ON DISCRETE FRAMES ASSOCIATED WITH SEMIDIRECT PRODUCTS

P. ANIELLO, G. CASSINELLI, E. DE VITO, AND A. LEVRERO

Abstract. For groups that are semidirect products of a vector group and a unimodular group, we prove that the existence of a discrete frame associated with a unitary representation implies the square-integrability of the representation.

1. Introduction

In this paper we consider a unimodular locally compact second countable (lcsc) topological group H and a continuous representation Λ of H acting in $\mathbb{R}^n$. We define G as the semidirect product of $\mathbb{R}^n$ and H with respect to the action given by Λ, i.e. $G = \mathbb{R}^n \times' H$ and

$$(x, h)(x', h') = (x + \Lambda_h x', hh'),$$

where $(x, h) \in G$ and $(x', h') \in G$.

Moreover, we consider a unitary continuous representation U of G acting in a complex separable Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$, linear in the first argument. With slight abuse of notation, the representation U is said to be *square-integrable* if there is $\psi \in \mathcal{H}$, $\psi \neq 0$ such that, for all $\phi \in \mathcal{H}$

$$\int_G |\langle \phi, U_g \psi \rangle|^2 \, d\mu_G(g) < \infty,$$

and such a ψ is said to be an *admissible vector* (usually, a square-integrable representation is assumed to be irreducible, whereas we do not assume that U is so).

Given an at most countable subset $\{h_j\}_{j\in J}$ of H, a maximal discrete subgroup $\{x_{\mathbf{k}}\}_{\mathbf{k}\in I}$ of $\mathbb{R}^n$ and a vector $\psi \in \mathcal{H}$, we define the at most countable family of vectors in $\mathcal{H}$

$$\mathbb{F}_\psi := \{U_{h_j\, x_{\mathbf{k}}}\, \psi : \ j \in J, \, \mathbf{k} \in I\}.$$

In the following, we assume that $\mathbb{F}_\psi$ is a frame with frame bounds α and β, *i.e.*, for all $\phi \in \mathcal{H}$,

$$\alpha \left\|\phi\right\|^2 \leq \sum_{j\in J, \mathbf{k}\in\mathbb{Z}^n} |\left\langle \phi, U_{h_j x_{\mathbf{k}}} \psi \right\rangle|^2 \leq \beta \left\|\phi\right\|^2,$$

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with $\alpha,\ \beta \in]0,\infty[$.

We prove that, under some further assumptions on the set $\{h_j\}_{j\in J}$, the fact that $\mathbb{F}_\psi$ is a frame forces the representation U to be square-integrable. This result was known for the $ax+b$ group, [3], and [2] (see, also, Th. 3.3.1 of [4]), but it was still an open problem for the more general class of groups we are considering.

2. Main result

In the following we let μ be the Lebesgue measure on $\mathbb{R}^n$ and μ_H a Haar measure on H, so that a left invariant Haar measure on G, which is obviously a lcsc topological group, is

$$\mu_G = \mu \otimes \rho^{-1}\mu_H,$$

where $\rho(h) = \det(\Lambda_h)$. Moreover, we denote simply by

$$h[x] := \Lambda_h x \qquad h \in H, x \in \mathbb{R}^n$$

the action of the matrix Λ_h on the vector x.

Since $\{x_{\mathbf{k}}\}_{\mathbf{k}\in I}$ is a maximal discrete subgroup of $\mathbb{R}^n$, there is a basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ such that the index set $I=\mathbb{Z}^n$ and, for all $\mathbf{k}\in\mathbb{Z}^n$,

$$x_{\mathbf{k}} = k_1 e_1 + \cdots + k_n e_n.$$

Moreover, we denote by $D = |e_1\wedge\ldots\wedge e_n|^{-1}$, where $e_1\wedge\ldots\wedge e_n$ is the determinant of the matrix having the vectors e_i as its rows.

The main result of the paper is the following one.

Proposition 1. *Let $\{e_1,\dots,e_n\}$ be a basis of $\mathbb{R}^n$, $\{h_j\}_{j\in J}$ an at most countable set in H and $\psi\in\mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame. If there is a compact set $K\subset H$ such that $\bigcup_{j\in J} h_j^{-1}K = H$, then U is a square-integrable cyclic representation of G and ψ is an admissible vector for U. In particular,*

$$\int_G |\langle\phi, U_g\psi\rangle|^2 d\mu_G(g) \le \frac{\beta}{D}\mu_H(K)\,\|\phi\|^2 \qquad \phi\in\mathcal{H}.$$

We obtain also an estimate from below of $\int_G |\langle\phi,U_g\psi\rangle|^2 d\mu_G(g)$. Indeed

Proposition 2. *Let $\{e_1,\dots,e_n\}$ be a basis of $\mathbb{R}^n$, $\{h_j\}_{j\in J}$ an at most countable set in H and $\psi\in\mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame. Let C be a compact subset of H such that there is a finite partition $\{J_1,\dots J_N\}$ of the index set J such that, for any $p=1\dots N$,*

$$i,j\in J_p \Longrightarrow h_i^{-1}C\cap h_j^{-1}C=\emptyset,$$

then

$$\frac{\alpha}{DN}\mu_H(C)\,\|\phi\|^2 \le \int_G |\langle\phi,U_g\psi\rangle|^2 d\mu_G(g),$$

for all $\phi\in\mathcal{H}$.

Remark 1. Obviously, the above inequality is interesting only if $\mu_H(C)>0$ and U is square-integrable. Sufficient conditions for the existence of such a compact set C are given at the end of the paper.

The proof of the above results is based on the following theorem, which holds also in the case that H is not unimodular.

Theorem 1. *Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{R}^n$, $\{h_j\}_{j\in J}$ an at most countable set in H and $\psi \in \mathcal{H}$ such that $\mathbb{F}_\psi$ is a frame. Then, for any ϕ in $\mathcal{H}$ and for any compact set $K \subset H$, we have*

$$\alpha \|\phi\|^2 \mu_H(K) \leq D \sum_{j\in J} \int_{\mathbb{R}^n \times h_j^{-1}K} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x)\, d\mu_H(h) \leq \beta \|\phi\|^2 \mu_H(K).$$

Proof. For simplicity of notations, we consider only the case $\|\phi\| = 1$. The general statement follows by homogeneity. Since $\mathbb{F}_\psi$ is a frame, for any unit vector $\phi \in \mathcal{H}$ and for any $g \in G$,

$$\alpha \leq \sum_{j\in J, \mathbf{k}\in\mathbb{Z}^n} |\langle U_g\phi, U_{h_j x_\mathbf{k}}\psi\rangle|^2 \leq \beta,$$

hence, for any $f \in L^1(G, \mu_G)$, $f \geq 0$, we have

$$\alpha \|f\|_1 \leq \int_G \sum_{j\in J, \mathbf{k}\in\mathbb{Z}^n} |\langle U_g\phi, U_{h_j x_\mathbf{k}}\psi\rangle|^2 f(g) d\mu_G(g) \leq \beta \|f\|_1,$$

Applying the monotone convergence theorem and performing on the $(j, \mathbf{k})$-th term the change of variables $g \mapsto h_j x_\mathbf{k} g$ we obtain

$$\alpha \|f\|_1 \leq \int_G |\langle U_g\phi, \psi\rangle|^2 \sum_{j,\mathbf{k}} f(h_j x_\mathbf{k} g) d\mu_G(g) \leq \beta \|f\|_1. \tag{1}$$

Now, let $K \subset H$ be a compact set and consider $\theta \in C_c(\mathbb{R}^n)$ such that $\theta \geq 0$ and $\int_{\mathbb{R}^n} \theta(x) d\mu(x) = 1$. For any $m \in \mathbb{N}$ the function

$$G \ni (x, h) \mapsto f_m(xh) = \frac{1}{m^n} \theta\left(\frac{1}{m} h^{-1}[x]\right) \chi_K(h)$$

is in $L^1(G, \mu_G)$ and we have

$$\begin{aligned}
\|f_m\|_1 &= \int_{\mathbb{R}^n \times H} \frac{1}{m^n} \theta\left(\frac{1}{m} h^{-1}[x]\right) \chi_K(h) \rho(h)^{-1} d\mu(x)\, d\mu_H(h) \\
&= \int_{\mathbb{R}^n \times H} \frac{1}{m^n} \theta\left(\frac{x}{m}\right) \chi_K(h) d\mu(x)\, d\mu_H(h) \\
&= \mu_H(K) \int_{\mathbb{R}^n} \theta(x) d\mu(x) \\
&= \mu_H(K).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{j,\mathbf{k}} f_m(h_j x_\mathbf{k} x h) &= \sum_{j,\mathbf{k}} f_m(h_j[x_\mathbf{k} + x] h_j h) \\
&= \frac{1}{m^n} \sum_{\mathbf{k}} \theta\left(\frac{1}{m} h^{-1}[x + x_\mathbf{k}]\right) \sum_j \chi_K(h_j h)
\end{aligned}$$

and

$$\theta\left(\frac{1}{m}h^{-1}[x+x_{\mathbf{k}}]\right)=\theta\left(h^{-1}\left[\frac{x+x_{\mathbf{k}}}{m}\right]\right)=\theta^h\left(\frac{x+x_{\mathbf{k}}}{m}\right).$$

Let us set:

$$S_m(x,h):=\rho(h)^{-1}\frac{1}{m^n}\sum_{\mathbf{k}}\theta^h\left(\frac{x+x_{\mathbf{k}}}{m}\right).$$

Since θ is compactly supported and continuous, for any compact subset B of $\mathbb{R}^n\times H$ the function S_m restricted to B is in fact a *finite sum* of continuous addenda, so S_m is continuous. Moreover, we have

$$\rho(h)^{-1}\sum_{j,\mathbf{k}}f_m(h_jx_{\mathbf{k}}xh)=S_m(x,h)\sum_j\chi_K(h_jh).$$

Hence, from (1) applied to f_m we obtain

$$\begin{aligned}\alpha\mu_H(K) &\leq \sum_j\int_{\mathbb{R}^n\times H}|\langle U_{xh}\phi,\psi\rangle|^2S_m(x,h)\chi_K(h_jh)d\mu(x)\,d\mu_H(h)\\ &\leq \beta\mu_H(K),\end{aligned}\tag{2}$$

where we applied again the monotone convergence theorem to take the sum over j out of the integral.

Now, since $\theta\in C_c(\mathbb{R}^n)$, for all $(x,h)\in G$,

$$\begin{aligned}\lim_{m\to\infty}S_m(x,h) &= D\,\rho(h)^{-1}\int_{\mathbb{R}^n}\theta^h(y)d\mu(y)\\ &= D\int_{\mathbb{R}^n}\theta(y)d\mu(y)\\ &= D.\end{aligned}$$

In Lemma 1 we will prove that in fact $S_m(x,h)\to D$ uniformly on any subset of G of the form $\mathbb{R}^n\times C$, with C compact in H. Let us assume this fact for the moment. Then we are in a position to prove the desired inequality

$$\alpha\mu_H(K)\leq D\sum_{j\in J}\int_{\mathbb{R}^n\times h_j^{-1}K}|\langle U_{xh}\phi,\psi\rangle|^2d\mu(x)\,d\mu_H(h)\leq\beta\mu_H(K).$$

Indeed, fix $j\in J$ and observe that, according to (2),

$$\int_{\mathbb{R}^n\times h_j^{-1}K}|\langle U_{xh}\phi,\psi\rangle|^2S_m(x,h)d\mu(x)\,d\mu_H(h)\leq\beta\mu_H(K).$$

Since $S_m(x,h)\to D$ and all quantities are positive, due to Fatou's lemma we see that the function

$$\mathbb{R}^n\times h_j^{-1}K\ni(x,h)\mapsto|\langle U_{xh}\phi,\psi\rangle|^2$$

is integrable on $\mathbb{R}^n\times h_j^{-1}K$. Suppose now that

$$D\sum_{j\in J}\int_{\mathbb{R}^n\times h_j^{-1}K}|\langle U_{xh}\phi,\psi\rangle|^2d\mu(x)\,d\mu_H(h)>M>\beta\mu_H(K).$$

Then, for some $N \in \mathbb{N}$,

$$D \sum_{j=1}^{N} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x)\, d\mu_H(h) > M.$$

On the other hand, from Eq. (2),

$$\begin{aligned}
\beta\mu_H(K) &\geq \sum_{j\in J} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 S_m(x,h) d\mu(x)\, d\mu_H(h) \\
&\geq \sum_{j=1}^{N} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 S_m(x,h) d\mu(x)\, d\mu_H(h) \\
&= \sum_{j=1}^{N} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 [S_m(x,h) - D] d\mu(x)\, d\mu_H(h) \\
&+ D \sum_{j=1}^{N} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x)\, d\mu_H(h).
\end{aligned}$$

Now, due to Lemma 1, given $\epsilon > 0$, we can choose m in such a way that, for all $(x,h) \in \bigcup_{j=1}^{N} \mathbb{R}^n \times h_j^{-1} K$,

$$|S_m(x,h) - D| \leq \frac{\epsilon}{\kappa}$$

where $\kappa = \sum_{j=1}^{N} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x)\, d\mu_H(h)$ and we obtain $\beta\mu_H(K) \geq M - \epsilon$. In the limit we have $\beta\mu_H(K) \geq M$, which is a contradiction. Hence we conclude that

$$D \sum_{j\in J} \int_{\mathbb{R}^n \times h_j^{-1} K} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x)\, d\mu_H(h) \leq \beta\mu_H(K).$$

The other inequality is proved in an analogous way and this concludes the proof. □

We now prove a technical result we used in the above proof.

Lemma 1. *Let C be a compact subset of H. Then we have that*

$$S_m(x,h) \overset{m\to\infty}{\longrightarrow} D$$

uniformly on the subset $\mathbb{R}^n \times C$ of G.

Proof. We will first prove that $S_m(x,h) \to D$ uniformly on any compact subset B of G. Due to Ascoli's theorem, since the constant function D is continuous, it is sufficient to prove that the set $\{S_m\}_{m\geq 1} \subset C(B)$ is equicontinuous. We can freely assume that $B = B_1 \times B_2$, where B_2 is a compact set in H and B_1 is a closed ball centred at 0 in $\mathbb{R}^n$. Since the action of H on $\mathbb{R}^n$ is continuous, the set $B_1 \cup (B_2[\operatorname{supp}\theta] + B_1)$ is compact so that we can choose a closed cube A in $\mathbb{R}^n$ containing it. Since the function

$$G \ni (x,h) \mapsto \theta^h(x)$$

is continuous and A is compact, then, given $h \in B_2$ and $\epsilon > 0$, we can find a neighbourhood $V \subset H$ of h and $\delta > 0$ such that if $h' \in V$, $x, x' \in A$ and $|x - x'| \leq \delta$, then $|\theta^{h'}(x') - \theta^h(x)| \leq \frac{\epsilon}{\mu(A)M}$, where $M = \sup\{\rho(\hat{h})^{-1} \; : \; \hat{h} \in B_2\}$. Let now $h' \in V \cap B_2$, $x, x' \in B_1$ and $|x - x'| \leq \delta$, then

$$\begin{aligned} |S_m(x', h') - S_m(x, h)| \; &= \\ &= \frac{1}{m^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \rho(h')^{-1} \theta^{h'} \left(\frac{x' + x_{\mathbf{k}}}{m} \right) - \rho(h)^{-1} \theta^h \left(\frac{x + x_{\mathbf{k}}}{m} \right) \right| \\ &\leq \frac{M}{m^n} \sum_{\mathbf{k} \in I} \left| \theta^{h'} \left(\frac{x' + x_{\mathbf{k}}}{m} \right) - \theta^h \left(\frac{x + x_{\mathbf{k}}}{m} \right) \right|, \end{aligned}$$

where

$$I = \{\mathbf{k} \in \mathbb{Z}^n \; : \; \frac{x_{\mathbf{k}}}{m} \in \left(\operatorname{supp} \theta^h - \frac{x}{m} \right) \cup \left(\operatorname{supp} \theta^{h'} - \frac{x'}{m} \right) \}.$$

At this point, since $x, x' \in B_1$, then $\frac{x}{m}, \frac{x'}{m} \in B_1$ so that, if $\mathbf{k} \in I$, then $\frac{x_{\mathbf{k}}}{m} \in A$. Moreover, $\frac{x + x_{\mathbf{k}}}{m}$ and $\frac{x' + x_{\mathbf{k}}}{m}$ are in A and

$$\left| \frac{x + x_{\mathbf{k}}}{m} - \frac{x' + x_{\mathbf{k}}}{m} \right| = \frac{1}{m} |x - x'| \leq \delta.$$

Hence,

$$\begin{aligned} |S_m(x', h') - S_m(x, h)| \; &\leq \; \frac{1}{m^n} \sum_{\mathbf{k} \in I} \frac{\epsilon}{\mu(A)} \\ &\leq \; \frac{1}{m^n} \sum_{\frac{x_{\mathbf{k}}}{m} \in A} \frac{\epsilon}{\mu(A)}. \end{aligned}$$

Since A is a cube, the last term has limit ϵ as m tends to infinity and this proves that $(S_m)_{m \geq 1}$ is equicontinuous, so that $S_m(x, h) \to D$ uniformly on every compact subset of G.

Taking into account that, for any $\mathbf{k} \in \mathbb{Z}^n$, $S_m(x, h) = S_m(x + x_{\mathbf{k}}, h)$, we see that in fact $S_m(x, h) \to D$ uniformly on $x \in \mathbb{R}^n$ and h in any compact subset of H. This concludes the proof of the lemma. □

We are now in position to prove the propositions.

Proof of Proposition 1. The fact the U is cyclic is a well-known consequence of the fact that ψ is a frame. We prove that U is square-integrable. Since

H is unimodular, μ_H is both left- and right-invariant, so that

$$\begin{aligned}
\int_G |\langle \phi, U_g\psi\rangle|^2 d\mu_G(g) &= \int_{\mathbb{R}^n\times' H} |\langle \phi, U_{xh}\psi\rangle|^2 \rho(h)^{-1} d\mu(x) d\mu_H(h) \\
&= \int_{\mathbb{R}^n\times' H} |\left\langle U_{h^{-1}[x]h^{-1}}\phi, \psi\right\rangle|^2 \rho(h)^{-1} d\mu(x) d\mu_H(h) \\
&= \int_{\mathbb{R}^n\times' H} |\langle U_{xh^{-1}}\phi, \psi\rangle|^2 d\mu(x) d\mu_H(h) \\
&= \int_{\mathbb{R}^n\times' H} |\langle U_{xh}\phi, \psi\rangle|^2 d\mu(x) d\mu_H(h).
\end{aligned}$$

Applying Theorem. 1 and the assumption on the compact set K, we have the thesis. $\square$

Proof of Proposition 2. Use the same arguments of the proof of Proposition 1. $\square$

As a final remark, we notice that the existence of compact subsets of H satisfying the hypotheses of the previous propositions is assured in many concrete cases. For example, suppose that the set $\{h_j\}j \in J$ is a discrete *subgroup* Γ of H.

1. If the quotient space H/Γ is compact (for example, if H is a semisimple Lie group, there exists such a group Γ, see Theorem C of [1]), there exists a compact subset K of H such that $\pi(K) = H/\Gamma$, where π is the canonical projection from H onto H/Γ. It is clear that $\bigcup_{j\in J} h_j^{-1}K = H$.
2. Since Γ is discrete, there exist a neighbourhood U of the identity e in H such that $\Gamma\cap U = \{e\}$. Let C be a compact neighbourhood of e such that $CC^{-1} \subset U$. Then $\mu_H(C) > 0$ and it is clear that $h_iC \cap h_jC = \emptyset$ if $i \neq j$.

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Paolo Aniello, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: aniello@ge.infn.it

Gianni Cassinelli, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: cassinelli@ge.infn.it

Ernesto De Vito, Dipartimento di Matematica, Università di Modena, Via Campi 213/B, 41100 Modena, Italy and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: devito@unimo.it

Alberto Levrero, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: levrero@ge.infn.it