\vec{p}) &= e^{i\vec{a}\cdot\vec{p}} \\ \big(V_{\vec{u}}\phi\big)(\vec{p}) &= \phi(\vec{p}+h\vec{u}) \end{aligned}$$

where $p \in \mathbb{P}_n$, $\phi \in \mathcal{H}$, $\vec{u} \in \mathcal{V}_n$ and $\vec{a} \in \mathbb{R}^n$, and observe that the Lebesgue measure $d\vec{p}$ is clearly invariant with respect to the action (23).

Hence, since $\mathcal{V}_n$ is unimodular and its Haar measure is $d\vec{u}$, it follows from Corollary 2 and Corollary 1 that for any nonzero vector $\psi \in \mathcal{H}$, the map

$$\mathbb{F}^h_\psi : A_n \times \mathcal{V}_n \ni (\vec{a},\vec{u}) \mapsto V^h_{-\vec{u}} U_{-\vec{a}}\,\psi \in \mathcal{H}_n$$

is a tight frame in $\mathcal{H}$, over the space $(A_n \times \mathcal{V}_n, d\vec{a} \otimes d\vec{u})$ with frame bound $\|\psi\|$.

We remark that, given the unitary operator $\mathfrak{U}_h$, $h \in \mathbb{R}-\{0\}$, in $\mathcal{H}$ defined by:

$$\big(\mathfrak{U}_h\,\phi\big)(\vec{p}) = |h|^{-n/2}\,\phi(h^{-1}\vec{p}), \quad \phi \in \mathcal{H},$$

then, for any $\psi \in \mathcal{H}$ and $h_1, h_2 \in \mathbb{R}-\{0\}$, we have:

$$\mathbb{F}^{h_1}_\psi = \mathfrak{U}_{h_1/h_2} \circ \mathbb{F}^{h_2}_{\psi'} \quad \text{if} \quad \psi' = \mathfrak{U}^*_{h_1/h_2}\,\psi,$$

namely, the frames $\mathbb{F}^{h_1}_\psi$ and $\mathbb{F}^{h_2}_{\psi'}$ are unitarily equivalent. Moreover we notice explicitly that, for $n = 3$ and $h = m \in \mathbb{R}^+$, the frame $\mathbb{F}^h_\psi$ is exactly the frame (22) over the coset space $G/(\mathbb{T} \times SU(2))$, where G is the Galilei group, with the choice $j = 0$.

Still in this case, the frames $\mathbb{F}^h_\psi$ can be defined in terms of a representation of the group H_n that is square-integrable modulo a coset space. Indeed, to sake of simplicity, fix $h = 1$ and let W be the irreducible representation of H_n acting on $\mathcal{H} = L^2(\mathbb{P}^n, d\vec{p})$ as

$$\big(W_{(\tau,\vec{a},\vec{u})}\,\phi\big)(\vec{p}) = e^{\imath(\tau+\frac{h}{2}\vec{a}\cdot\vec{u}-\vec{a}\cdot\vec{p})}\phi(\vec{p}-\vec{u}), \quad \phi \in \mathcal{H}.$$

The left coset space H_n/T is clearly diffeomorphic to $A_n \times \mathcal{V}_n$, so that $A_n \times \mathcal{V}_n$ turns out to be a H_n-space where the action is explicitly given by

$$(\tau,\vec{a},\vec{u})\,[(\vec{a}',\vec{u}')] = (\vec{a}+\vec{a}',\vec{u}+\vec{u}') \qquad (\tau,\vec{a},\vec{u}) \in H_n,\ (\vec{a}',\vec{u}') \in A_n \times \mathcal{V}_n.$$

One easily check that the map

$$\widetilde{q} : A_n \times \mathcal{V}_n \ni (\vec{a},\vec{u}) \mapsto (\vec{u}\,[(0,\vec{a})],\vec{u}) = (\vec{a}\cdot\vec{u},\vec{a},\vec{u}) \in H_n$$

is a smooth section and $d\vec{a} \otimes d\vec{u}$ is an invariant measure on $A_n \times \mathcal{V}_n$. Finally, for any $\psi \in \mathcal{H}$, we have:

$$\mathbb{F}_\psi(\vec{a}, \vec{u}) = W_{q(\vec{a},\vec{u})}\psi.$$

Thus, since, for any nonzero vector $\psi \in \mathcal{H}$, $\mathbb{F}_\psi$ is a frame over the measured space $(A_n \times \mathcal{V}_n, d\vec{a} \otimes d\vec{u})$, it follows that W is square-integrable mod.$(A_n \times \mathcal{V}_n, q; \psi)$.

Moreover, one can show that, for n=3, if the analysing vector ψ is chosen to be a *Gaussian function*, the family of vectors

$$\mathfrak{C} = \{W_{q(\vec{a},\vec{u})} | \ \vec{a} \in \mathbb{R}^3, \vec{u} \in \mathbb{R}^3\}$$

is nothing but the classical canonical family of coherent states associated to the quantum harmonic oscillator.

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