WAVELET TRANSFORMS AND DISCRETE FRAMES ASSOCIATED TO SEMIDIRECT PRODUCTS

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ABSTRACT. We consider a semidirect product $G = \mathbb{R}^n \times' H$ and its unitary representations U of the form $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ where Ind is the unitary induction, p_0 is in the dual group of $\mathbb{R}^n$, G_0 is the stability group of p_0 and m is a unitary representation of $G_0 \cap H$. We give sufficient conditions such that U defines a wavelet transform and a discrete frame.

1. INTRODUCTION

In the framework of *wavelet analysis* it is well known that the groups that are semidirect products with normal abelian factor play an essential role in the construction both of continuous wavelet transforms and of discrete frames.

The classical example, [8], is the group $ax + b$ and its quasi-regular representation U acting on $L^2(\mathbb{R})$ as

$$(U_{ab}f)(x) = a^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right) \qquad x \in \mathbb{R},\ f \in L^2(\mathbb{R}),\ a \in \mathbb{R}^+,\ b \in \mathbb{R}.$$

We refer for this to the treatise [4] and to the review paper [9].

In the last years there has been an interest to extend this kind of results to other groups. On one side, for example [2], [6], [7], obtaining results valid for a fairly general class of groups, on the other side discussing specific cases. For example, in [12], the group $\mathbb{R}^n \times' (\mathbb{R}^+ \times SO(n))$ has been considered in order to define wavelet transforms on $L^2(\mathbb{R}^n)$.

In this paper we consider the case of groups G that are semidirect product of $\mathbb{R}^n$ and a topological group H and their representations U of the form $U = \mathrm{Ind}_{G_0}^{G}(p_0 m)$ where Ind is the unitary induction *à la* Mackey, $p_0 \in \mathbb{P}^n$, the dual of $\mathbb{R}^n$, G_0 is the stability group of p_0 with respect to the dual action of G on $\mathbb{P}^n$ and m is a unitary representation of $G_0 \cap H$. For such groups we consider two kinds of problems.

First we describe the conditions under which the representations $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ are square-integrable and characterise for them the formal degree operator. Moreover we discuss a concrete example that extends the results of [12] to vector valued functions.

Second we consider the problem of the existence of a countable set $(g_i)_{i\in I}$ of G and a vector $\psi \in \mathcal{H}$ such that $(U_{g_i}\psi)_{i\in I}$ is a *frame* in $\mathcal{H}$, denoting by $\mathcal{H}$

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the Hilbert space where U acts. In particular we show that, if H is the direct product of $\mathbb{R}^+$ (the multiplicative group of positive numbers) and a semisimple connected Lie group, and m is a cyclic finite dimensional representation, then there are discrete frames associated to $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ provided that the stability group $H \cap G_0$ is compact and that the orbit $G[p_0]$ is open. These results extend the ones obtained by [2] and [7] since we consider wavelet transforms for vector valued functions and we *do not* assume the existence either of a *fundamental domain*, [7], or of a *frame generator*, [2].

2. Preliminaries and notations

Let G be a *locally compact second countable* (lcsc) topological group. We denote by μ_G and Δ_G respectively a left Haar measure on G and its modular function.

We review the basic definition and properties of square integrable representations and of discrete frames. In this paper we will use the word *representation* to mean a continuous unitary representation of G acting in a complex separable Hilbert space $\mathcal{H}$.

A representation U of G in $\mathcal{H}$ is *square-integrable* if it is irreducible and there exist two nonzero vectors $\phi, \psi \in \mathcal{H}$ such that the map $c_{\phi,\psi}$ from G to $\mathbb{C}$

$$c_{\psi,\phi}(g) = \langle U_g \psi, \phi \rangle_{\mathcal{H}}$$

is in $L^2(G, \mu_G)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product (linear in the second variable). We recall this well known result of Duflo and Moore (Theorem 3 of [5]).

Theorem 1 (Duflo and Moore)**.** *Let U be a square-integrable representation of G acting in $\mathcal{H}$. There exists a unique positive selfadjoint operator K_U in $\mathcal{H}$ such that*

1. *if $\phi, \psi \in \mathcal{H}$, $\phi \neq 0$ the map $c_{\psi,\phi}$ is in $L^2(G, \mu_G)$ if and only if $\psi \in \mathrm{Dom}\, K_U^{-1/2}$;*
2. *if $\phi, \phi' \in \mathcal{H}$ and if $\psi, \psi' \in \mathrm{Dom}\, K_U^{-1/2}$, then*

$$\left\langle c_{\psi,\phi}, c_{\psi',\phi'} \right\rangle_{L^2(G,\mu_G)} = \left\langle \phi, \phi' \right\rangle_{\mathcal{H}} \left\langle K_U^{-1/2} \psi', K_U^{-1/2} \psi \right\rangle_{\mathcal{H}}.$$

The operator K_U is called the *formal degree* of the representation and the vectors $\psi \in \mathrm{Dom}\, K_U^{-\frac{1}{2}}$ are called *admissible.*

Let $\mathcal{H}$ be a (complex separable) Hilbert space and $\{\psi_i \,:\, i \in I\}$ a countable family of nonzero vectors in $\mathcal{H}$. We will say that $\{\psi_i\}_{i\in I}$ is a *frame* in $\mathcal{H}$ if there exist two positive numbers α and β such that the following condition holds

$$\alpha \left\|\phi\right\|^2 \leq \sum_{i\in I} | \left\langle \psi_i, \phi \right\rangle |^2 \leq \beta \left\|\phi\right\|^2 \qquad \forall \phi \in \mathcal{H}.$$

The numbers α and β are called *frame bounds* of the frame $\{\psi_i\}_{i\in I}$.

If there exists a representation of a group G acting in $\mathcal{H}$, then it is of particular interest to determine frames of the following form

$$\psi_i = U_{g_i}\psi,$$

where $\{g_i\}_{i\in I}$ is a countable family of elements of G and ψ is a fixed element in $\mathcal{H}$.

From now on let G be the semidirect product $\mathbb{R}^n \times' H$, where H is a lcsc topological group acting continuously on $\mathbb{R}^n$ (G is in a natural way a lcsc topological group). Explicitly, let Ω be the representation of H that defines the action of H on $\mathbb{R}^n$. Denoting $g = (x, h)$ the elements of G, with $x \in \mathbb{R}^n$ and $h \in H$, we have

$$(x, h)(x', h') = (x + \Omega(h)x', hh')$$

and the inner action of G on $\mathbb{R}^n$ is

$$g[x] := gxg^{-1} = \Omega(h)x, \qquad x \in \mathbb{R}^n,\ g = (y, h) \in G.$$

We denote by $d^n x$ the Lebesgue measure on $\mathbb{R}^n$. Let $\mathbb{P}^n$ be the dual group of $\mathbb{R}^n$, which we identify with $\mathbb{R}^n$ in a natural way, and $d^n p$ be the corresponding Lebesgue measure. By duality, G acts continuously on $\mathbb{P}^n$, explicitly

$$g[p] \cdot x = p \cdot g^{-1}[x], \qquad x \in \mathbb{R}^n,\ p \in \mathbb{P}^n,\ g \in G,$$

where $\cdot$ denotes the natural scalar product in $\mathbb{R}^n$. We fix an element p_0 of $\mathbb{P}^n$ and we denote by $\mathcal{O}$ the corresponding orbit

$$\mathcal{O} = \{p \in \mathbb{P}^n \ :\ p = g[p_0],\ g \in G\}$$

and by G_0 the stability subgroup of p_0,

$$G_0 = \{g \in G \ :\ g[p_0] = p_0\}.$$

Let $H_0 = G_0 \cap H$, so that $G_0 = \mathbb{R}^n \times' H_0$. We choose a representation m of H_0 acting in a complex separable Hilbert space $\mathcal{K}$ and we consider the representation $p_0 m$ of G_0 in $\mathcal{K}$

$$(p_0 m)(xh) := e^{ip_0\cdot x} m(h) \qquad x \in \mathbb{R}^n, h \in H_0.$$

We denote by $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ the representation unitarily induced by $p_0 m$ from G_0 to G, [11].

3. Square-integrability

The following theorem solves the problem of the square-integrability of the induced representations of G.

Theorem 2. *The representation $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ is square-integrable if and only if $\mathcal{O}$ has positive Lebesgue measure in $\mathbb{P}^n$ and m is a square-integrable representation.*

This result was proven by Kleppner and Lipsman (Corollary 11.1 of [10]) as a consequence of the characterisation of the Plancherel measure for a locally compact group with a closed normal Type I subgroup. In paper [1] we give an elementary proof of this theorem, based only on the properties of the commutative Fourier transform.

Let us assume that $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ is square-integrable. Since the square-integrability of a representation depends only on its equivalence class, we can freely choose the explicit form of $\mathrm{Ind}_{G_0}^{G}(p_0 m)$. Due to Theorem 2 the measure of $\mathcal{O}$ has to be positive, hence the restriction of the Lebesgue measure $d^n p$ to $\mathcal{O}$ is a non-zero σ-finite measure defined on $\mathcal{O}$. In fact it is H-quasi-invariant, due to the following elementary lemma (the proof is given, for example, in [1]). Let $\rho(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$.

Lemma 1. *For any $h \in H$ and $E \in \mathcal{B}(\mathbb{P}^n)$ we have*

$$(d^n p)(h[E]) = \rho(h^{-1})(d^n p)(E).$$

With the above notations, a suitable choice is the representation U acting in $L^2(\mathcal{O}, \mathcal{K})$ as

$$(U_{xh}\phi)(p) = \rho(h)^{-1/2} e^{ip\cdot x}\, m(q(p)^{-1} h q(h^{-1}[p]))\phi(h^{-1}[p]) \tag{1}$$

where $p \in \mathcal{O}$, $x \in \mathbb{R}^n$, $h \in H$, $\phi \in L^2(\mathcal{O}, \mathcal{K})$ and q is a measurable section for the action of H on $\mathcal{O}$, that is, a measurable map $q : \mathcal{O} \to H$ such that

$$\begin{aligned} q(p_0) &= e \qquad \text{where } e \text{ is the identity,} \\ q(p)[p_0] &= p, \qquad p \in \mathcal{O} \end{aligned}$$

(we recall that the orbit $\mathcal{O}$ is a Borel subset of $\mathbb{R}^n$).

With this choice the condition of admissibility is given by the following corollary (for the proof see Corollary 2 of [1]). Let K_m be the formal degree of m.

Corollary 1. *With the above assumptions, $\psi \in L^2(\mathcal{O}, \mathcal{K})$ is an admissible vector for U if and only if the following two conditions hold*

$$\psi(p) \in \mathrm{Dom}\, K_m^{-\frac{1}{2}} \qquad \textit{for almost all } p \in \mathcal{O}$$

$$\int_{\mathcal{O}} \Delta_G(q(p))^{-1} \left\| K_m^{-\frac{1}{2}} \psi(p) \right\|_{\mathcal{K}}^2 d^n p \;\; < \infty.$$

If H_0 is compact, all its irreducible representations are square-integrable, and one can easily show that $\mathrm{Dom}\, K_m^{-\frac{1}{2}} = \mathcal{K}$, K_m is a multiple of the identity and $\Delta_G(q(p))$ is *independent* of the choice of the section q, precisely

$$\Delta_G(q(p)) = \Delta_G(g_p) \quad \forall g_p \in G \text{ such that } g_p[p_0] = p.$$

Going back to the general case, if U is square-integrable, we can define a corresponding *wavelet transform.* Let $\psi \in L^2(\mathcal{O}, \mathcal{K})$ be an admissibile vector normalised in a such way that

$$\int_{\mathcal{O}} \Delta_G(q(p))^{-1} \left\| K_m^{-\frac{1}{2}} \psi(p) \right\|_{\mathcal{K}}^2 d^n p = 1.$$

Let $\mathcal{W}_\psi$ be the map from $L^2(\mathcal{O}, \mathcal{K})$ to $L^2(G, \mu_G)$ defined by

$$(\mathcal{W}_\psi \phi)(g) = \langle U_g \psi, \phi \rangle_{\mathcal{H}} \qquad g \in G,\ \phi \in L^2(\mathcal{O}, \mathcal{K}).$$

As a consequence of Theorem 1, one can easily prove that $\mathcal{W}_\psi$ is an isometry and the projection on its range is a convolution operator, precisely

$$F \in \mathrm{Ran}\mathcal{W}_\psi \Leftrightarrow F * \mathcal{W}_\psi \psi = F,$$

where $*$ denotes the convolution in $L^2(G, \mu_G)$. Moreover the inverse map of $\mathcal{W}_\psi$ from $\mathrm{Ran}(\mathcal{W}_\psi)$ onto $\mathcal{H}$ is given by

$$F \mapsto \int_G F(g)(\mathcal{W}_\psi \psi)(g)\, d\mu_G(g),$$

where the integral is in the weak sense.

To present this kind of results, we consider a specific example. Let $G = \mathbb{R}^n \times' (\mathbb{R}^+ \times SO(n))$, where $SO(n)$ is the n-dimensional special orthogonal group. The direct product $H = \mathbb{R}^+ \times SO(n)$ acts on $\mathbb{R}^n$ as

$$(aR)[x] = aRx$$

where Rx is the usual matrix action and a acts as a scalar. The group H is unimodular and the Haar measure is $a^{-1} da\, dR$, where dR is the normalised Haar measure on $SO(n)$. The modular function of G is $\Delta_G(xaR) = a^{-n}$. The dual action is

$$(aR)[p] = a^{-1} Rp.$$

With respect to this action, $\mathbb{P}^n$ is the union of two orbits $\{0\}$ and $\mathcal{O} = \mathbb{P}^n \backslash \{0\}$. The first orbit has zero Lebesgue measure, so that the corresponding representations are not square-integrable. We consider the orbit $\mathcal{O}$ and we fix $p_0 = (0, \cdots, 0, 1) \in \mathcal{O}$. The stability group is $G_0 = \mathbb{R}^n \times' SO(n-1)$ where $SO(n-1)$ is regarded as a subgroup of $SO(n)$ in the following way

$$SO(n-1) \ni R \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \in SO(n).$$

Since $SO(n-1)$ is compact, every representation of G induced on the orbit $\mathcal{O}$ is square-integrable. The usual way to construct wavelets in $L^2(\mathbb{R}^n)$ is to choose m as the trivial representation. We suggest an alternative way in order to obtain vector valued wavelets.

Let m be a representation of $SO(n-1)$. As a consequence of the Frobenious reciprocity theorem and of the fact that $SO(n)$ is compact, there exists an (in general non unique) representation D of $SO(n)$ such that m is a subrepresentation of D, when D is restricted to $SO(n-1)$. This allows to use a *covariant realisation* of $\mathrm{Ind}_{G_0}^G(p_0 m)$, instead of the formula (1). It is defined in the following way. Let D act in W and m act in the subspace $\mathcal{K}$ of W.

For all $p \in \mathcal{O}$ let $\mathcal{K}_p = D(R_p)\mathcal{K}$ where $R_p \in SO(n)$ is such that $R_p p_0 = \frac{p}{|p|}$. Define the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \{\phi \in L^2(\mathbb{P}^n, W) \ : \ \phi(p) \in \mathcal{K}_p \text{ for almost all } p\}$$

and the representation V acting in $\mathcal{H}$ as

$$(V_{xaR}\phi)(p) = a^{\frac{n}{2}} e^{ip\cdot x} D(R)\phi(aR^{-1}p).$$

It is possible to show, [13], that V is equivalent to $\mathrm{Ind}_{G_0}^{G}(p_0 m)$. Moreover a vector $\psi \in \mathcal{H}$ is admissible if and only if

$$\int |p|^{-n} \, \|\psi(p)\|_W^2 \, d^n p < \infty.$$

The Fourier-Plancherel transform intertwines V with a *geometric* representation $\widehat{V}$ acting on a closed subspace $\widehat{\mathcal{H}}$ of $L^2(\mathbb{R}^n, W)$ as

$$(\widehat{V}_{xaR} f)(y) = a^{-\frac{n}{2}} D(R) f\left(\frac{R^{-1}(y-x)}{a}\right).$$

To give an explicit description of the subspace $\widehat{\mathcal{H}}$ let us choose, for example, $n = 3$ and D be the representation of $SO(3)$

$$D(R) = R$$

acting in $W = \mathbb{C}^3$ in a natural way. Let $\mathcal{K} = \{(0,0,c) \ : \ c \in \mathbb{C}\}$, hence the restriction m to $SO(2)$ of D leaves invariant $\mathcal{K}$ (m is the trivial representation). An easy calculation shows that

$$\widehat{\mathcal{H}} = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^3) \ : \ (\nabla \wedge f) = 0\}$$

where $\nabla\wedge$ is the curl operator (in the sense of distributions). Moreover we observe that the orthogonal complement of $\widehat{\mathcal{H}}$

$$(\widehat{\mathcal{H}})^{\perp} = \{f \in L^2(\mathbb{R}^n, \mathbb{C}^3) \ : \ \nabla \cdot f = 0\},$$

where $\nabla\cdot$ is the divergence operator (in the sense of distributions), is the direct sum of two closed subspaces where $\widehat{V}$ acts irreducibly. These subspaces correspond to the choice of the subspaces of W

$$\begin{aligned} \mathcal{K}_1 &= \{(c, ic, 0) \ : \ c \in \mathbb{C}\} \\ \mathcal{K}_2 &= \{(c, -ic, 0) \ : \ c \in \mathbb{C}\} \end{aligned}$$

and of the representations of $SO(2)$

$$\begin{aligned} m_1(\theta) &= e^{i\theta} \\ m_2(\theta) &= e^{-i\theta}, \end{aligned}$$

where $\theta \in [0, 2\pi]$ is the usual parameter of $SO(2)$.

4. Construction of discrete frames

In this section we consider the problem of the existence of discrete frames associated to the induced representations of $G = \mathbb{R}^n \times' H$, when $H = \mathbb{R}^+ \times S$ and S is a semisimple connected Lie group.

The action of H on $\mathbb{R}^n$ is

$$h[x] = (as)[x] := a(s[x])$$

where $x \mapsto s[x]$ is a smooth action of S on $\mathbb{R}^n$. Let $\mathbb{P}^n$ denote the dual group of $\mathbb{R}^n$ and fix $p_0 \in \mathbb{P}^n$. Assume that the orbit $\mathcal{O} = G[p_0]$ for the dual action of G on $\mathbb{P}^n$ is *open* and that the subgroup $H_0 = H \cap G_0$ is compact (notice that $\mathbb{R}^+ \cap H_0 = \emptyset$). In these hypotheses $\mathcal{O}$ is homeomorphic to G/G_0 and to H/H_0.

We will prove the existence of frames associated to the representation $\mathrm{Ind}_{G_0}^{G}(p_0 m)$ where m is any unitary finite dimensional *cyclic* representation of H_0.

We need to establish some preliminary topological results.

Lemma 2. *Let Γ be a discrete subgroup of H. Then, for any compact subset C in $\mathcal{O}$ there exists a finite partition $\Gamma_1, \dots, \Gamma_d$ of Γ such that*

$$h', h'' \in \Gamma_j \Rightarrow h'[C] \cap h''[C] = \emptyset.$$

Proof. Let $\pi : H \to \mathcal{O}$ be the canonical projection. First of all, we prove that $\pi^{-1}(C)$ is compact. By Theorem 5.11 of [13], there exists a section $q : \mathcal{O} \to H$ such that the set $q(C)$ is relatively compact in H. Since H_0 is compact also $\pi^{-1}(C) = q(C)H_0$ is relatively compact. Since $\pi^{-1}(C)$ is closed, it is compact.

By lemma 3.3 of [6], there exists a finite partition $\Gamma_1, \dots, \Gamma_d$ of Γ such that

$$h', h'' \in \Gamma_j \Rightarrow h'\pi^{-1}(C) \cap h''\pi^{-1}(C) = \emptyset.$$

Hence

$$\pi^{-1}(h'[C] \cap h''[C]) = \emptyset$$

and $h'[C] \cap h''[C] = \emptyset$, since π is surjective. □

Lemma 3. *There exist a discrete subgroup Γ of H and a compact C in $\mathcal{O}$ such that*

$$\mathcal{O} = \bigcup \{h[C] \ : \ h \in \Gamma\}.$$

Proof. Since H is the direct product of $\mathbb{R}^+$ and a semisimple connected Lie group, as a consequence of Theorem C of [3] H has a discrete subgroup Γ such that H/Γ is compact. Denote by π' the canonical projection of H onto H/Γ. There exists a compact $K' \subset H$ such that $\pi'(K') = H/\Gamma$, hence

$$H = \bigcup \{hK \ : \ h \in \Gamma\}$$

where $K = (K')^{-1}$ is compact in H. Define $C = \pi(K)$, then C is compact in $\mathcal{O}$ and

$$\bigcup \{h[C] \ : \ h \in \Gamma\} = \bigcup \{\pi(hK) \ : \ h \in \Gamma\} = \mathcal{O}.$$

□

Lemma 4. *For any non void open set $X \subset \mathcal{O}$ there exists a finite set $\{h_1, \dots h_L\}$ in H, such that*

$$\bigcup \{hh_k[X] \ : \ h \in \Gamma, \ 1 \leq k \leq L\} = \mathcal{O},$$

where Γ is the discrete subgroup of H given by the previous lemma.

Proof. The family of sets $\{h[X] \ : \ h \in H\}$ is an open covering of the compact subset C of $\mathcal{O}$ singled out by the previous lemma. Hence there exists a finite subcovering $\{h_k[X] \ : \ 1 \leq k \leq L\}$ and we have the thesis. □

Having established these results, let m be a unitary cyclic finite dimensional representation of the compact group H_0 acting in the Hilbert space $\mathcal{K}$ of dimension M. Let v be a cyclic vector for m, then there exist M elements $s_1, \dots, s_M$ of H_0 such that $\{m(s_i)v \ : \ i = 1, \dots, s_M\}$ is an algebraic basis of $\mathcal{K}$. Fix these elements. Let $c : \mathcal{O} \to H$ be a section for the action of H on $\mathcal{O}$. Since H is a Lie group acting smoothly on $\mathcal{O}$ we can assume that c is locally continuous around p_0. For all $h \in H$ and $p \in \mathcal{O}$ define $m(h,p) := m(c(p)^{-1}hc(h^{-1}[p]))$. Then we have:

Lemma 5. *There exist a closed ball B with center p_0, contained in $\mathcal{O}$ and two positive numbers α, β such that*

$$\alpha \|w\|^2_{\mathcal{K}} \leq \sum_{i=1}^{M} |\langle m(s_i,p)v, w\rangle_{\mathcal{K}}|^2 \leq \beta \|w\|^2_{\mathcal{K}}$$

for all $w \in \mathcal{K}$ and $p \in B$.

Proof. Let $\{e_i\}_{i=1}^M$ be an orthonormal basis of $\mathcal{K}$ and, for all $p \in \mathcal{O}$, $A(p)$ be the operator

$$A(p)w = \sum_i \langle m(s_i,p)v, w\rangle \, e_i, \qquad w \in \mathcal{K}.$$

Since the vectors $m(s_i)v = m(s_i,p_0)v$ form a basis of $\mathcal{K}$, $A(p_0)$ is invertible and

$$\left\|A(p_0)^{-1}\right\|^{-2} \|w\|^2_{\mathcal{K}} \leq \sum_{i=1}^{M} |\langle m(s_i,p)v, w\rangle_{\mathcal{K}}|^2 \leq \|A(p_0)\|^2 \, \|w\|^2_{\mathcal{K}}.$$

By the properties of c there exists a neighbourhood of p_0 where the function $p \mapsto A(p)$ is continuous. The thesis easily follows. □

We consider the explicit realisation of $\mathrm{Ind}_{G_0}^G(p_0 m)$ acting in $L^2(\mathcal{O},\mathcal{K})$ as

$$(U_{xh}f)(p) = \rho(h)^{-1/2} e^{ix\cdot p} m(h,p) f(h^{-1}[p]).$$

We notice that, since S is semisimple and connected, ρ has the following expression

$$\rho(as) = a^n \qquad a \in \mathbb{R}^+, \ s \in S.$$

Fix the closed ball B with center p_0 as in Lemma 5 and the discrete subgroup Γ of H as in Lemma 3. Let Y be an open subset of $\mathcal{O}$ such that $p_0 \in Y$ and $s_i[Y] \subset B$ for all $i = 1, \ldots, M$.

The set $X := \bigcap\{s_i[Y] \ : \ 1 \leq i \leq M\}$ is not empty (since it contains p_0) and open, hence, by Lemma 4, there exists a finite family $\{h_1, \ldots, h_L\} \subset H$ such that

$$\bigcup\{hh_k[X] \ : \ h \in \Gamma,\, 1 \leq k \leq L\} = \mathcal{O}.$$

Since the set $\bigcup\{h_k[B] \ : \ 1 \leq k \leq L\}$ is compact, due to Lemma 2, there exists a finite partition $\Gamma_1, \ldots, \Gamma_d$ of Γ such that

$$h', h'' \in \Gamma_j \Rightarrow h'\left[\bigcup_{k=1}^{L} h_k[B]\right] \cap h''\left[\bigcup_{k=1}^{L} h_k[B]\right] = \emptyset. \tag{2}$$

Let ν be an integer such that $B \subset R$ where

$$R := \{x \in \mathbb{R}^n \ : \ -\pi\nu \leq x_i \leq \pi\nu \ \ 1 \leq i \leq n\}$$

and, for all $l \in \mathbb{Z}^n$, $x_l = 2\pi l \in \mathbb{R}^n$.

We observe that the countable family of functions

$$\{e_l : l \in \mathbb{Z}^n\}$$

defined as

$$e_l(p) = (2\pi\nu)^{-n/2} e^{ix_l \cdot p}, \qquad p \in R$$

is a Hilbert basis of $L^2(R)$.

Now we are in a position to describe a covariant frame in $L^2(\mathcal{O}, \mathcal{K})$. Let $\psi \in L^2(\mathcal{O}, \mathcal{K})$ be the vector

$$\psi(p) := \chi_Y(p)v, \qquad p \in \mathcal{O}$$

and consider the countable set of vectors

$$\psi_{h,k,i,l} := U_h U_{h_k} U_{s_i} U_{x_l} \psi$$

where $h \in \Gamma$, $1 \leq k \leq L$, $1 \leq i \leq M$ and $l \in \mathbb{Z}^n$ (Γ is countable, since it is discrete). With these notations we have the following result.

Theorem 3. *The set $\{\psi_{h,k,i,l}\}$ is a frame in $L^2(\mathcal{O}, \mathcal{K})$.*

Proof. Given $f \in L^2(\mathcal{O}, \mathcal{K})$, we have

$$\begin{aligned}
\langle \psi_{h,k,i,l}, f \rangle_{L^2(\mathcal{O},\mathcal{K})} &= \left\langle U_{x_l}\psi, U_{(hh_ks_i)^{-1}} f \right\rangle_{L^2(\mathcal{O},\mathcal{K})} \\
&= \int_{\mathcal{O}} e^{-ix_l \cdot p} \left\langle \psi(p), (U_{(hh_ks_i)^{-1}} f)(p) \right\rangle_{\mathcal{K}} d^n p \\
&= (2\pi\nu)^{\frac{n}{2}} \int_R \overline{e_l(p)} \left\langle \psi(p), (U_{(hh_ks_i)^{-1}} f)(p) \right\rangle_{\mathcal{K}} d^n p.
\end{aligned}$$

Since ψ is bounded the function

$$\mathcal{O} \ni p \mapsto \left\langle \psi(p), (U_{(hh_ks_i)^{-1}} f)(p) \right\rangle_{\mathcal{K}}$$

is in $L^2(R)$, hence

$$\sum_{hkil} |\langle \psi_{h,k,i,l}, f\rangle_{L^2(\mathcal{O},\mathcal{K})}|^2 =$$
$$= (2\pi\nu)^n \sum_{hki} \int_{\mathcal{O}} \left|\left\langle \psi(p), (U_{(hh_k s_i)^{-1}} f)(p)\right\rangle_{\mathcal{K}}\right|^2 d^n p$$
$$= (2\pi\nu)^n \sum_{hki} \int_{\mathcal{O}} \left|\left\langle m(s_i, s_i[p])\psi(p), (U_{(hh_k)^{-1}} f)(s_i[p])\right\rangle_{\mathcal{K}}\right|^2 d^n p =$$

(changing variable $p \mapsto s_i^{-1}[p]$)

$$\begin{aligned} &= (2\pi\nu)^n \sum_{hki} \int_{\mathcal{O}} \left|\left\langle m(s_i, p)\psi(s_i^{-1}[p]), (U_{(hh_k)^{-1}} f)(p)\right\rangle_{\mathcal{K}}\right|^2 d^n p \\ &= (2\pi\nu)^n \sum_{hki} \int_{s_i[Y]} \left|\left\langle m(s_i, p)v, (U_{(hh_k)^{-1}} f)(p)\right\rangle_{\mathcal{K}}\right|^2 d^n p. \end{aligned}$$

Recalling that $X = \bigcap\{s_i[Y] \,:\, 1 \le i \le M\}$, for any positive function ϕ we have

$$\int_X \phi\, d^n p \le \int_{s_i[Y]} \phi\, d^n p \le \int_B \phi\, d^n p$$

for all $1 \le i \le M$. Hence, since the sum on the index i is finite,

$$\begin{aligned} &(2\pi\nu)^n \sum_{hki} \int_{s_i[Y]} \left|\left\langle m(s_i, p)v, (U_{(hh_k)^{-1}} f)(p)\right\rangle_{\mathcal{K}}\right|^2 d^n p \ge \\ &\ge (2\pi\nu)^n \sum_{hk} \int_X \sum_i \left|\left\langle m(s_i, p)v, (U_{(hh_k)^{-1}} f)(p)\right\rangle_{\mathcal{K}}\right|^2 d^n p \\ &\ge \alpha(2\pi\nu)^n \sum_{hk} \int_X \left\|(U_{(hh_k)^{-1}} f)(p)\right\|_{\mathcal{K}}^2 d^n p \\ &= \alpha(2\pi\nu)^n \sum_{hk} \int_X \rho(hh_k)^{-1} \left\|f(hh_k[p])\right\|_{\mathcal{K}}^2 d^n p \end{aligned}$$

(changing variable $p \mapsto (hh_k)^{-1}[p]$)

$$\begin{aligned} &= \alpha(2\pi\nu)^n \sum_{hk} \int_{hh_k[X]} \|f(p)\|_{\mathcal{K}}^2 d^n p \\ &\ge \alpha(2\pi\nu)^n \|f\|_{L^2(\mathcal{O},\mathcal{K})}^2, \end{aligned}$$

taking into account that

$$\bigcup\{hh_k[X] \,:\, h \in \Gamma,\, 1 \le k \le L\} = \mathcal{O}.$$

On the other hand

$$(2\pi\nu)^n \sum_{hki} \int_{s_i[Y]} \left| \left\langle m(s_i,p)v, (U_{(hh_k)^{-1}} f)(p) \right\rangle_{\mathcal{K}} \right|^2 d^n p \leq$$

$$\leq (2\pi\nu)^n \sum_{hk} \int_B \sum_i \left| \left\langle m(s_i,p)v, (U_{(hh_k)^{-1}} f)(p) \right\rangle_{\mathcal{K}} \right|^2 d^n p$$

(reasoning in the same way as before)

$$\leq \beta(2\pi\nu)^n \sum_{hk} \int_{hh_k[B]} \|f(p)\|^2_{\mathcal{K}} \, d^n p$$

(by property (2))

$$\begin{aligned} &= \beta(2\pi\nu)^n \sum_{j=1}^{d} \sum_{h\in\Gamma_j} \sum_{k=1}^{L} \int_{hh_k[B]} \|f(p)\|^2_{\mathcal{K}} \, d^n p \\ &\leq d\beta(2\pi\nu)^n \int_{\mathcal{O}} \|f(p)\|^2_{\mathcal{K}} \, d^n p \\ &= d\beta(2\pi\nu)^n \, \|f\|^2_{L^2(\mathcal{O},\mathcal{K})} \, . \end{aligned}$$

Summarising we have

$$\alpha(2\pi\nu)^n \, \|f\|^2_{L^2(\mathcal{O},\mathcal{K})} \leq \sum_{hkil} | \left\langle \psi_{h,k,i,l}, f \right\rangle_{L^2(\mathcal{O},\mathcal{K})} |^2 \leq d\beta(2\pi\nu)^n \, \|f\|^2_{L^2(\mathcal{O},\mathcal{K})} \, .$$

□

References

[1] P. Aniello, G. Cassinelli, E. De Vito and A. Levrero, to appear on *Rev.Math.Phys.*

[2] D. Bernier and K.F. Taylor, *SIAM J. Math. Anal.* **27**, 594 (1996).

[3] A. Borel, *Topology* **2**, 111 (1963).

[4] See, for example, I. Daubechies, *Ten lectures on wavelets*, SIAM 1992 and references therein.

[5] M. Duflo and C.C. Moore, *J. Funct. Anal.* **21**, 209 (1976).

[6] H. Feichtinger and K. Grochenig, *J. Funct. Anal* **86**, 307(1989).

[7] H. Führ, *J. Math. Phys.* **37**, 6353 (1996).

[8] A. Grossmann, J. Morlet, *SIAM J. Math. Anal.* **15**, 723 (1984).

[9] C.E. Heil and D.F. Walnut, *SIAM Review* **31**, 628 (1989).

[10] A. Kleppner and R.L. Lipsman, *Ann. Sci. École Norm. Sup.* **5**, 459 (1972).

[11] G.W. Mackey, *Ann. of Math.* **55**, 101 (1952).

[12] R. Murenzi, in J.M. Combes, A. Grossmann and P. Tchamitchian (eds.) *Wavelets: Time-Frequency Methods and Phase Space (Proc. Marseille 1987)* Springer-Verlag 1989.

[13] V.S. Varadarajan, *Geometry of Quantum Theory*, Second Edition, Springer-Verlag, New York, 1985.

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