SYMMETRIES OF THE QUANTUM STATE SPACE AND GROUP REPRESENTATIONS

G. CASSINELLI, E. DE VITO, P. LAHTI, AND A. LEVRERO

Dedicated to Francesca

Abstract. The homomorphisms of a connected Lie group G into the symmetry group of a quantum system are classified in terms of unitary representations of a simply connected Lie group associated with G. Moreover, an explicit description of the $\mathbf{T}$-multipliers of G is obtained in terms of the $\mathbb{R}$-multipliers of the universal covering G^* of G and the characters of G^*. As an application, the Poincaré group and the Galilei group, both in $3+1$ and $2+1$ dimensions, are considered.

1. Introduction

In the standard framework of Quantum Mechanics, the physical properties of a quantum system are described in terms of a Hilbert space. In particular, the one dimensional projectors are the *pure states* of the system and the physical content of the theory is given by the *transition probabilities* between such states.

In this context, a *symmetry* is a bijective map from the set of pure states onto itself preserving the transition probabilities; the set $\mathcal{S}$ of all symmetries is a group under the composition of maps. A group G is a *symmetry group* for a quantum system if there exists a group homomorphism from G to $\mathcal{S}$. We call such homomorphisms *symmetry actions* of G.

Given a symmetry group G (for example, the covariance group of space-time is a symmetry group for the free particles), one can pose the mathematical problem of describing all possible symmetry actions of G. For the problem to be well-posed, one has to specify a suitable notion of equivalence between symmetry actions. Taking into account the physical meaning of the theory, a symmetry action α_1, acting on a Hilbert space $\mathcal{H}_1$, is *equivalent* to another symmetry action α_2, acting on a Hilbert space $\mathcal{H}_2$, if there exists a bijective map β from the set of pure states of $\mathcal{H}_1$ onto the set of pure states of $\mathcal{H}_2$ such that β preserves the transition probabilities and intertwines α_1 and α_2.

Using the Wigner theorem on the characterisation of symmetries in terms of unitary operators, [1], one can analyse the symmetry actions of G in the framework of *projective representations*, i.e., maps $g \mapsto U_g$ from G to the

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group of unitary operators satisfying

$$U_g U_h = \mu(g,h) U_{gh} \qquad g,h \in G,$$

where $\mu(g,h)$ belongs to the circle group $\mathbf{T}$ and the map $(g,h) \mapsto \mu(g,h)$ is called a *multiplier* of G. However, the natural notion of unitary equivalence among projective representations does not correspond to the one among symmetry actions.

The classification of projective representations for finite groups, up to unitary equivalence, was given by Schur, [2], [3] who first introduced the concept of projective representation. In particular, he showed the existence of a finite group with the property that there is a one to one correspondence between its irreducible unitary representations and the irreducible projective representations of G and this correspondence preserves the unitary equivalence, (see, for example, [4], section 14.2, for a modern exposition of these results).

The first to solve this problem for an infinite group was Wigner in his celebrated paper [5]. He classified the projective representations of the Poincaré group considering the unitary representations of its universal covering group.

Bargmann, in his seminal paper [6], considered the case of a connected Lie group G, reducing the problem of classifying its projective representations to the one of classifying the projective representations of the universal covering G^*. He proved that the multipliers of G^* can always be chosen smooth and that its projective representations are in one to one correspondence with a family of unitary representations of a set of Lie groups associated with G^*.

The fundamental contribution of Mackey to this problem was a complete analysis in the case of G being a locally compact topological group, [7]. In particular, he showed that there is a one to one correspondence between the projective representations of G and the unitary representations of a family of locally compact topological groups, namely the central extensions of $\mathbf{T}$ by G, parametrised by the group $H^2(G, \mathbf{T})$ of multipliers of G.

The structure of $H^2(G, \mathbf{T})$ was exploited completely by Moore in the context of the cohomology of locally compact topological groups, [8], [9], [10], [11]. In particular, he proved that there exists a central extension H of an abelian group by G, called *splitting group*, such that there is a surjective map from the set of irreducible representations of H and the set of irreducible projective representations of G, preserving again the notion of unitary equivalence, [9].

The construction of the splitting group given by Moore requires a careful analysis of the cohomology groups $H^2(G, \mathbb{Z})$ and $H^2(G, \mathbb{R})$ (see section 2 of [9]). Since G, in general, is not simply connected, one cannot use linear methods to study these cohomology groups.

A clear and complete exposition of the theory of projective representations and multipliers can be found in Varadarajan's book [12], which we use in the following as a standard reference.

In this paper we consider the case of a connected Lie group G and we prove that there exist a Lie group $\overline{G}$, namely the *universal extension* of G, and a notion of equivalence among unitary representations of $\overline{G}$, called *physical equivalence*, such that there is a one to one correspondence between the equivalence classes of irreducible unitary representations of $\overline{G}$ (with respect to physical equivalence) and the equivalence classes of irreducible symmetry actions of G.

The explicit construction of $\overline{G}$, which is a splitting group in the sense of Moore, requires the knowledge of $H^2(G^*, \mathbb{R})$, where G^* is the universal covering group of G. Moreover, $\overline{G}$ is a central extension of an abelian Lie group K by G and $H^2(G, \mathbf{T})$ is isomorphic, as a topological group, to the quotient group $\widehat{K}/V$, where V is a subgroup (not necessarily closed) of the dual group $\widehat{K}$ of K and V is completely defined in terms of $H^1(G^*, \mathbf{T})$, i.e the group of characters of G^*. Since G^* is simply connected, the study of $H^2(G^*, \mathbb{R})$ and $H^1(G^*, \mathbf{T})$ can be done by the use of linear methods.

Finally, we show that every symmetry action of G is induced by a representation of $\overline{G}$ satisfying an *admissibility condition* and *vice versa*. This admissibility condition is at the root of the existence of superselection rules in the case of reducible representations (this topic has been recently considered also by Divakaran [13]).

In the paper we assume that G is connected, otherwise one can consider only the connected component of the identity of G. We do not consider the problem of discrete symmetries.

2. Preliminary results

In this section we introduce the notations and we review some results on the theory of multipliers for simply connected Lie groups.

2.1. **Notations.**

1. By *Hilbert space* we mean a complex separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ linear in the second argument.
2. If $\mathcal{H}$ is a Hilbert space, we denote by $\mathcal{P}_{\mathcal{H}}$ (or $\mathcal{P}$) the set of one dimensional projectors on $\mathcal{H}$.
3. We use the word *representation* to mean a unitary representation of a topological group, acting on a Hilbert space and continuous with respect to the strong operator topology.
4. If α is a Lie group homomorphism we denote by $\dot{\alpha}$ the differential of α at the identity.

Let $\mathcal{H}$ be the Hilbert space associated with a quantum system. Then $\mathcal{P}$ is the set of pure states of the system and the transition probability between two pure states $P_1, P_2 \in \mathcal{P}$ is given by $\operatorname{tr}\left[P_1 P_2\right]$, where $\operatorname{tr}\left[\cdot\right]$ denotes the trace.

Definition 1. *A map $\alpha : \mathcal{P} \to \mathcal{P}$ is a* symmetry *of $\mathcal{P}$ if α is bijective and*

$$\operatorname{tr}\left[P_1 P_2\right] = \operatorname{tr}\left[\alpha(P_1)\alpha(P_2)\right], \qquad P_1, P_2 \in \mathcal{P}.$$

We denote by $\mathcal{S}$ the set of symmetries of $\mathcal{P}$. The set $\mathcal{S}$ is a group with respect to the usual composition of maps and it is a topological space with respect to the initial topology given by the family of functions

$$\mathcal{S} \ni \alpha \mapsto \operatorname{tr}\bigl[P_1\alpha(P_2)\bigr] \in \mathbb{R},$$

labelled by $P_1, P_2 \in \mathcal{P}$.

Using the Wigner theorem, [1], we can obtain a useful characterisation of $\mathcal{S}$. In fact, let $\mathcal{U}$ denote the set of unitary operators on $\mathcal{H}$, $\overline{\mathcal{U}}$ the set of antiunitary operators on $\mathcal{H}$, and let

$$\mathbf{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$$

denote the circle group. We identify $\mathbf{T}$ with $\{zI \ : \ z \in \mathbf{T}\} \subset \mathcal{U}$. The set $\mathcal{U} \cup \overline{\mathcal{U}}$ is a group under the usual composition of operators and it is a topological space with respect to the restriction of the strong operator topology. Then we have the following results, whose proofs can be found, for example, in [12] (for a review see, also, [14]).

Proposition 1. *With the above notations:*

1. *the group $\mathcal{U} \cup \overline{\mathcal{U}}$ is a second countable metrisable topological group, $\mathcal{U}$ is the connected component of the identity and $\mathbf{T}$ is its centre;*
2. *the group $\mathcal{S}$ is a second countable metrisable topological group;*
3. *the map $\pi : \mathcal{U} \cup \overline{\mathcal{U}} \to \mathcal{S}$ defined as*

$$\pi(U)(P) = UPU^{-1}, \qquad P \in \mathcal{P},$$

   is a continuous surjective open group homomorphism and its kernel is the group $\mathbf{T}$;

4. *there exists a measurable map s from $\mathcal{S}_0$, the connected component of the identity of $\mathcal{S}$, to $\mathcal{U}$ such that*

$$\begin{aligned} s(I) &= I, \\ \pi(s(\alpha)) &= \alpha, \qquad \alpha \in \mathcal{S}_0. \end{aligned}$$

   We call such a map a section *for π.*

We are now in a position to define a symmetry action. Let G be a connected Lie group.

Definition 2. *A* symmetry action *of G on $\mathcal{H}$ is a continuous group homomorphism $\alpha : G \to \mathcal{S}$.*

Two symmetry actions α^1 and α^2 of G acting, respectively, on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent *if there is a bijective map β from $\mathcal{P}_{\mathcal{H}_1}$ onto $\mathcal{P}_{\mathcal{H}_2}$ such that*

$$\begin{aligned} \operatorname{tr}\bigl[PQ\bigr]_{\mathcal{H}_1} &= \operatorname{tr}\bigl[\beta(P)\beta(Q)\bigr]_{\mathcal{H}_2}, \\ \beta\alpha^1_g &= \alpha^2_g\beta, \end{aligned}$$

for all $P, Q \in \mathcal{P}_{\mathcal{H}_1}$ and for all $g \in G$.

A symmetry action α is irreducible *if for all $P_1, P_2 \in \mathcal{P}$ there is $g \in G$ satisfying*

$$\operatorname{tr}\big[P_2 \alpha_g(P_1)\big] \neq 0.$$

Since G is connected all symmetry actions take values in $\mathcal{S}_0$. Moreover, if α is an irreducible symmetry action, then every symmetry action equivalent to α is irreducible too. Finally, the condition on the continuity of the symmetry actions can be weakened, using the standard result that a group homomorphism $\alpha : G \to \mathcal{S}$ is continuous if and only if it is measurable (see, for example, Lemma 5.28 of [12]).

2.2. **Multipliers for Lie groups: a review.** In this section we give a brief review of the theory of multipliers for a connected, simply connected Lie group. For this class of groups the problem of the classification (up to equivalence) of multipliers can be reduced to a finite-dimensional linear problem on the Lie algebra of the group. We stress that the theory works only in the case of simply connected Lie groups; nevertheless, as we are going to show in the next section, this is enough to study the symmetry actions also for Lie groups which are not simply connected. All the proofs of this section can be found, for example, in chapter 7 of [12] where a systematic study of multipliers is presented.

Let H be a connected Lie group and A an abelian Lie group (for the moment we do not assume that H is simply connected). We denote by e and 1 the unit elements of H and A, respectively.

Definition 3. *An A-*multiplier *of H is a measurable map τ from $H \times H$ to A such that*

$$\begin{aligned} \tau(e,g) = \tau(g,e) &= 1, && g \in H, \\ \tau(g_1, g_2g_3)\tau(g_2,g_3) &= \tau(g_1,g_2)\tau(g_1g_2,g_3), && g_1,g_2,g_3 \in H. \end{aligned}$$

Two A-multipliers τ_1 and τ_2 of H are equivalent *if there is a measurable map b from H to A such that*

$$\tau_2(g_1,g_2) = \frac{b(g_1g_2)}{b(g_1)b(g_2)}\tau_1(g_1,g_2), \qquad g_1,g_2 \in H.$$

An A-multiplier τ is exact *if it is equivalent to the multiplier 1, that is,*

$$\tau(g_1,g_2) = \frac{b(g_1g_2)}{b(g_1)b(g_2)}, \qquad g_1,g_2 \in H$$

for some measurable map $b : H \to A$.

The set of A-multipliers is an abelian group under the pointwise multiplication and the set of exact A-multipliers is a subgroup. We denote by $H^2(H,A)$ the corresponding quotient group.

Remark 1. There is a natural topology on $H^2(H,A)$ converting it into a locally compact group (in general not Hausdorff), see [11]. If $A = \mathbf{T}$ it coincides with the (quotient) topology of uniform convergence on compact sets defined on the set of $\mathbf{T}$-multipliers (see Theorem 6 of [11]). We will always consider $H^2(H,\mathbf{T})$ endowed with this topology.

From now on, assume that H is simply connected. If A is the group $\mathbf{T}$, we have the following result.

Lemma 1. *Each $\mathbf{T}$-multiplier of H is similar to one of the form $e^{i\tau}$, where τ is a smooth $\mathbb{R}$-multiplier of H. Moreover, τ is exact if and only if $e^{i\tau}$ is exact.*

If A is the vector group $\mathbb{R}^n$, we denote it additively. The set of $\mathbb{R}^n$-multipliers is a real vector space under the pointwise operations and the set of exact $\mathbb{R}^n$-multipliers is a subspace of it, so that $H^2(H,\mathbb{R}^n)$ is a vector space. Moreover, we have

Lemma 2. *Any $\mathbb{R}^n$-multiplier of H is equivalent to a smooth one.*

Let $\mathrm{Lie}\,(H)$ be the Lie algebra of H.

Definition 4. *A bilinear skew symmetric map F from $Lie\,(H)\times Lie\,(H)$ to $\mathbb{R}^n$ such that*

$$F(X,[Y,Z])+F(Z,[X,Y])+F(Y,[Z,X])=0, \qquad X,Y,Z\in Lie\,(H),$$

is called a closed $\mathbb{R}^n$-form*. A closed $\mathbb{R}^n$-form F is* exact *if there is a linear map q from $Lie\,(H)$ to $\mathbb{R}^n$ such that*

$$F(X,Y)=q([X,Y]), \qquad X,Y\in Lie\,(H).$$

The set of closed $\mathbb{R}^n$-forms is a finite dimensional real vector space and the set of exact $\mathbb{R}^n$-forms is a subspace. We denote by $H^2(\mathrm{Lie}\,(H),\mathbb{R}^n)$ the corresponding quotient space.

Theorem 1. *The vector spaces $H^2(H,\mathbb{R}^n)$ and $H^2(Lie\,(H),\mathbb{R}^n)$ are canonically isomorphic.*

We exhibit explicitly the above isomorphism. Let F be a closed $\mathbb{R}^n$-form, and denote by $\mathbb{R}^n\oplus_F \mathrm{Lie}\,(H)$ the Lie algebra defined by the following Lie bracket

$$\big[(v_1,X_1),(v_2,X_2)\big]:=(F(X_1,X_2),\big[X_1,X_2\big]),$$

for all $X_1,X_2\in\mathrm{Lie}\,(H)$ and $v_1,v_2\in\mathbb{R}^n$. Let $\alpha:\mathbb{R}^n\to\mathbb{R}^n\oplus_F\mathrm{Lie}\,(H)$ be the natural injection and $\beta:\mathbb{R}^n\oplus_F\mathrm{Lie}\,(H)\to\mathrm{Lie}\,(H)$ be the natural projection. Both these maps are Lie algebra homomorphisms and $\mathrm{Ker}\,\beta=\mathrm{Im}\,\alpha$. By the general theory of Lie groups, there exist a unique (up to an isomorphism) connected, simply connected Lie group H_F, such that $\mathrm{Lie}\,(H_F)=\mathbb{R}^n\oplus_F\mathrm{Lie}\,(H)$, and two Lie group homomorphisms $a:\mathbb{R}^n\to H_F$, $b:H_F\to H$ such that $\dot a=\alpha$ and $\dot b=\beta$. Moreover, one can prove that a is a homeomorphism from $\mathbb{R}^n$ onto $a(\mathbb{R}^n)$ and $H_F/a(\mathbb{R}^n)$ is isomorphic to H. By a lemma of Malčev (see, for example, lemma 7.26 of [12]) there exists a smooth map c from H to H_F such that $c(e)=e$ and $b(c(h))=h$ for all $h\in H$. If we define

$$\tau_F(h_1,h_2)=c(h_1)c(h_2)c(h_1h_2)^{-1}, \qquad h_1,h_2\in H,$$

then τ_F is a smooth $\mathbb{R}^n$-multiplier. The equivalence class of τ_F is the image of the equivalence class of F under the isomorphism of the above theorem. Since τ_F is smooth, one can easily check that H_F is isomorphic, as a Lie group, to $\mathbb{R}^n \times_{\tau_F} H$, which is a Lie group with respect to the product

$$(v_1, g_1)(v_2, g_2) = (v_1 + v_2 + \tau_F(g_1, g_2), g_1 g_2), \qquad v_1, v_2 \in \mathbb{R}^n,\ g_1, g_2 \in H.$$

3. Main results

In this section we introduce the notion of *universal central extension* $\overline{G}$ for a connected Lie group G and the notions of *physical equivalence* and *admissibility* for representations of $\overline{G}$. By the use of these concepts, we then state the main results of the paper. Moreover, we discuss the physical equivalence for induced representations in the case that $\overline{G}$ is a semidirect product with abelian normal factor.

3.1. **Universal central extension.**

Let G be a connected Lie group. We denote by G^* its universal covering group and by δ the corresponding covering homomorphism.

Let $H^2(G^*, \mathbb{R})_\delta$ be the set of equivalence classes $[\tau] \in H^2(G^*, \mathbb{R})$ such that

$$\tau(k, g^*) = \tau(g^*, k), \qquad k \in \operatorname{Ker} \delta,\ g^* \in G^*. \tag{1}$$

Since $\operatorname{Ker} \delta$ is central in G^*, equation (1) holds for all $\mathbb{R}$-multipliers equivalent to τ and, hence, the definition of $H^2(G^*, \mathbb{R})_\delta$ is well-posed. Moreover, $H^2(G^*, \mathbb{R})_\delta$ is a subspace of $H^2(G^*, \mathbb{R})$, so that, due to Theorem 1, it has finite dimension N.

By Lemma 2, we can fix N smooth $\mathbb{R}$-multipliers of G^*, $\tau_1, \ldots, \tau_N$, such that the equivalence classes $[\tau_1], \ldots, [\tau_N]$ form a basis of $H^2(G^*, \mathbb{R})_\delta$. Let $\overline{\tau} : G^* \times G^* \to \mathbb{R}^N$ be defined as

$$\overline{\tau}(g_1^*, g_2^*)_i = \tau_i(g_1^*, g_2^*), \qquad g_1^*, g_2^* \in G^*,\ i = 1, \ldots, N,$$

then $\overline{\tau}$ is a smooth $\mathbb{R}^N$-multiplier of G^*. The restriction of $\overline{\tau}$ to $\operatorname{Ker} \delta \times \operatorname{Ker} \delta$ is an $\mathbb{R}^N$-multiplier of the discrete group $\operatorname{Ker} \delta$, hence it is exact (see Proposition 2, Sec. 4, ch.1 of [15]). Then, without loss of generality, we can always assume that $\overline{\tau}$ is smooth and that

$$\overline{\tau}(k_1, k_2) = 0, \qquad k_1, k_2 \in \operatorname{Ker} \delta. \tag{2}$$

Definition 5. *Let $\overline{G} = \mathbb{R}^N \times_{\overline{\tau}} G^*$ be the product manifold. Since $\overline{\tau}$ is smooth, $\overline{G}$ is a Lie group with the product*

$$(v_1, g_1^*)(v_2, g_2^*) = (v_1 + v_2 + \overline{\tau}(g_1^*, g_2^*), g_1^* g_2^*), \qquad g_1^*, g_2^* \in G^*,\ v_1, v_2 \in \mathbb{R}^n.$$

We call it the universal central extension *of G and we denote by σ the smooth map from $\overline{G}$ to G given by*

$$\sigma(v, g^*) = \delta(g^*), \qquad v \in \mathbb{R}^N,\ g^* \in G^*.$$

We denote by K the closed subgroup of $\overline{G}$ defined as

$$K = \{(v,k) \in \overline{G} \ : \ v \in \mathbb{R}^N, \ k \in \mathrm{Ker}\,\delta\}.$$

The main properties of $\overline{G}$ are stated in the following lemma.

Lemma 3. *Let $\overline{G}$ be the universal central extension of G.*

1. *The restriction of a character of $\overline{G}$ to the subgroup $\mathbb{R}^N$ is the identity character. Any character of G^* extends naturally to a character of $\overline{G}$.*
2. *The map σ is a surjective group homomorphism whose kernel is K, which is central in $\overline{G}$. Moreover, the group K is the direct product of $\mathbb{R}^N$ and Ker δ.*
3. *There is a measurable map $c : G \to \overline{G}$ such that $c(e) = e$ and $\sigma(c(g)) = g$ for all $g \in G$ (we call such a map a* section *for σ).*
4. *Given a section c for σ, let Γ_c from $G \times G$ to K be the map*
$$\Gamma_c(g_1, g_2) = c(g_1)c(g_2)c(g_1g_2)^{-1}, \qquad g_1, g_2 \in G,$$
then Γ_c is a K-multiplier of G and its equivalence class does not depend on the choice of the section c.
5. *If we consider $\mathbb{R}^N$ as a subgroup of K in a natural way, then the K-multiplier $\Gamma_c \circ (\delta \times \delta)$ of G^* is equivalent to $\overline{\tau}$.*

Proof.

1. Let χ be a character of $\overline{G}$. The restriction of χ to $\mathbb{R}^N$ is of the form
$$\chi(v, e^*) = e^{iw\cdot v} \qquad v \in \mathbb{R}^N,$$
for some $w \in \mathbb{R}^N$. Then, if $g_1^*, g_2^* \in G^*$,
$$\begin{aligned} \chi((0, g_1^*))\chi((0, g_2^*)) &= \chi((\overline{\tau}(g_1^*, g_2^*), e^*))\chi((0, g_1^*g_2^*)) \\ &= e^{iw\cdot\overline{\tau}(g_1^*,g_2^*)}\chi((0, g_1^*g_2^*)). \end{aligned}$$
Hence, by Lemma 1, the $\mathbb{R}$-multiplier $w \cdot \overline{\tau}$ is exact, so that $w = 0$. The other statement is evident.
2. The fact that K is central in $\overline{G}$ follows taking into account that, by definition of $H^2(G^*, \mathbb{R})_\delta$, $\overline{\tau}(k, g^*) = \overline{\tau}(g^*, k)$ for all $k \in$ Ker δ and $g^* \in G^*$. Using equality (2), one has $K = \mathbb{R}^N \times \mathrm{Ker}\,\delta$. The other facts are evident.
3. By the previous item, G is isomorphic, as a Lie group, to the quotient $\overline{G}/K$. The existence of a section is thus a standard result (see, for example, Theorem 5.11 of [12]).
4. If $g_1, g_2 \in G$, then $\sigma(\Gamma_c(g_1, g_2)) = e$, so that $\Gamma_c(g_1, g_2) \in K$. By direct computation one checks that Γ_c is a K-multiplier. Let c' be another section, then, for all $g \in G$, $c(g) = b(g)c'(g)$ for some measurable map b from G to K. Hence, for all $g_1, g_2 \in G$
$$\Gamma_{c'}(g_1, g_2) = \frac{b(g_1g_2)}{b(g_1)b(g_2)}\Gamma_c(g_1, g_2).$$

5. Let $i : G^* \to \overline{G}$ be the natural immersion and a be the measurable map from G^* to $\overline{G}$
$$a(g^*) = c(\delta(g^*))i(g^*)^{-1}, \qquad g^* \in G^*.$$
Since $\sigma(a(g^*)) = e$, then a takes values in K. Then, if $g_1^*, g_2^* \in G^*$,
$$\begin{aligned}\Gamma_c(\delta(g_1^*), \delta(g_2^*)) &= c(\delta(g_1^*))c(\delta(g_1^*))c(\delta(g_1^*)\delta(g_2^*))^{-1}\\ &= a(g_1^*)i(g_1^*)a(g_2^*)i(g_2^*)i(g_1^*g_2^*)^{-1}a(g_1^*g_2^*)^{-1}\\ &= a(g_1^*)a(g_2^*)a(g_1^*g_2^*)^{-1}i(g_1^*)i(g_2^*)i(g_1^*g_2^*)^{-1}\\ &= a(g_1^*)a(g_2^*)a(g_1^*g_2^*)^{-1}(\overline{\tau}(g_1^*, g_2^*), e^*),\end{aligned}$$
i.e., $\Gamma_c \circ (\delta \times \delta)$ is equivalent to $\overline{\tau}$.

□

The following theorem describes the group $H^2(G, \mathbf{T})$ in terms of the characters of K and the characters of G^*. We first observe that

Lemma 4. *Let $c : G \to \overline{G}$ be a section for σ and Γ_c the corresponding K-multiplier of G defined in statement 4 of Lemma 3. Let χ be a character of K, then the map μ_χ from $G \times G$ to $\mathbf{T}$ defined as*
$$\mu_\chi(g_1, g_2) = \chi\left(\Gamma_c(g_1, g_2)\right), \qquad g_1, g_2 \in G,$$
is a $\mathbf{T}$-multiplier of G and its equivalence class $[\mu_\chi]$ does not depend on the choice of the section c.

Proof. It is a simple consequence of the properties of Γ_c given in statement 4 of Lemma 3. □

Let $\widehat{K}$ be the dual group of K (with the topology of uniform convergence on compact sets) and V the subgroup of characters that extend to characters of $\overline{G}$, which, due to the statement 1 of Lemma 3, can be identified with characters of G^*.

Theorem 2. *The mapping*
$$\widehat{K} \ni \chi \mapsto [\mu_\chi] \in H^2(G, \mathbf{T})$$
is a surjective homomorphism whose kernel is V. Moreover, $H^2(G, \mathbf{T})$ is isomorphic, as a topological group, to the quotient group $\widehat{K}/V$.

Proof. By direct computation, one can check that $\chi \mapsto [\mu_\chi]$ is a group homomorphism. To show its surjectivity, we notice that, since the equivalence class $[\mu_\chi]$ does not depend on the specific form of the section c, we can choose for c the particularly simple form
$$c(g) = (0, \tilde{c}(g)) \qquad g \in G,$$
where $\tilde{c} : G \to G^*$ is measurable and satisfies $\tilde{c}(e) = e^*$ and $\delta(\tilde{c}(g)) = g$ for all $g \in G$. With this choice, a straightforward calculation shows that
$$\Gamma_c(g_1, g_2) = \left(\ \overline{\tau}(\tilde{c}(g_1), \tilde{c}(g_2)) - \overline{\tau}(\gamma(g_1, g_2), \tilde{c}(g_1 g_2))\ ,\ \gamma(g_1, g_2)\ \right), \tag{3}$$

where $g_1, g_2 \in G$ and

$$\gamma(g_1, g_2) \quad = \quad \tilde{c}(g_1)\tilde{c}(g_2)\tilde{c}(g_1 g_2)^{-1} \in \mathrm{Ker}\ \delta.$$

Let now μ be a $\mathbf{T}$-multiplier of G and μ^* the $\mathbf{T}$-multiplier of G^*

$$\mu^*(g_1^*, g_2^*) = \mu(\delta(g_1^*), \delta(g_2^*)), \qquad g_1^*, g_2^* \in G^*.$$

Applying Lemma 1 to μ^*, we have that

$$\mu^*(g_1^*, g_2^*) = \frac{a(g_1^* g_2^*)}{a(g_1^*)a(g_2^*)} e^{i\tau(g_1^*, g_2^*)}, \qquad g_1^*, g_2^* \in G^*, \tag{4}$$

for some smooth $\mathbb{R}$-multiplier τ of G^* and some measurable function a from G^* to $\mathbf{T}$.

We claim that

$$\tau(k, g^*) = \tau(g^*, k) \qquad k \in \mathrm{Ker}\ \delta,\ g^* \in G^*. \tag{5}$$

In fact, let $k \in \mathrm{Ker}\ \delta$ and $g^* \in G^*$, since $\mu^*(k, g^*) = \mu^*(g^*, k) = 1$, then

$$e^{i\tau(k,g^*)} = \frac{a(k)a(g^*)}{a(kg^*)} = \frac{a(k)a(g^*)}{a(g^*k)} = e^{i\tau(g^*,k)}.$$

Hence $\tau(k, g^*) = \tau(g^*, k) + 2\pi n(k, g^*)$ where $n(k, g^*)$ is an integer. By continuity of $\tau(k, \cdot)$ and since G^* is connected, the map $n(\cdot, \cdot)$ depends only on k, and, choosing $g^* = k$, we conclude that $n(k, g^*) = 0$ for all $k \in \mathrm{Ker}\ \delta$, $g^* \in G^*$.

Due to (5), the equivalence class of τ belongs to $H^2(G^*, \mathbb{R})_\delta$ and, by definition of $\overline{\tau}$, there is $w \in \mathbb{R}^N$ such that, up to equivalence, $\tau = w \cdot \overline{\tau}$. Hence (4) becomes

$$\mu^*(g_1^*, g_2^*) = \frac{a(g_1^* g_2^*)}{a(g_1^*)a(g_2^*)} e^{iw\cdot\overline{\tau}(g_1^*, g_2^*)}, \qquad g_1^*, g_2^* \in G^*. \tag{6}$$

The previous equality implies that the map χ from K to $\mathbf{T}$

$$\chi(v, k) := e^{iw\cdot v} a(k), \qquad v \in \mathbb{R}^N, k \in \mathrm{Ker}\ \delta,$$

is, in fact, a character of K. Hence, by Lemma 4, χ defines a $\mathbf{T}$-multiplier μ_χ of G.

We will show that μ_χ is equivalent to μ. In fact, using equation (3), one has

$$\begin{aligned} \mu_\chi(g_1, g_2) &= \chi(\Gamma_c(g_1, g_2)) \\ &= e^{iw\cdot(\ \overline{\tau}(\tilde{c}(g_1), \tilde{c}(g_2)) - \overline{\tau}(\gamma(g_1, g_2), \tilde{c}(g_1 g_2))\ )} a(\gamma(g_1, g_2)) \end{aligned}$$

Using twice equation (6) we obtain

$$\begin{aligned} e^{iw\cdot\overline{\tau}(\tilde{c}(g_1), \tilde{c}(g_2))} &= \frac{a(\tilde{c}(g_1))a(\tilde{c}(g_2))}{a(\tilde{c}(g_1)\tilde{c}(g_2))} \mu(g_1, g_2) \\ e^{-iw\cdot\overline{\tau}(\gamma(g_1, g_2), \tilde{c}(g_1 g_2))} &= \frac{a(\tilde{c}(g_1)\tilde{c}(g_2))}{a(\gamma(g_1, g_2))a(\tilde{c}(g_1 g_2))} \end{aligned}$$

so that

$$\mu_\chi(g_1,g_2)=\frac{a(\tilde{c}(g_1))a(\tilde{c}(g_2))}{a(\tilde{c}(g_1g_2))}\mu(g_1,g_2),$$

which shows the equivalence of μ and μ_χ.

Suppose now that χ is a character of K that extends to a character of $\overline{G}$ (still denoted by χ). Then

$$\begin{aligned}\mu_\chi(g_1,g_2) &= \chi(c(g_1)c(g_2)c(g_1g_2)^{-1})\\ &= \chi(c(g_1))\chi(c(g_2))\chi(c(g_1g_2)^{-1}),\end{aligned}$$

showing that μ_χ is exact. Conversely, assume that

$$\mu_\chi(g_1,g_2)=\frac{a(g_1g_2)}{a(g_1)a(g_2)}$$

for some measurable function $a: G\to \mathbf{T}$. Observe that, for all $h\in\overline{G}$, $hc(\sigma(h))^{-1}\in K$ and define $\chi':\overline{G}\to\mathbf{T}$ as

$$\chi'(h)=\chi(hc(\sigma(h))^{-1})a(\sigma(h))^{-1}\qquad h\in\overline{G}.$$

Then χ' is a character of $\overline{G}$. Indeed, χ' is measurable, and if $h_1,h_2\in\overline{G}$,

$$\begin{aligned}\chi'(h_1)\chi'(h_2) &= \frac{\chi(h_1c(\sigma(h_1))^{-1}h_2c(\sigma(h_2))^{-1})}{a(\sigma(h_1))a(\sigma(h_2))}\\ &= \frac{\chi\left(h_1h_2c(\sigma(h_2))^{-1}c(\sigma(h_1))^{-1}\right)\mu_\chi(g_1,g_2)}{a(\sigma(h_1h_2))}\\ &= \frac{\chi\left(h_1h_2c(\sigma(h_2))^{-1}c(\sigma(h_1))^{-1}c(\sigma(h_1))c(\sigma(h_2))c(\sigma(h_1h_2))^{-1}\right)}{a(\sigma(h_1h_2))}\\ &= \chi\left(h_1h_2c(\sigma(h_1h_2))^{-1}\right)a(\sigma(h_1h_2))^{-1}\\ &= \chi'(h_1h_2).\end{aligned}$$

Moreover, since $a(e)=1$, $\chi'(k)=\chi(k)$ for all $k\in K$.

Hence, $H^2(G,\mathbf{T})$ is isomorphic, as an abstract group, to the quotient group K/V. This completes the proof but to observe that $\overline{G}$ is a splitting group for $\mathbf{T}$ in the sense of Moore (see Definition 3 of [11]) so that, using Theorem 6 of [11], $H^2(G,\mathbf{T})$ is isomorphic, as a topological group, to the quotient group K/V. □

As a consequence of the previous result, one has that $H^2(G,\mathbf{T})$ is Hausdorff if and only if V is closed in $\widehat{K}$. The following example, inspired by Moore [9], shows that this is not always the case.

Example 1. Let $G=\mathbf{T}^2\times\mathbb{R}^2$ be the Lie group with product

$$(z,\zeta,x,y)(z',\zeta',x',y')=\left(zz'e^{i\frac{2\pi}{\alpha}(xy'-yx')},\zeta\zeta'e^{i2\pi(xy'-yx')},x+x',y+y'\right),$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The universal covering group G^* of G is $G^* = \mathbb{R}^2 \times \mathbb{R}^2$ with product

$$(v, w, x, y)(v', w', x', y') = \\ \left(v + v' + (xy' - yx'), w + w' + (xy' - yx'), x + x', y + y'\right),$$

and with covering homomorphism δ from G^* to G

$$\delta(v, w, x, y) = (e^{i\frac{2\pi}{\alpha}v}, e^{i2\pi w}, x, y).$$

We have

$$\text{Ker } \delta = \{(\alpha n, m, 0, 0) \ : \ n, m \in \mathbb{Z}\}.$$

A simple algebraic calculation shows that any $\mathbb{R}$-multiplier of G^* is equivalent to one of the form

$$\tau\left((v, w, x, y), (v', w', x', y')\right) = Q((v, w), (x', y'))$$

where $Q(\,\cdot\,,\,\cdot\,)$ is a bilinear form on $\mathbb{R}^2 \times \mathbb{R}^2$. It follows that $H^2(G^*, \mathbb{R})_\delta$ is $\{0\}$ so that the universal central extension of G is G^* and $K = \text{Ker } \delta$. Then $\widehat{K}$ is isomorphic to $\mathbf{T} \times \mathbf{T}$.

Moreover, one checks that the characters of G^* are of the form

$$(v, w, x, y) \mapsto e^{ia(v-w)+bx+cy},$$

where $a, b, c \in \mathbb{R}$. Hence we have

$$V = \{(e^{i\alpha a}, e^{-ia}) \in \mathbf{T} \times \mathbf{T} \ : \ a \in \mathbb{R}\},$$

so that V is closed in $\mathbf{T} \times \mathbf{T}$ if and only if α is rational.

If V is closed, we can give a better description of $H^2(G, \mathbf{T})$. Define

$$K_0 = \{(v, k) \in K \ : b(k) = 1 \ \text{ for any character } b \text{ of } G^*\},$$

then K_0 is an abelian closed subgroup of K. Since V is closed, a standard result on abelian locally compact groups (see, for example, Theorem 4.39 of [16]) shows that $\widehat{K}/V$ is isomorphic to the dual group $\widehat{K_0}$ of K_0. In particular, any element $\chi \in \widehat{K_0}$ extends to an element $\widehat{\chi} \in \widehat{K}$ and $\widehat{\chi}$ is uniquely defined by χ, up to an element of V. Let μ_χ be the $\mathbf{T}$-multiplier of G defined by

$$\mu_\chi(g_1, g_2) = \widehat{\chi}(\Gamma_c(g_1, g_2)), \qquad g_1, g_2 \in g,$$

where Γ_c is defined in statement 4 of Lemma 3. As a consequence of Theorem 2, the equivalence class $[\mu_\chi]$ depends only on χ and not on the particular extension chosen.

Corollary 1. *If V is closed, the mapping*

$$\widehat{K_0} \ni \chi \mapsto [\mu_\chi] \in H^2(G, \mathbf{T})$$

is an isomorphism of topological groups.

We now turn to the representations of $\overline{G}$.

Definition 6. *A representation U of $\overline{G}$ satisfying the condition*

$$U_h \in \mathbf{T} \qquad \text{for all } h \in K, \tag{7}$$

is called admissible*.*

Definition 7. *Let U be an admissible representation, then its restriction to K is a character of K and, due to statement 2 of Lemma 2, is of the form*

$$U_{(v,k)} = e^{iw\cdot v}\epsilon(k), \qquad v \in \mathbb{R}^N,\ k \in Ker\,\delta,$$

for some $w \in \mathbb{R}^N$ and some character ϵ of $Ker\,\delta$. We call w the algebraic charge *of U and ϵ the* topological charge*.*

In the above definition we have followed the terminology of Divakaran, [13].

Definition 8. *Let U and U' be two representations of $\overline{G}$ acting respectively in $\mathcal{H}$ and $\mathcal{H}'$. We say that U and U' are* physically equivalent *if there exist a unitary or antiunitary operator $B : \mathcal{H} \to \mathcal{H}'$ and a measurable map $b : \overline{G} \to \mathbf{T}$ such that*

$$BU_h = b(h)U'_h B, \qquad h \in \overline{G}. \tag{8}$$

We notice the following facts concerning these definitions.

1. The notion of physical equivalence preserves condition (7) and the usual notion of irreducibility of representations. The case of unitary equivalence is a particular instance of the physical equivalence.
2. Since K is central in $\overline{G}$, every irreducible representation of $\overline{G}$ is admissible.
3. Since in (8) U and U' are representations, the map b is, in fact, a character of $\overline{G}$ and, by statement 1 of Lemma 3, a character of G^*.

We are now in a position to state the main property of $\overline{G}$. Given an admissible representation U of $\overline{G}$, define, for all $g \in G$,

$$\alpha^U_g = \pi(U_h), \tag{9}$$

where π is defined in Proposition 1 and $h \in \overline{G}$ is such that $\sigma(h) = g$. The following theorem is then obtained.

Theorem 3. *With the above notations, α^U is a symmetry action of G and the correspondence*

$$[U] \mapsto [\alpha^U]$$

between the physical equivalence classes of admissible representations of $\overline{G}$ and the equivalence classes of symmetry actions of G is a bijection. The representation U of $\overline{G}$ is irreducible if and only if α^U is an irreducible symmetry action of G.

Proof. In the following we fix a section $c : G \to \overline{G}$ for σ (cf. item 3 of Lemma 3) and a section $s : \mathcal{S}_0 \to \mathcal{U}$ for $\pi : \mathcal{U} \to \mathcal{S}_0$ (cf. item 4 of Proposition 1).

Due to condition (7), if $h_1, h_2 \in \overline{G}$ are such that $\sigma(h_1) = \sigma(h_2) = g$, then $\pi(U_{h_1}) = \pi(U_{h_2})$, showing that α^U_g is well-defined. In particular, we have

$$\alpha^U_g = \pi(U_{c(g)}) \qquad g \in G.$$

First we show that $g \mapsto \alpha^U_g$ is a symmetry action of G. Indeed, if $g_1, g_2 \in G$ then

$$\begin{aligned}
\alpha^U_{g_1}\alpha^U_{g_2} &= \pi(U_{c(g_1)})\pi(U_{c(g_2)}) \\
&= \pi(U_{c(g_1)}U_{c(g_2)}) \\
&= \pi(U_{c(g_1)c(g_2)c(g_1g_2)^{-1}})\pi(U_{c(g_1g_2)}) \\
&= \pi(U_{c(g_1g_2)}) = \alpha^U_{g_1g_2},
\end{aligned}$$

where we used the fact that $c(g_1)c(g_2)c(g_1g_2)^{-1} \in K$ and condition (7). Since c is measurable, α^U is measurable too, and $\alpha^U_e = I$. Hence α^U is a symmetry action of G.

Now let U and U' be two physically equivalent admissible representations of $\overline{G}$ acting on $\mathcal{H}$ and $\mathcal{H}'$, respectively, then the corresponding symmetry actions α^U and $\alpha^{U'}$ are equivalent too. Indeed, in this case

$$BU_h = b(h)U'_hB, \qquad h \in \overline{G},$$

for some unitary or antiunitary $B : \mathcal{H} \to \mathcal{H}'$ and a character $b : \overline{G} \to \mathbf{T}$. Define β from $\mathcal{P}_{\mathcal{H}}$ to $\mathcal{P}_{\mathcal{H}'}$ as $\beta(P) = BPB^{-1}$. Then β is bijective, preserves the transition probabilities and satisfies $\beta\alpha^U_g = \alpha^{U'}_g\beta$ for all $g \in G$, which is just to say that α^U and $\alpha^{U'}$ are equivalent. This shows that the map $[U] \mapsto [\alpha^U]$ is well-defined.

We now show its surjectivity. Let α be a symmetry action of G, define $\mu : G \times G \to \mathcal{U}$ as

$$\mu(g_1, g_2) := s(\alpha_{g_1})s(\alpha_{g_2})s(\alpha_{g_1g_2})^{-1}, \qquad g_1, g_2 \in G.$$

Since $\pi(\mu(g_1, g_2)) = I$ then $\mu(g_1, g_2) \in \mathbf{T}$. Moreover, μ is measurable and by a direct computation one confirms that μ is, in fact, a $\mathbf{T}$-multiplier of G. By Theorem 2, there are a character χ of K and a measurable function $a : G \to \mathbf{T}$ such that

$$\mu(g_1, g_2) = \frac{a(g_1g_2)}{a(g_1)a(g_2)}\mu_\chi(g_1, g_2) \qquad g_1, g_2 \in G. \tag{10}$$

Define a map $U^\alpha : \overline{G} \to \mathcal{U}$ as

$$U^\alpha_h := \chi(hc(\sigma(h))^{-1})a(\sigma(h))s(\alpha_{\sigma(h)}), \qquad h \in \overline{G}.$$

Then U^α is a representation of $\overline{G}$. Indeed,

1. U^α is measurable as a composition of measurable maps;
2. $U^\alpha_{(0,e^*)} = I$, since $a(e) = 1$ and $s(I) = I$;

3. for any $h_1, h_2 \in \overline{G}$,

$$\begin{aligned}
U_{h_1}U_{h_2} &= \chi(h_1c(\sigma(h_1))^{-1}h_2c(\sigma(h_2))^{-1})a(\sigma(h_1))a(\sigma(h_2))s(\alpha_{\sigma(h_1)})s(\alpha_{\sigma(h_2)}) \\
&= \chi\left(h_1h_2c(\sigma(h_2))^{-1}c(\sigma(h_1))^{-1}\right) \\
&\quad\ a(\sigma(h_1))a(\sigma(h_2))\mu(g_1,g_2)s(\alpha_{\sigma(h_1h_2)}) \\
&= \chi\left(h_1h_2c(\sigma(h_2))^{-1}c(\sigma(h_1))^{-1}\right)\chi\left(c(\sigma(h_1))c(\sigma(h_2))c(\sigma(h_1h_2))^{-1}\right) \\
&\quad\ a(\sigma(h_1h_2))s(\alpha_{\sigma(h_1h_2)}) \\
&= \chi\left(h_1h_2c(\sigma(h_1h_2))^{-1}\right)a(\sigma(h_1h_2))s(\alpha_{\sigma(h_1h_2)}) \\
&= U_{h_1h_2}.
\end{aligned}$$

Since $\pi \circ s = id_{\mathcal{S}_0}$ and $\sigma \circ c = id_G$, one readily verifies that $\alpha^{U^\alpha} = \alpha$, proving the surjectivity of the map $[U] \mapsto [\alpha^U]$.

Assume next that α^U and $\alpha^{U'}$ are equivalent symmetry actions. By the definition of equivalence of symmetry actions, there is a bijective map β, preserving the transition probabilities, such that, for all $g \in G$,

$$\pi(U'_{c(g)})\beta = \beta\pi(U_{c(g)}).$$

Applying the Wigner theorem, [1], we deduce that for some unitary or antiunitary operator B and for some measurable map $b : G \to \mathbf{T}$,

$$U'_{c(g)} = b(g)BU_{c(g)}B^{-1}.$$

Let $h \in \overline{G}$, $g = \sigma(h)$, and $k = hc(g)^{-1}$, then $k \in K$ and

$$\begin{aligned}
U'_h &= U'_kU'_{c(g)} = U'_kb(c(g))BU_{c(g)}B^{-1} \\
&= U'_kb(c(g))BU_{k^{-1}}U_hB^{-1} = \hat{b}(h)BU_hB^{-1},
\end{aligned}$$

taking into account that, due to (7), U'_k and $U_{k^{-1}}$ are phase factors that we have collected in $\hat{b}$. This shows that U and U' are physically equivalent representations of $\overline{G}$, proving the injectivity of the map $[U] \mapsto [\alpha^U]$.

Finally, due to (7), an admissible representation U is irreducible if and only if

$$\left(\langle\phi_1, U_{c(g)}\phi_2\rangle = 0 \qquad \forall g \in G\right) \implies (\phi_1 = 0 \text{ or } \phi_2 = 0)\,.$$

This last condition is equivalent to the fact that α^U is irreducible. $\square$

This theorem shows that the equivalence classes of admissible representations of $\overline{G}$ classify the different (with respect to the given symmetry group G) quantum systems. In particular, the irreducible representations of $\overline{G}$ are always admissible and describe the elementary systems.

Let us now consider the case of reducible representations of $\overline{G}$. For the sake of simplicity, let $U = U_1 \oplus U_2$ where U_1 and U_2 are irreducible representations. Since the representations U_i are irreducible, they are admissible and we denote by w_i and ϵ_i the corresponding algebraic and topological charges. However, in general, U is not admissible. A simple calculation shows that U is admissible if and only if $w_1 = w_2$ and $\epsilon_1 = \epsilon_2$. This fact is at the root of the existence of superselection rules for non-elementary systems, as it will be discussed in more detail in the examples.

Furthermore, the relation between the decomposition into irreducible representations and the notion of physical equivalence requires some special care. One can easily show that, if b is a nontrivial character of $\overline{G}$ that is 1 on K, then $U_1 \oplus U_2$ and $U_1 \oplus b\,U_2$ are physically inequivalent admissible representations, even though U_2 and bU_2 are physically equivalent. In the same way, if the algebraic charge w of U_1 and U_2 is zero and their topological charge ϵ is such that $\epsilon^2 \in V$, then $U_1 \oplus U_2$ and $U_1 \oplus bBU_2B^{-1}$ (where b is any character of $\overline{G}$ that extends ϵ^2 and B is any antiunitary operator) are physically inequivalent admissible representations, even though U_2 and bBU_2B^{-1} are physically equivalent. This kind of phenomenon does not occur if one considers the unitary equivalence instead of the physical one.

We add some comments about the relation between the admissible representations of $\overline{G}$ and the projective representations of G and G^*. Let U be an admissible representation of $\overline{G}$, w and ϵ its algebraic and topological charges.

1. The map $G^* \ni g^* \mapsto U_{(0,g^*)} \in \mathcal{U}$ is a projective representation of G^* with $\mathbf{T}$-multiplier $\mu^*(g_1^*, g_2^*) = e^{iw\cdot\bar{\tau}(g_1^*,g_2^*)}$.
2. If $c : G \to \overline{G}$ is a section for σ then the map $G \ni g \mapsto U_{c(g)} \in \mathcal{U}$ is a projective representation of G and its $\mathbf{T}$-multiplier is μ_χ where $\chi(v, k) = e^{iw\cdot v}\epsilon(k)$ and μ_χ is defined in Lemma 4. As a consequence of statement 5 of Lemma 3, μ^* and $\mu_\chi\circ(\delta\times\delta)$ are equivalent, nevertheless, even if μ^* is exact, μ_χ could be *nonexact* (see remarks 1 in Section 4.1 and 1 in Section 4.2).

3.2. **The physical equivalence for semidirect products.** According to Theorem 3, the irreducible inequivalent symmetry actions of a group G are completely described by the irreducible physically inequivalent representations of its universal central extension $\overline{G}$. In the examples we consider in the next section, the universal central extension is a regular semidirect product with abelian normal subgroup, so that any irreducible representation is unitarily equivalent to some induced one, [17]. In this way, the problem of characterising physically inequivalent irreducible representations is reduced to the analogous problem for the induced ones. In the present section we describe the solution in terms of properties of the orbits in the dual space and of the inducing representations.

Let $\overline{G} = A \times' H$ be a Lie group with A an abelian normal closed subgroup and H a closed subgroup. We denote by $\hat{A}$ the dual group of A and by $(g, \cdot) \mapsto g[\cdot]$ both the inner action of $\overline{G}$ on A and the dual action of $\overline{G}$ on $\hat{A}$. If $x \in \hat{A}$, let $\overline{G}_x$ be the stability subgroup of $\overline{G}$ at x and $\overline{G}[x]$ the corresponding orbit. We assume that each orbit in $\hat{A}$ is locally closed (i.e. the semidirect product is regular) and, to simplify the exposition, that it has a $\overline{G}$-invariant σ-finite measure.

Moreover, given $x \in \hat{A}$ and a representation D of $\overline{G}_x \cap H$ acting in a Hilbert space $\mathcal{K}$, we denote by $U = \mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD)$ the representation of $\overline{G}$ unitarily induced by the representation xD of $\overline{G}_x$,

$$(xD)_{ah} = x_a D_h, \qquad a \in A,\ h \in \overline{G}_x \cap H.$$

Explicitly, let ν be a $\overline{G}$-invariant σ-finite measure on $\overline{G}[x]$ and c a measurable map from $\overline{G}[x]$ to $\overline{G}$ such that $c(x) = e$ and $c(y)[x] = y$ for all $y \in \overline{G}[x]$ (we call such a map a *section* for $\overline{G}[x]$), then U acts on the Hilbert space $L^2(\overline{G}[x], \nu, \mathcal{K})$ as

$$(U_g f)(y) = (xD)_{(c(y)^{-1} g c(g^{-1}[y]))} f(g^{-1}[y]),$$

where $y \in \overline{G}[x]$, $f \in L^2(\overline{G}[x], \nu, \mathcal{K})$, and $g \in \overline{G}$.

We are now in a position to classify all the equivalence classes (with respect to the notion of physical equivalence) of irreducible representations of $\overline{G}$ in the case of regular semidirect products.

Let $\hat{A}_s$ be the set of singleton $\overline{G}$-orbits in $\hat{A}$, i.e.,

$$\hat{A}_s = \{y \in \hat{A} \ : \ g[y] = y, \quad g \in \overline{G}\}.$$

Define for all $x \in \hat{A}$ the *orbit class*

$$\widetilde{\mathcal{O}}_x := \{yg[x^\epsilon] \ : \ y \in \hat{A}_s, \quad g \in \overline{G},\ \epsilon = \pm 1\}.$$

Obviously, for all $x' \in \widetilde{\mathcal{O}}_x$, $\overline{G}[x'] \subset \widetilde{\mathcal{O}}_x$ and $\widetilde{\mathcal{O}}_x = \widetilde{\mathcal{O}}_{x'}$, so that we can choose a family $\{x_i\}_{i \in I}$ of elements in $\hat{A}$ such that $\hat{A}$ is the disjoint union of the sets $\widetilde{\mathcal{O}}_{x_i}$.

Theorem 4. *Let $\overline{G} = A \times' H$ be a regular semidirect product.*

1. *Every irreducible representation of $\overline{G}$ is physically equivalent to one of the form $\mathrm{Ind}_{\overline{G}_{x_i}}^{\overline{G}}(x_i D)$ for some index i and some irreducible representation D of $\overline{G}_{x_i} \cap H$.*
2. *If $i \neq j$ and D, D' are two representations of $\overline{G}_{x_i} \cap H$ and $\overline{G}_{x_j} \cap H$, respectively, then $\mathrm{Ind}_{\overline{G}_{x_i}}^{\overline{G}}(x_i D)$ and $\mathrm{Ind}_{\overline{G}_{x_j}}^{\overline{G}}(x_j D')$ are physically inequivalent.*
3. *Let $x \in \hat{A}$ and D, D' be two representations of $\overline{G}_x \cap H$. Then $\mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD)$ and $\mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD')$ are physically equivalent if and only if one of the following two conditions is satisfied:*
   (a) *there exist $y \in \hat{A}_s$, a character χ of H, and a unitary operator M such that*

$$\begin{aligned} \overline{G}[x] &= y\overline{G}[x], \\ D'_{hsh^{-1}} &= \chi_s M D_s M^{-1}, \qquad s \in \overline{G}_x \cap H, \end{aligned}$$

   where $h \in H$ is such that $x = yh[x]$;

(b) *there exist $y \in \hat{A}_s$, a character χ of H, and an antiunitary operator M such that*

$$\begin{aligned} \overline{G}[x] &= y\overline{G}[x]^{-1}, \\ D'_{hsh^{-1}} &= \chi_s M D_s M^{-1}, \qquad s \in \overline{G}_x \cap H, \end{aligned}$$

where $h \in H$ is such that $x = yh[x^{-1}]$.

Motivated by the above theorem, if U is a representation of $\overline{G}$ physically equivalent to some induced representation $\mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD)$ we say, with slight abuse of terminology, that U *lives* on the orbit class $\widetilde{\mathcal{O}}_x$.

The proof of the theorem is based on the following lemma.

Lemma 5. *Let $x, x' \in \hat{A}$. Let D be a representation of $\overline{G}_x \cap H$ acting in $\mathcal{K}$ and D' a representation of $\overline{G}_{x'} \cap H$ acting in $\mathcal{K}'$. The induced representations $Ind_{\overline{G}_x}^{\overline{G}}(xD)$ and $Ind_{\overline{G}_{x'}}^{\overline{G}}(x'D')$ are physically equivalent if and only if there exist $h \in \overline{G}$, a character $\widetilde{\chi}$ of $\overline{G}$ and a unitary or antiunitary operator M from $\mathcal{K}$ onto $\mathcal{K}'$ such that*

1. $\overline{G}_{x'} = h\overline{G}_x h^{-1}$;
2. $(x'D')_{hgh^{-1}} = \widetilde{\chi}_g M (xD)_g M^{-1}$ *for all* $g \in \overline{G}_x$.

Moreover, every character of $\overline{G}$ is of the form

$$(a, h) \mapsto \hat{\chi}_a \chi_h \qquad a \in A, \ h \in H$$

where $\hat{\chi} \in \hat{A}_s$ and χ is a character of H.

Proof. First we prove the statement on the characters of $\overline{G}$. If $\widetilde{\chi}$ is a character of $\overline{G}$, let $\hat{\chi}$ and χ be its restrictions to A and H, respectively. Then χ is a character of H and, by definition of dual action, $\hat{\chi} \in \hat{A}_s$. The proof of the converse implication is similar.

We now turn to the first statement. To simplify the notations, denote $U = \mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD)$ and $U' = \mathrm{Ind}_{\overline{G}_{x'}}^{\overline{G}}(x'D')$. The representations U and U' are physically equivalent if and only if there exist a character $\widetilde{\chi}$ of $\overline{G}$ and a unitary or antiunitary operator B such that

$$U' = \widetilde{\chi} B^{-1} U B.$$

As a first step we define in terms of U and $\widetilde{\chi}$ two induced representations U^+ and U^- of $\overline{G}$ such that

$$U^{\pm} = \widetilde{\chi} W_{\pm}^{-1} U W_{\pm},$$

where W_+ [resp. W_-] is unitary [resp. antiunitary]. In particular, U^+ and U^- are physically equivalent to U.

By the previous result $\widetilde{\chi} = \hat{\chi}\chi$, where $\hat{\chi} \in \hat{A}_s$ and χ is a character of H. Define the maps ψ_+ and ψ_- from $\hat{A}$ onto $\hat{A}$ as $\psi_{\pm}(x) := \hat{\chi} x^{\pm 1}$. The maps $\psi_{\pm}$ are measurable isomorphisms that commute with the action of $\overline{G}$, so that $\psi_{\pm}$ maps the orbit $\overline{G}[x]$ onto the orbit $\overline{G}[\psi_{\pm}(x)]$ and one has $\overline{G}_x = \overline{G}_{\psi_{\pm}(x)}$. If ν is an invariant measure on $\overline{G}[x]$, the image measure $\nu^{\pm}$ with respect to

$\psi_\pm$ is an invariant measure on $\overline{G}[\psi_\pm(x)]$ and if c is a section for the orbit $\overline{G}[x]$, then $c^\pm = c\circ\psi_\pm^{-1}$ is a section for the action of $\overline{G}$ on the orbit $\overline{G}[\psi_\pm(x)]$.

Fix a unitary operator L_+ and an antiunitary operator L_- on $\mathcal{K}$. Consider the representations of $\overline{G}_x$,

$$g \mapsto \widetilde{\chi}_g L_\pm (xD)_g L_\pm^{-1},$$

and observe that their restriction to A are exactly the elements $x_\pm := \psi_\pm(x)$. Since $\overline{G}_{x_\pm} = \overline{G}_x$ we can define the induced representations of $\overline{G}$,

$$U^\pm := \mathrm{Ind}_{\overline{G}_{x_\pm}}^{\overline{G}} (\widetilde{\chi} L_\pm x D L_\pm^{-1}),$$

acting in $L^2(\overline{G}[x_\pm], \nu^\pm, \mathcal{K})$.

Moreover, define the operators $W_\pm$ from $L^2(\overline{G}[x_\pm], \nu^\pm, \mathcal{K})$ onto $L^2(\overline{G}[x], \nu, \mathcal{K})$

$$(W_\pm f)(y) = \widetilde{\chi}^{\pm 1}_{c(y)} L_\pm^{-1} f(\psi_\pm(y)), \qquad y \in \overline{G}[x].$$

It is easy to show that W_+ [resp. W_-] is unitary [resp. antiunitary].

We have

$$U^\pm = \widetilde{\chi} W_\pm^{-1} U W_\pm.$$

In fact, let $g \in \overline{G}$, $f \in L^2(\overline{G}[x_\pm], \nu^\pm, \mathcal{K})$, and $y \in \overline{G}[x_\pm]$

$$\begin{aligned}
\widetilde{\chi}_g \left(W_\pm^{-1} U_g W_\pm f\right)(y) &= \widetilde{\chi}_g \widetilde{\chi}^{-1}_{c^\pm(y)} L_\pm \left(U_g W_\pm f\right)(\psi_\pm^{-1}(y)) \\
&= \widetilde{\chi}_g \widetilde{\chi}^{-1}_{c^\pm(y)} L_\pm (xD)_{\gamma^\pm(g,y)} (W_\pm f)(g^{-1}[\psi_\pm^{-1}(y)]) \\
&= \widetilde{\chi}_g \widetilde{\chi}^{-1}_{c^\pm(y)} L_\pm (xD)_{\gamma^\pm(g,y)} \widetilde{\chi}^{\pm 1}_{c^\pm(g^{-1}[y])} L_\pm^{-1} f(g^{-1}[y]) \\
&= (\widetilde{\chi} L_\pm x D L_\pm^{-1})_{\gamma^\pm(g,y)} f(g^{-1}[y]) \\
&= (U_g^\pm f)(y).
\end{aligned}$$

where $\gamma^\pm(g,y) = c^\pm(y)^{-1} g c^\pm(g^{-1}[y]) = c(\psi_\pm^{-1}(y))^{-1} g c(g^{-1}[\psi_\pm^{-1}(y)])$.

To conclude the proof of the lemma, observe first that there always exist a unitary operator V such that either $B = W_+ V$ or $B = W_- V$, according to the fact that B is unitary or antiunitary. Hence U and U' are physically equivalent if and only if U' is unitarily equivalent either to U^+ or to U^-. Due to a theorem of Mackey (see, for example, Theorem 6.42 of [16]), this is possible if and only if there exist $h \in \overline{G}$ such that $\overline{G}_{x'} = h\overline{G}_x h^{-1}$ and a unitary or antiunitary operator M (depending on the fact that B is unitary or antiunitary) such that $(x'D')_{hgh^{-1}} = \widetilde{\chi}_g M (xD)_g M^{-1}$ for all $g \in \overline{G}_x$. □

We turn to the proof of Theorem 4.

Proof of Theorem 4.

1. Since the semidirect product is regular, a theorem of Mackey (see, for example, Theorem 6.42 of [16]) implies that each irreducible unitary representation of $\overline{G}$ is unitarily (hence physically) equivalent to one of the form $\mathrm{Ind}_{\overline{G}_x}^{\overline{G}}(xD')$ for some $x \in \hat{A}$ and some irreducible representation D' of $\overline{G}_x \cap H$. There is an index i such that $x \in \widetilde{\mathcal{O}}_{x_i}$ and, by definition of orbit class, there exist $y \in \hat{A}_s$ and $h \in \overline{G}$ such that

$x = yh[x_i^\epsilon]$ where $\epsilon = \pm 1$. Hence $G_x = hG_{x_i}h^{-1}$ and we can define a representation D of $G_{x_i} \cap H$ either as

$$D_g = D'_{h^{-1}gh}, \qquad g \in G_{x_i} \cap H,$$

if $\epsilon = 1$, or as

$$D_g = MD'_{h^{-1}gh}M^{-1}, \qquad g \in G_{x_i} \cap H,$$

if $\epsilon = -1$, where M is a fixed antiunitary operator. Then, by Lemma 5, $\mathrm{Ind}_{\overline{G_x}}^{\overline{G}}(xD')$ is physically equivalent to $\mathrm{Ind}_{\overline{G_{x_i}}}^{\overline{G}}(x_iD)$.

2. If $\mathrm{Ind}_{\overline{G_{x_i}}}^{\overline{G}}(x_iD)$ and $\mathrm{Ind}_{\overline{G_{x_j}}}^{\overline{G}}(x_jD')$ are physically equivalent, the condition 2 of Lemma 5 with the choice $g = a \in A$ implies that $x_j = yh[x_i^\epsilon]$ for some $y \in \hat{A}_s$ and $\epsilon = \pm 1$, so that, by definition of x_i, $i = j$.
3. Apply Lemma 5 with $x = x'$, taking into account the form of the characters of $\overline{G}$.

□

We observe that if D' is unitarily equivalent to D, the conditions (a) of item 3 of Theorem 4 are satisfied with $y = 1$, $\widetilde{\chi} = 1$, and $h = e$ and this is exactly the case of unitary equivalence of the induced representations. However, in general, there are other possibilities apart from the unitary equivalence. There are even situations in which both conditions (a) and (b) hold.

4. Examples

In this section we give a brief review of the classification of the *free quantum particles* for the Poincaré group and for the Galilei group using the framework introduced in the previous section. We consider the case of the Galilei group in $2+1$ dimensions and we confront our results with the ones obtained by Bose, [19].

4.1. **The Poincaré group.**

Let G be the connected component of the Poincaré group, which is the semidirect product of $A = \mathbb{R}^4$ and the connected component H of the Lorentz group. The covering group G^* of G is the semidirect product of $A = \mathbb{R}^4$ and $SL(2, \mathbb{C})$. It is a standard result (see, for example, [12]) that each multiplier of G^* is exact. Hence, the universal central extension of G is its universal covering group and our Theorem 3 reduces, in this case, to Theorem 7.40 of [12]. The classification of relativistic free quantum particles is thus traced back to the problem of classifying the irreducible representations of G^*. This problem was first solved by Wigner, [5], in terms of two parameters: the mass, labelling the orbits in the dual group of A, and the spin, labelling the irreducible representations of the stability group at the origin of each orbit.

We add some comments.

1. Since every multiplier of G^* is exact, we have that

$$K = \mathrm{Ker}\,\delta = \mathbb{Z}_2.$$

Moreover, G^* has only the trivial character, since the only singleton orbit in $\hat{A}$ is the origin and the semisimple group $SL(2,\mathbb{C})$ has no nontrivial characters. Hence, by Theorem 2, $H^2(G,\mathbf{T})$ is isomorphic to $\widehat{K} \simeq \mathbb{Z}_2$. Explicitly, any $\mathbf{T}$-multiplier of G is either exact or equivalent to

$$(g,g') \mapsto \epsilon(c(g)c(g')c(gg')^{-1}),$$

where c is a section for the covering homomorphism δ and ϵ is the nontrivial character of K.

2. Since G^* has only the trivial character, two representations of G^* are physically equivalent if and only if they are either unitarily or antiunitarily equivalent.
3. For reducible representations the admissibility condition (7) gives rise to the superselection rule that does not allow the superposition among fermions and bosons.

4.2. **The Galilei group.** We discuss this case in more details since it presents a nontrivial application of the notion of universal central extension.

Let $\mathcal{V} := (\mathbb{R}^3,+)$ be the group of velocity transformations and let $SO(3)$ be the rotation group in $\mathbb{R}^3$. The group $SO(3)$ acts on $\mathcal{V}$ in a natural way and we can consider the corresponding semidirect product, which is the *homogeneous Galilei group*,

$$G_0 := \mathcal{V} \times' SO(3).$$

The elements of G_0 are denoted by $(\vec{v},R)$. Let $\mathcal{T}_s := (\mathbb{R}^3,+)$ be the group of space translations and $\mathcal{T}_t := (\mathbb{R},+)$ the group of time translations; we denote the group of space-time translations by $\mathcal{T} := \mathcal{T}_s \times \mathcal{T}_t$ and its elements by $(\vec{a},b)$. The action of G_0 on $\mathcal{T}$ is defined by

$$(\vec{v},R)[(\vec{a},b)] := (R\vec{a}+b\vec{v},b), \qquad (\vec{v},R)\in G_0,\ (\vec{a},b)\in\mathcal{T},$$

and the corresponding semidirect product

$$G := \mathcal{T} \times' G_0$$

is the *Galilei group*. For any $g \in G$ we write $g = (\vec{a},b,\vec{v},R)$.

The covering group of G is

$$G^* = \mathcal{T} \times' (\mathcal{V} \times' SU(2)),$$

where $SU(2)$ acts on $\mathcal{V}$ and $\mathcal{V} \times' SU(2)$ acts on $\mathcal{T}$ in a natural way using the covering homomorphism δ from $SU(2)$ onto $SO(3)$. We denote again by δ the covering homomorphism $G^* \to G$ (this is a small abuse of notation that does not cause any confusion) and we notice that Ker $\delta = \{(\vec{0},0,\vec{0},\pm I)\}$. The corresponding Lie algebra is, as a vector space,

$$\begin{aligned}\mathrm{Lie}\,(G^*) &= \mathrm{Lie}\,(\mathcal{T}) \oplus \mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(SU(2)) \\ &= \mathbb{R}^4 \oplus \mathbb{R}^3 \oplus su\,(2),\end{aligned}$$

and we denote its elements by $(\vec{a}, b, \vec{v}, A)$, with $b \in \mathbb{R}$, $\vec{a}, \vec{v} \in \mathbb{R}^3$ and $A \in su(2)$.

We apply the results of Section 2.2 to compute the multipliers of G^*.

A classical result of Bargmann, [6], shows that $H^2(\mathrm{Lie}\,(G^*), \mathbb{R})$ is one dimensional and a nonexact closed $\mathbb{R}$-form is given by

$$F\left((\vec{a}_1, b_1, \vec{v}_1, A_1), (\vec{a}_2, b_2, \vec{v}_2, A_2)\right) = \vec{v}_1 \cdot \vec{a}_2 - \vec{v}_2 \cdot \vec{a}_1,$$

where $\cdot$ denotes the scalar product on $\mathbb{R}^3$. To compute the $\mathbb{R}$-multiplier τ_F corresponding to F, we have to exhibit the simply connected Lie group G^*_F such that

$$\mathrm{Lie}\,(G^*_F) \quad = \quad \mathbb{R} \oplus_F \mathrm{Lie}\,(G^*).$$

Denote the elements of $\mathrm{Lie}\,(G^*_F)$ by $(c, \vec{a}, b, \vec{v}, A)$. By a direct computation one can confirm that

$$\{(\vec{v}, A) \equiv (0, \vec{0}, 0, \vec{v}, A) \ : \ (\vec{v}, A) \in \mathrm{Lie}\,(\mathcal{V}) \oplus su\,(2)\}$$

is a subalgebra of $\mathrm{Lie}\,(G^*_F)$ isomorphic to $\mathrm{Lie}\,(\mathcal{V} \times' SU\,(2))$, and that

$$\{(c, \vec{a}, b) \equiv (c, \vec{a}, b, \vec{0}, 0) \ : \ (c, \vec{a}, b) \in \mathbb{R} \oplus \mathrm{Lie}\,(\mathcal{T})\}$$

is an abelian ideal of $\mathrm{Lie}\,(G^*_F)$ isomorphic to $\mathrm{Lie}\,(\mathbb{R} \times \mathcal{T})$. Hence, $\mathrm{Lie}\,(G^*_F)$ is isomorphic to the semidirect sum of $\mathrm{Lie}\,(\mathbb{R} \times \mathcal{T})$ and $\mathrm{Lie}\,(\mathcal{V} \times' SU\,(2))$.

Explicitly, if $(\vec{v}, A) \in \mathrm{Lie}\,(\mathcal{V} \times' SU\,(2))$ and $(c, \vec{a}, b) \in \mathrm{Lie}\,(\mathbb{R} \times \mathcal{T})$, one has

$$\begin{aligned}
[(\vec{v}, A), (c, \vec{a}, b)] &= (\vec{v} \cdot \vec{a}, \dot{\delta}(A)\vec{a} + b\vec{v}, 0, \vec{0}, 0) \\
&=: \dot{\rho}(\vec{v}, A)(c, \vec{a}, b),
\end{aligned}$$

with $\dot{\rho}(\vec{v}, A)$ denoting the 5×5 real matrix

$$\dot{\rho}(\vec{v}, A) = \begin{pmatrix} 0 & \vec{v} & 0 \\ \vec{0} & \dot{\delta}(A) & \vec{v} \\ 0 & \vec{0} & 0 \end{pmatrix},$$

which acts on the (column) vector $(c, \vec{a}, b) \in \mathrm{Lie}\,(\mathbb{R} \times \mathcal{T}) \simeq \mathbb{R} \times \mathcal{T}$.

Let ρ be the representation of $\mathcal{V} \times' SU(2)$ on $\mathbb{R} \times \mathcal{T}$ such that its differential at the identity is $\dot{\rho}$. Then G^*_F is the semidirect product of $\mathbb{R} \times \mathcal{T}$ and $\mathcal{V} \times' SU(2)$ with respect to the action defined by ρ. We denote the elements of G^*_F by $(c, \vec{a}, b, \vec{v}, h)$ where $b, c \in \mathbb{R}$, $\vec{a}, \vec{v} \in \mathbb{R}^3$ and $h \in SU\,(2)$.

To compute explicitly ρ, if $A \in su\,(2)$, then

$$\begin{aligned}
\rho(\vec{0}, e^A) &= e^{\dot{\rho}(\vec{0}, A)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n\,!}\, \dot{\rho}(\vec{0}, A)^n \\
&= \begin{pmatrix} 1 & \vec{0} & 0 \\ \vec{0} & e^{\dot{\delta}(A)} & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix}.
\end{aligned}$$

Thus, for all $h \in SU(2)$

$$\rho(\vec{0}, h) = \begin{pmatrix} 1 & \vec{0} & 0 \\ \vec{0} & \delta(h) & \vec{0} \\ 0 & \vec{0} & 1 \end{pmatrix}.$$

In a similar way, if $\vec{v} \in \mathcal{V}$, one gets

$$\rho(\vec{v}, I) = \begin{pmatrix} 1 & \vec{v} & \frac{1}{2}\vec{v}^2 \\ \vec{0} & I & \vec{v} \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, the action of $\mathcal{V} \times' SU(2)$ on $\mathbb{R} \times \mathcal{T}$ is explicitly given by

$$\rho(\vec{v}, h)[(c, \vec{a}, b)] = (c + \vec{v} \cdot \delta(h)\vec{a} + \tfrac{1}{2}b\vec{v}^2, \delta(h)\vec{a} + b\vec{v}, b),$$

with $(\vec{v}, h) \in \mathcal{V} \times' SU(2)$ and $(c, \vec{a}, b) \in \mathbb{R} \times \mathcal{T}$, and the multiplication law in G_F^* is

$$g_1g_2 = (c_1+c_2+\vec{v}_1\cdot\delta(h_1)\vec{a}_2+\tfrac{1}{2}b_2\vec{v}_1^2, \vec{a}_1+\delta(h_1)\vec{a}_2+b_2\vec{v}_1, b_1+b_2, \vec{v}_1+\delta(h_1)\vec{v}_2, h_1h_2),$$

for all $g_i = (c_i, \vec{a}_i, b_i, \vec{v}_i, h_i) \in G_F^*$, $i = 1, 2$. The corresponding $\mathbb{R}$-multiplier τ_F for G^* is

$$\tau_F(g_1^*, g_2^*) = \vec{v}_1 \cdot \delta(h_1)\vec{a}_2 + \tfrac{1}{2}b_2\vec{v}_1^2 \qquad g_1^*, g_2^* \in G^*.$$

We notice that the usual way to deduce τ_F from F is not so direct and requires more computations.

It is evident that, if $k \in \mathrm{Ker}\,\delta$ and $g^* \in G^*$,

$$\tau_F(k, g^*) = \tau_F(g^*, k),$$

so that $\dim H^2(G^*, \mathbb{R})_\delta = 1$. Since

$$\tau_F(k_1, k_2) = 0 \qquad k_1, k_2 \in \mathrm{Ker}\,\delta,$$

we can choose $\overline{\tau} = \tau_F$, and we have $\overline{G} = G_F^*$ and $K = \mathbb{R} \times \mathbb{Z}_2$.

Since $\overline{G}$ is the semidirect product of $A := \mathbb{R} \times \mathcal{T}$ and $H = \mathcal{V} \times' SU(2)$, we can use the results of Section 3.2 to classify the physically inequivalent irreducible representations of $\overline{G}$.

We identify the dual group $\hat{A}$ of A with $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ by the pairing

$$\langle (m, \vec{p}, E), (c, \vec{a}, b) \rangle = mc - \vec{p} \cdot \vec{a} + Eb.$$

With this identification, the dual action of $\overline{G}$ on $\hat{A}$ is

$$g[(m, \vec{p}, E)] = (m, \delta(h)\vec{p} + m\vec{v}, E + \tfrac{1}{2}m\vec{v}^2 + \vec{v} \cdot \delta(h)\vec{p}),$$

with $g = (c, \vec{a}, b, \vec{v}, h) \in \overline{G}$.

With respect to this action $\hat{A}$ splits into three kinds of orbits:

1. for each $E_0 \in \mathbb{R}, m \in \mathbb{R}, m \neq 0$,

$$\overline{G}[(m, \vec{0}, E_0)] = \{(m, \vec{p}, E) \,:\, \vec{p} \in \mathbb{R}^3, E = E_0 + \frac{\vec{p}^2}{2m}\};$$

2. for each $r \in \mathbb{R}, r > 0$,
$$\overline{G}[(0, \vec{p}_r, 0)] = \{(0, \vec{p}, E) \, : \, \vec{p} \in \mathbb{R}^3, \vec{p}^2 = r^2, E \in \mathbb{R}\},$$
where $\vec{p}_r = (0, 0, r)$;
3. for each $E_0 \in \mathbb{R}$,
$$\overline{G}[(0, \vec{0}, E_0)] = \{(0, \vec{0}, E_0)\}.$$

Since these orbits are closed in $\hat{A}$, the semidirect product is regular and we can apply Theorem 4.

To do this, we observe that the set of singleton orbits is
$$\hat{A}_s = \{(0, \vec{0}, E_0) \ : \ E_0 \in \mathbb{R}\},$$
and the orbit classes are the following:

1. for any $m > 0$,
$$\widetilde{\mathcal{O}}_{(m,\vec{0},0)} = \bigcup_{\vec{p} \in \mathbb{R}^3, E \in \mathbb{R}} \left(\overline{G}[(m, \vec{p}, E)] \cup \overline{G}[(-m, \vec{p}, E)] \right);$$
2. for any $r > 0$,
$$\widetilde{\mathcal{O}}_{(0,(0,0,r),0)} = \overline{G}[(0, (0, 0, r), 0)];$$
3. $$\widetilde{\mathcal{O}}_{(0,\vec{0},0)} = \bigcup_{E \in \mathbb{R}} \overline{G}[(0, \vec{0}, E)].$$

We consider only the set of orbit classes $\widetilde{\mathcal{O}}_{(m,\vec{0},0)}$, that have a direct physical interpretation. Define, for each $m > 0$,
$$p_m := (m, \vec{0}, 0) \in \widetilde{\mathcal{O}}_{(m,\vec{0},0)},$$
then the stability subgroup at p_m is $A \times' SU(2)$ and the irreducible unitary representations of $SU(2)$ are unitarily equivalent to those of the form D^j acting on the Hilbert space $\mathbb{C}^{2j+1}$, with $2j \in \mathbb{N}$. Moreover,

1. if $y \in \hat{A}_s$, $y \neq 0$, then $y\overline{G}[p_m] \neq \overline{G}[p_m]$;
2. $\overline{G}[p_m] \neq \overline{G}[p_m]^{-1}$
3. H has only the character 1,
4. the representations D^j and $D^{j'}$, with $j \neq j'$, act on Hilbert spaces with different dimension, so that they are unitarily inequivalent.

Applying Theorem 4, we conclude that every irreducible representation of $\overline{G}$ living on an orbit class $\widetilde{\mathcal{O}}_{p_m}$ is physically equivalent to an induced representation of the form
$$U^{m,j} := \mathrm{Ind}_{A \times' SU(2)}^{\overline{G}} (p_m D^j),$$
where the inducing representation $p_m D^j$ of $A \times' SU(2)$ is
$$(a, h) \mapsto e^{i \langle p_m, a \rangle} D^j(h).$$
Moreover, the set $\{U^{m,j} \ : \ m \in \mathbb{R}, \ m > 0, \ 2j \in \mathbb{N}\}$ is a family of physically inequivalent irreducible representations of $\overline{G}$.

Hence, by Theorem 3, the inequivalent irreducible symmetry actions of G are classified by two parameters $m > 0$ and $2j \in \mathbb{N}$. This result was obtained by Bargmann, [6].

We end with some comments.

1. The characters of G^* are of the form
$$G^* \ni (\vec{a}, b, \vec{v}, h) \mapsto e^{iEb} \in \mathbf{T},$$
where $E \in \mathbb{R}$. When restricted to K, any character is trivial. Hence, by Theorem 2, the group $H^2(G, \mathbf{T})$ is isomorphic to $\widehat{K} \simeq \mathbb{R} \times \mathbb{Z}_2$. The elements of $\widehat{K}$ are the maps
$$\mathbb{R} \times \mathbb{Z}_2 \ni (c, \xi) \mapsto e^{imc}\epsilon(\xi) \in \mathbf{T}$$
where $m \in \mathbb{R}$ and ϵ is a character of $\mathbb{Z}_2$. If s is a section for $\delta : SU(2) \to SO(3)$, we have that any $\mathbf{T}$-multiplier of G is equivalent to one of the form
$$\big((\vec{a}, b, \vec{v}, R), (\vec{a}', b', \vec{v}', R')\big) \mapsto e^{im(\vec{v}\cdot R\vec{a}' + \frac{1}{2}b'\vec{v}^2)}\epsilon(s(R)s(R')s(RR')^{-1}).$$
2. Let U be an admissible representation and $m \in \mathbb{R}$, $\epsilon \in \mathbb{Z}_2$ be the corresponding algebraic and topological charges. Then the algebraic charge m parametrises the orbits in the dual group (as in the Poincaré case) and it has the physical meaning of a mass. On the other hand, the topological charge is connected with the spin of the particles: the case $\epsilon = 1$ characterises the bosonic representations, while $\epsilon = -1$ corresponds to the fermionic ones.
3. If we consider a direct sum of irreducible representations, the admissibility condition (7) gives rise to two superselection rules. Namely, it does not allow superposition among particles with different masses (Bargmann superselection rule) and superposition among bosons and fermions.

4.3. **The Galilei group in $2 + 1$ dimensions.** From the physical point of view, the interest in the Galilei group in $2+1$ dimensions arises in solid state physics where some genuine examples of two dimensional systems can be found. The analysis of the multipliers of this group has been done by Bose [18]. The classification of the representations of the corresponding central extensions has been done in [19]. In the latter paper no discussion of the physical equivalence is given and this leads to misleading conclusions regarding the spin of elementary particles. For these reasons we consider anew this case here as a nontrivial application of our theory.

The Galilei group in $2 + 1$ dimensions is
$$G = \mathcal{T} \times' (\mathcal{V} \times' SO(2)),$$
where $\mathcal{T} = \mathcal{T}_s \times \mathcal{T}_t$, $\mathcal{T}_s = \mathbb{R}^2$, $\mathcal{T}_t = \mathbb{R}$, and $\mathcal{V} = \mathbb{R}^2$. The semidirect product structure is the analogous of the $3+1$ dimensional case. The covering group

is $G^* = \mathcal{T} \times' (\mathcal{V} \times' \mathbb{R})$ and we denote its element as $(\vec{a}, b, \vec{v}, r)$, where $\vec{a}, \vec{v} \in \mathbb{R}^2$, $b \in \mathbb{R}$ and $r \in \mathbb{R}$. The kernel of the covering homomorphism δ is

$$\{(\vec{0}, 0, \vec{0}, 2\pi k) \ : \ k \in \mathbb{Z}\}.$$

The Lie algebra of G^* is, as a vector space,

$$\begin{aligned} \mathrm{Lie}\,(G^*) &= \mathrm{Lie}\,(\mathcal{T}) \oplus \mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(\mathbb{R}) \\ &= \mathbb{R}^3 \oplus \mathbb{R}^2 \oplus \mathbb{R}, \end{aligned}$$

and we denote its elements by $(\vec{a}, b, \vec{v}, r)$, with $b, r \in \mathbb{R}$, $\vec{a}, \vec{v} \in \mathbb{R}^2$.

A result of Bose, [18], shows that $H^2(\mathrm{Lie}\,(G^*), \mathbb{R})$ is a three dimensional vector space and a basis is given by the equivalence classes of the following closed $\mathbb{R}$-forms:

$$\begin{aligned} F_1\left((\vec{a}_1, b_1, \vec{v}_1, r_1), (\vec{a}_2, b_2, \vec{v}_2, r_2)\right) &= r_1 b_2 - r_2 b_1, \\ F_2\left((\vec{a}_1, b_1, \vec{v}_1, r_1), (\vec{a}_2, b_2, \vec{v}_2, r_2)\right) &= \vec{v}_1 \cdot \vec{a}_2 - \vec{v}_2 \cdot \vec{a}_1, \\ F_3\left((\vec{a}_1, b_1, \vec{v}_1, r_1), (\vec{a}_2, b_2, \vec{v}_2, r_2)\right) &= \vec{v}_1 \wedge \vec{v}_2, \end{aligned}$$

where $\vec{v}_1 \wedge \vec{v}_2$ is a shorthand notation for $v_{1x} v_{2y} - v_{2x} v_{1y}$. Define F as the closed $\mathbb{R}^3$-form $F = (F_1, F_2, F_3)$. To compute the corresponding $\mathbb{R}^3$-multiplier τ_F of G^*, we have to determine the simply connected Lie group G^*_F with Lie algebra

$$\mathrm{Lie}\,(G^*_F) \ = \ \mathbb{R}^3 \oplus_F \mathrm{Lie}\,(G^*).$$

The algebra $\mathrm{Lie}\,(G^*_F)$ is, in fact, a semidirect sum. This can be shown as follows. Write $\mathrm{Lie}\,(G^*_F) = \mathbb{R}^2 \oplus \mathrm{Lie}\,(G^*) \oplus \mathbb{R}$ and its elements as (c_1, c_2, X, x) with $c_1, c_2, x \in \mathbb{R}$ and $X \in \mathrm{Lie}\,(G^*)$ in a such a way that

$$[(c_1, c_2, X, x), (c_1, c_2, X', x')] = \left(F_1(X, X'), F_2(X, X'), [X, X'], F_3(X, X') \right).$$

By direct computation, the set

$$\{(\vec{v}, r, x) \equiv (0, 0, \vec{0}, 0, \vec{v}, r, x) \ : \ (\vec{v}, r, x) \in \mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(\mathbb{R}) \oplus \mathbb{R}\}$$

is a subalgebra of $\mathrm{Lie}\,(G^*_F)$ with Lie brackets

$$[(\vec{v}, r, x), (\vec{v}', r', x')] = (\dot{\delta}(r)\vec{v}, 0, \vec{v} \wedge \vec{v}')$$

where $(\vec{v}, r, x), (\vec{v}', r', x') \in \mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(\mathbb{R}) \oplus \mathbb{R}$. By this equation, $\mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(\mathbb{R}) \oplus \mathbb{R}$ is isomorphic to the Lie algebra of the covering group H of the *diamond group*, i.e. the Lie group $H = \mathcal{V} \times \mathbb{R} \times \mathbb{R}$ with product

$$(\vec{v}, r, x)(\vec{v}', r', x') = (\vec{v} + \delta(r)\vec{v}', r + r', x + x' + \vec{v} \wedge \delta(r)\vec{v}')$$

with $(\vec{v}, r, x), (\vec{v}', r', x') \in H$.

Moreover, the set

$$\{(c_1, c_2, \vec{a}, b) \equiv (c_1, c_2, \vec{a}, b, \vec{0}, 0, 0) \ : \ (c_1, c_2, \vec{a}, b) \in \mathbb{R}^2 \oplus \mathrm{Lie}\,(\mathcal{T})\}$$

is an abelian ideal of $\mathrm{Lie}\,(G^*_F)$ isomorphic to $\mathrm{Lie}\,(\mathbb{R}^2 \times \mathcal{T})$.

Taking into account the previous results and the fact that, as a vector space,

$$\mathrm{Lie}\,(G_F^*) = \left(\mathbb{R}^2 \oplus \mathrm{Lie}\,(\mathcal{T})\right) \oplus \left(\mathrm{Lie}\,(\mathcal{V}) \oplus \mathrm{Lie}\,(\mathbb{R}) \oplus \mathbb{R}\right),$$

then $\mathrm{Lie}\,(G_F^*)$ is isomorphic to the semidirect sum of $\mathrm{Lie}\,(\mathbb{R}^2 \times \mathcal{T})$ and $\mathrm{Lie}\,(H)$.

Explicitly, if $(\vec{v}, r, x) \in \mathrm{Lie}\,(H)$ and $(c_1, c_2, \vec{a}, b) \in \mathrm{Lie}\,(\mathbb{R}^2 \times \mathcal{T})$ one has

$$\begin{aligned}[(\vec{v}, r, x), (c_1, c_2, \vec{a}, b)] &= (rb, \vec{v}\cdot\vec{a}, \dot{\delta}(r)\vec{a} + b\vec{v}, 0)\\ &=: \dot{\rho}(\vec{v}, r, x)(c_1, c_2, \vec{a}, b),\end{aligned}$$

where $\dot{\rho}(\vec{v}, r, x)$ is the 5×5 matrix

$$\dot{\rho}(\vec{v}, r, x) = \begin{pmatrix} 0 & 0 & \multicolumn{2}{c}{\vec{0}} & r \\ 0 & 0 & \multicolumn{2}{c}{\vec{v}} & 0 \\ 0 & 0 & 0 & -r & \multirow{2}{*}{$\vec{v}$} \\ 0 & 0 & r & 0 & \\ 0 & 0 & \multicolumn{2}{c}{\vec{0}} & 0 \end{pmatrix},$$

which acts on the column vector $(c_1, c_2, \vec{a}, b) \in \mathrm{Lie}\,(\mathbb{R}^2 \times \mathcal{T}) \simeq \mathbb{R}^2 \times \mathcal{T}$.

If ρ is the representation of H such that its differential at the identity is $\dot{\rho}$, then G_F^* is the semidirect product of $\mathbb{R}^2 \times \mathcal{T}$ and H with respect to ρ. A simple calculation shows that

$$\begin{aligned}\rho(\vec{v}, 0, 0) &= \begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{\vec{0}} & 0 \\ 0 & 1 & \multicolumn{2}{c}{\vec{v}} & \frac{1}{2}\vec{v}^2 \\ 0 & 0 & 1 & 0 & \multirow{2}{*}{$\vec{v}$} \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & \multicolumn{2}{c}{\vec{0}} & 1 \end{pmatrix}\\ \rho(\vec{0}, r, 0) &= \begin{pmatrix} 1 & 0 & \vec{0} & r \\ 0 & 1 & \vec{0} & 0 \\ 0 & 0 & \multirow{2}{*}{$\delta(r)$} & \multirow{2}{*}{$\vec{0}$} \\ 0 & 0 & & \\ 0 & 0 & \vec{0} & 1 \end{pmatrix}\\ \rho(\vec{0}, 0, x) &= \begin{pmatrix} 1 & 0 & \multicolumn{2}{c}{\vec{0}} & 0 \\ 0 & 1 & \multicolumn{2}{c}{\vec{0}} & 0 \\ 0 & 0 & 1 & 0 & \multirow{2}{*}{$\vec{0}$} \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & \multicolumn{2}{c}{\vec{0}} & 1 \end{pmatrix}.\end{aligned}$$

Hence the action of H on $\mathbb{R}^2 \times \mathcal{T}$ is given by

$$(\vec{v}, r, x)[(c_1, c_2, \vec{a}, b)] = (c_1 + br, c_2 + \vec{v}\cdot\delta(r)\vec{a} + \frac{b\vec{v}^2}{2}, \delta(r)\vec{a} + b\vec{v}, b).$$

If $g = (c_1, c_2, \vec{a}, b, \vec{v}, r, x)$ and $g' = (c_1', c_2', \vec{a}', b', \vec{v}', r', x')$ are in G_F^*, then

$$\begin{aligned}gg' &= (c_1 + c_1' + b'r, c_2 + c_2' + \vec{v}\cdot\delta(r)\vec{a}' + \frac{b'\vec{v}^2}{2}, \vec{a} + \delta(r)\vec{a}' + b'\vec{v}, b + b',\\ &\qquad \vec{v} + \delta(r)\vec{v}', r + r', x + x' + \vec{v}\wedge\delta(r)\vec{v}'),\end{aligned}$$

so that the explicit form of $\tau_F = (\tau_1, \tau_2, \tau_3)$ is

$$\begin{array}{rcl} \tau_1(g, g') & = & b' r \\ \tau_2(g, g') & = & \vec{v} \cdot \delta(r)\vec{a}' + b'\vec{v}^2/2 \\ \tau_3(g, g') & = & \vec{v} \wedge \delta(r)\vec{v}'. \end{array}$$

By Theorem 1, the equivalence classes $[\tau_1]$, $[\tau_2]$, $[\tau_3]$ form a basis of $H^2(G^*, \mathbb{R})$. Moreover τ_2 and τ_3 satisfy the condition

$$\tau_i(k, g^*) = \tau_i(g^*, k), \qquad k \in \text{Ker } \delta, \; g^* \in G^*,$$

while τ_1 does not. It follows that $\dim H^2(G^*, \mathbb{R})_\delta = 2$ and we can choose $\overline{\tau} = (\tau_2, \tau_3)$ (notice that $\overline{\tau}(k_1, k_2) = 0$ if $k_1, k_2 \in \text{Ker } \delta$) and the universal central extension $\overline{G}$ of G can be identified with the semidirect product of the vector group $A = \mathbb{R} \times \mathcal{T}$ and the Lie group H with respect to the action of H on A given by

$$(\vec{v}, r, x)[(c, \vec{a}, b)] = (c + \vec{v} \cdot \delta(r)\vec{a} + \frac{b\vec{v}^2}{2}, \delta(r)\vec{a} + b\vec{v}, b).$$

where the elements of A are denoted by $(c, \vec{a}, b)$, with $c \in \mathbb{R}$, $\vec{a} \in \mathcal{T}_s$ and $b \in \mathcal{T}_t$, and the ones of H by $(\vec{v}, r, x)$, with $x, r \in \mathbb{R}$ and $\vec{v} \in \mathcal{V}$. As usual, we denote the elements of $\overline{G}$ as $(c, \vec{a}, b, \vec{v}, r, x)$. Finally, one has that

$$K = \{(c, \vec{0}, 0, \vec{0}, 2\pi n, x) \; : \; c, x \in \mathbb{R}, \; n \in \mathbb{Z}\} \simeq \mathbb{R}^2 \times \mathbb{Z}.$$

Since $\overline{G}$ is a semidirect product we apply the results of Section 3.2 to classify the irreducible physically inequivalent representations of $\overline{G}$. Let $\hat{A}$ be the dual group of A. We identify $\hat{A}$ with $\mathbb{R}^4$ using the pairing

$$\langle (m, \vec{p}, p_0), (c, \vec{a}, b) \rangle = mc - \vec{p} \cdot \vec{a} + p_0 b.$$

The dual action of $\overline{G}$ on $\hat{A}$ is

$$g[(m, \vec{p}, p_0)] = (m, \delta(r)\vec{p} + m\vec{v}, p_0 + \delta(r)\vec{p} \cdot \vec{v} + \frac{1}{2} m\vec{v}^2),$$

where $g = (c, \vec{a}, b, \vec{v}, r, x) \in \overline{G}$. We have the following orbits for the dual action.

1. For each $l \in \mathbb{R}$, $l > 0$,
$$\overline{G}[(0, \vec{p}_l, 0)] = \{(0, \vec{p}, p_0) \; : \; \vec{p}^2 = l^2\},$$
where $\vec{p}_l = (0, l)$.
2. For each $E \in \mathbb{R}$,
$$\overline{G}[(0, \vec{0}, E)] = \{(0, \vec{0}, E)\}.$$
3. For each $m \in \mathbb{R}$, $E \in \mathbb{R}$, $m \neq 0$,
$$\overline{G}[(m, \vec{0}, E)] = \{(m, \vec{p}, p_0) \; : \; p_0 - \frac{\vec{p}^2}{2m} = E\}.$$

All the orbits are closed in $\hat{A}$, hence the semidirect product is regular and Theorem 4 holds.

The set of singleton orbits is

$$\hat{A}_s = \{(0, \vec{0}, E) \ : \ E \in \mathbb{R}\},$$

and the orbit classes of $\overline{G}$ are the following:

1. for each $l \in \mathbb{R}$, $l > 0$,
$$\widetilde{\mathcal{O}}_{(0,\vec{p}_l,0)} = \overline{G}[(0, \vec{p}_l, 0)];$$
2. $$\widetilde{\mathcal{O}}_{(0,\vec{0},0)} = \bigcup_{E \in \mathbb{R}} \overline{G}[(0, \vec{0}, E)];$$
3. for any $m > 0$,
$$\widetilde{\mathcal{O}}_{(m,\vec{0},0)} = \bigcup_{\vec{p} \in \mathbb{R}^2, E \in \mathbb{R}} \left(\overline{G}[(m, \vec{p}, E)] \cup \overline{G}[(-m, \vec{p}, E)] \right);$$

In the sequel we will exploit in detail only the third case, which presents some interesting physical features.

Let $m > 0$ and $p_m = (m, \vec{0}, 0) \in \widetilde{\mathcal{O}}_{(m,\vec{0},0)}$. We have that

$$G_{p_m} \cap H = \{(\vec{v}, r, x) \in H \ : \ \vec{v} = \vec{0}\}$$

is isomorphic to $\mathbb{R}^2$ and its irreducible representations are its characters. Explicitly, $\lambda, \mu \in \mathbb{R}$ define the character of $G_{p_m} \cap H$

$$(\vec{0}, r, x) \mapsto e^{i\lambda x} e^{i\mu r}.$$

Now we observe that

1. if $y \in \hat{A}_s$, $y \neq 0$, then $y\overline{G}[p_m] \neq \overline{G}[p_m]$;
2. $\overline{G}[p_m] \neq \overline{G}[p_m]^{-1}$;
3. the characters of H are of the form
$$(\vec{v}, r, x) \mapsto e^{i\mu r}.$$

According to Theorem 4, every irreducible representation of $\overline{G}$ living on an orbit class $\widetilde{\mathcal{O}}_{p_m}$ is equivalent to one of the form $U^{m,\lambda} = \mathrm{Ind}_{\overline{G}_{p_m}}^{\overline{G}} (D^{m,\lambda})$ where $D^{m,\lambda}$ is the representation of $\overline{G}_{p_m}$

$$(c, \vec{a}, b, \vec{0}, r, x) \mapsto e^{i(mc + \lambda x)}.$$

Moreover, the set $\{U^{m,\lambda} \ : \ m, \lambda \in \mathbb{R}, \ m > 0\}$ is a family of physically inequivalent representations of $\overline{G}$.

To compute explicitly $U^{m,\lambda}$, we observe that the orbit

$$\overline{G}[p_m] = \{(m, \vec{p}, p_0) \ : \ p_0 - \frac{\vec{p}^2}{2m} = 0\}$$

can be identified with $\mathbb{R}^2$ using the map

$$\mathbb{R}^2 \ni \vec{p} \longleftrightarrow \left(m, \vec{p}, \frac{\vec{p}^2}{2m} \right) \in \overline{G}[p_m].$$

With respect to this identification the action of $\overline{G}$ on the orbit becomes

$$(c,\vec{a},b,\vec{v},r,x)[\vec{p}] = \delta(r)\vec{p} + m\vec{v}$$

so that the Lebesgue measure $d\vec{p}$ on $\mathbb{R}^2$ is $\overline{G}$-invariant. We consider the section

$$\begin{aligned} \beta : \mathbb{R}^2 &\to \overline{G} \\ \vec{p} &\mapsto \left(0,\vec{0},0,\frac{\vec{p}}{m},0,0\right) \end{aligned}$$

for the action of $\overline{G}$ on $\mathbb{R}^2$. The representation $U^{m,\lambda}$ of $\overline{G}$ acts in $L^2(\mathbb{R}^2, d\vec{p})$ as

$$\left(U^{m,\lambda}_{(c,\vec{a},b,\vec{v},r,x)} f\right)(\vec{p}) = e^{i\left(\frac{b}{2m}\vec{p}^2 - \vec{p}\cdot\vec{a} + mc\right)} e^{i\lambda\left(x + \frac{1}{m}\vec{v}\wedge\vec{p}\right)} f(\delta(-r)(\vec{p}-m\vec{v})).$$

From the explicit form of $U^{m,\lambda}$ one readily gets that the angular momentum, i.e. the selfadjoint operator that generates the 1-parameter subgroup of rotations, has only the *orbital* part, so that the elementary particles in $2+1$ dimensions have no *spin*. However, they acquire a new *charge* λ which is not of a space-time origin, but arises from the structure of the multipliers. If $\lambda \neq 0$, the two linear momenta do not commute.

We add some final comments.

1. The characters of G^* are

   $$G^* \ni (\vec{a},b,\vec{v},r) \mapsto e^{iEb}e^{i\mu r} \in \mathbf{T},$$

   where $E,\mu \in \mathbb{R}$. The set V of characters of K that extend to G^* is

   $$V = \{(c,\vec{0},0,\vec{0},2\pi n,x) \mapsto z^n \ : \ z \in \mathbf{T}\} \simeq \mathbf{T}.$$

   The group V is a closed subgroup of $\widehat{K} = \mathbb{R}^2 \times \mathbf{T}$ and $K_0 = \mathbb{R}^2$. Applying Corollary 1, $H^2(G,\mathbf{T})$ is isomorphic to $\mathbb{R}^2$. In particular, any $\mathbf{T}$-multiplier of G is equivalent to one of the form

   $$\big((\vec{a},b,\vec{v},R),(\vec{a}',b',\vec{v}',R')\big) \mapsto e^{im(\vec{v}\cdot R\vec{a}' + \frac{1}{2}b'\vec{v}^2)}e^{i\lambda(\vec{v}\wedge R\vec{v}')}$$

   where $(m,\lambda) \in \mathbb{R}^2$.
2. From the explicit form of the characters of G^* one has that, for all $E,\mu \in \mathbb{R}$, the representation

   $$(c,\vec{a},b,\vec{v},r,x) \quad \mapsto \quad e^{i(Eb+\mu r)}U^{m,\lambda}_{(c,\vec{a},b,\vec{v},r,x)}$$

   is physically equivalent to $U^{m,\lambda}$. Hence the angular momentum and the energy are both defined up to an additive constant. For the energy this phenomenon is well known in $3+1$ dimensions, while it does not occur for the angular momentum.
3. The admissibility condition (7) gives rise to two superselection rules that do not allow superposition among states with different *mass* m and among states with different *charge* λ. However, there is no superselection rule connected with the *spin*.

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Gianni Cassinelli, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: cassinelli@genova.infn.it

Ernesto De Vito, Dipartimento di Matematica, Università di Modena, via Campi 213/B, 41100 Modena, Italy and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: devito@genova.infn.it

Pekka Lahti, Department of Physics, University of Turku, FIN-20014 Turku, Finland
E-mail address: pekka.lahti@utu.fi

Alberto Levrero, Dipartimento di Fisica, Università di Genova, I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
E-mail address: levrero@genova.infn.it