ON THE DECOMPOSITIONS OF A QUANTUM STATE

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Abstract. We classify all the decompositions of a quantum state as a weighted sum of one dimensional projectors. In particular we describe explicitly the set of irreducible decompositions. The physical interest in this problem rests on the possibility of interpreting the decomposition in terms of classical mixture.

1. Preliminaries

In the ordinary Hilbert space formulation of quantum mechanics the states of a quantum system are the positive trace one operators on $\mathcal{H}$, where $\mathcal{H}$ is the complex separable Hilbert space (with inner product $(\cdot,\cdot)$) associated to the quantum system.

The set $\mathcal{S}$ of states has the following properties:

1. it is a closed subset of the Banach space of trace class operators with respect to the trace norm;
2. it is convex, that is, if $W_1, W_2 \in \mathcal{S}$, then $W = wW_1 + (1-w)W_2$ is in $\mathcal{S}$ for all $0 < w < 1$. We call W *convex combination* of W_1 and W_2;
3. if $(W_i)_{i\in I}$ is a family of states and $(w_i)_{i\in I}$ is a family of weights, i.e. $0 < w_i < 1$ and $\sum_{i\in I} w_i = 1$, then $(w_i W_i)_{i\in I}$ is summable with respect to the trace norm topology and its sum is a state. Since $w_i > 0$ and $(w_i)_{i\in I}$ is summable, then the index set I is necessarily finite or countable. In this paper we use the word *family* to mean either a finite or countable family;
4. the extremal points of $\mathcal{S}$, that is the states that are not convex combinations of two different states, are exactly the one dimensional projectors (the *pure* states).

The previous properties are well known, classical facts in the theory of bounded and trace class operators in Hilbert spaces. In this paper we shall use freely some basic, elementary results of this theory.

Let W be a state and $\mathcal{P}$ a set of pure states. For the sake of clarity we label $\mathcal{P}$ with an index set I, so that $\mathcal{P} = \{P_i\}_{i\in I}$ and $P_i \neq P_j$ if $i \neq j$.

We say that $\mathcal{P}$ *decomposes* W if there exists a family of weights $(w_i)_{i\in I}$ such that the (summable) family $(w_i P_i)_{i\in I}$ has sum W (with respect to the trace norm); in this case the set $\{(P_i, w_i)\}_{i\in I}$ is a *decomposition* of W. We stress that since the family $(w_i P_i)_{i\in I}$ is summable, the fact the $\mathcal{P}$ decomposes W does not depend from the index set I labelling $\mathcal{P}$.

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The spectral theorem for selfadjoint compact operators assures that a decomposition exists for any state. However, if W is not a pure state, many decompositions correspond to the same state. This fact can be easily seen by direct computation in the elementary case $\mathcal{H} = \mathbb{C}^2$.

The study of the possible decompositions of a state has at least two physical motivations.

1. The central object of interest in quantum mechanics are the expectation values $\mathrm{Tr}\,(AW)$ where A ranges over the selfadjoint bounded operator representing the physical quantities and W is the state of the system. Given a decomposition $W = \sum_{i \in I} w_i P_i$ the previous expectation values can be expressed as
$$\mathrm{Tr}\,(AW) = \sum_{i \in I} w_i \mathrm{Tr}\,(AP_i);$$
different decompositions of W can be suggested by the concrete physical problem at hand, see, for example, remark 5.
2. The non uniqueness of the decomposition of a state W is at the root of (many of) the difficulties in the interpretation of quantum mechanics, as witnessed for example in two recent textbooks, [1],[2]; the problem of characterising the possible decompositions of a given state is thus an important issue in the foundation of quantum mechanics.

In this physical framework, given a state W we can single out three mathematical problems:

(a) classify all the decompositions of W;
(b) determine whether W can be decomposed on a given set of pure states;
(c) if a given set of pure states decomposes W, determine whether the family of weights is unique.

The present paper solves the first problem by giving a complete characterisation of all the decompositions of a given state; this is done in section 2. In section 3 we describe explicitly a particular class of decompositions that have relevance both from the physical and the mathematical point of view; for them we can answer also to the remaining two questions.

To the best of our knowledge the first author who posed the problem (a) considered in this paper was Jaynes, [5]. He considers only the case of *finite* decompositions of a state W with finite dimensional range. This case has been completely worked out by Hughston *et al.*, [4]. Nevertheless this result is not exhaustive, since a state with finite dimensional range can have countable, non-finite, decompositions. A first partial result on the general case (i.e. countable decompositions for any state) has been given by Hadjisavvas, [3].

2. The decompositions of a state

Let W be a state on $\mathcal{H}$, $\mathcal{K}$ the closure of the range of W and $\mathcal{N}$ its kernel. We denote by P the projector onto $\mathcal{K}$. As W is selfadjoint we have

$\mathcal{H} = \mathcal{N} \oplus \mathcal{K}$. Possibly embedding $\mathcal{H}$ in a bigger space, we can assume that $\mathcal{N}$ is infinite dimensional (see remark 3 *infra*). We recall that

$$\operatorname{Ran} W \subset \operatorname{Ran} W^{\frac{1}{2}} \subset \mathcal{K} = \overline{\operatorname{Ran}}\, W^{\frac{1}{2}},$$

where $\overline{\operatorname{Ran}}$ denotes the closure in $\mathcal{H}$ of the range. Since the restriction of $W^{\frac{1}{2}}$ to $\mathcal{K}$ is injective, there exists a unique (in general unbounded) selfadjoint operator T acting in $\mathcal{K}$, with domain $\operatorname{Ran} W^{\frac{1}{2}}$, such that

$$\begin{aligned} TW^{\frac{1}{2}}\phi &= \phi \quad \forall \phi \in \mathcal{K} \\ W^{\frac{1}{2}}T\phi &= \phi \quad \forall \phi \in \operatorname{Ran} W^{\frac{1}{2}}. \end{aligned}$$

Let $\{e_i\}_{i\in I}$ be a set of vectors in $\mathcal{H}$. We say that $\{e_i\}_{i\in I}$ is *nondegenerate with respect to* P when

1. $\{e_i\}_{i\in I}$ is orthonormal;
2. $\overline{\operatorname{span}}\{e_i : i \in I\} \supset \operatorname{Ran} P$ where $\overline{\operatorname{span}}$ denotes the closure in $\mathcal{H}$ of the subspace algebraically spanned;
3. $Pe_i \neq 0$ for all $i \in I$;
4. for all pairs $i, j \in I$ Pe_i is not collinear with Pe_j.

Theorem 1. *Let W be a state on $\mathcal{H}$ and let $\{e_i\}_{i\in I}$ be nondegenerate with respect to P. For all $i \in I$ define*

$$\begin{aligned} w_i &= (e_i, We_i) = \left\| W^{\frac{1}{2}} P e_i \right\|^2 > 0 \\ \phi_i &= w_i^{-\frac{1}{2}} W^{\frac{1}{2}} e_i, \end{aligned}$$

then $(w_i)_{i\in I}$ is a family of weights, ϕ_i are normalised vectors and $\{(P[\phi_i], w_i)\}_{i\in I}$ is a decomposition of W.
Conversely all decompositions of W can be obtained in this way (provided that the kernel of W is infinite dimensional).

Proof. Since W is a trace one operator and $\overline{\operatorname{span}}\{e_i : i \in I\} \supset \mathcal{K}$, then $(w_i)_{i\in I}$ is a family of weights and the family $(w_i P[\phi_i])_{i\in I}$ is summable. To prove that its sum is W it is sufficient to observe that for all $\phi \in \mathcal{K}$ $W^{\frac{1}{2}}\phi \in \mathcal{K}$, so that

$$\begin{aligned} (\phi, \textstyle\sum_{i\in I} w_i P[\phi_i]\phi) &= \sum_{i\in I} |(\phi, W^{\frac{1}{2}} e_i)|^2 \\ &= \sum_{i\in I} |(W^{\frac{1}{2}}\phi, e_i)|^2 \\ &= \left\| W^{\frac{1}{2}}\phi \right\|^2 = (\phi, W\phi) \end{aligned}$$

which proves the first claim.

Now we must prove that all decompositions of W are of this form. Let $\{(P[\phi_i], w_i)\}_{i\in I}$ be a decomposition of W, that is

$$W = \sum_{i\in I} w_i P[\phi_i]. \tag{1}$$

We prove that, for all $i \in I$, $\phi_i \in \operatorname{Ran} W^{\frac{1}{2}} = \operatorname{Dom} T$. In fact, for all $\phi \in \operatorname{Dom} T$ and $i \in I$ we have

$$\begin{aligned} w_i |(T\phi, \phi_i)|^2 &\leq (T\phi, WT\phi) \\ &= \|\phi\|^2. \end{aligned}$$

This shows that ϕ_i is in the domain of $T^* = T$. For all $i \in I$ let $\psi_i = w_i^{\frac{1}{2}} T\phi_i$. Let J be a finite subset of I, for all $\phi \in \operatorname{dom} T$ we have that

$$\begin{aligned} \sum_{i \in J} |(\psi_i, \phi)|^2 &= \sum_{i \in J} w_i |(T\phi_i, \phi)|^2 \\ &= \sum_{i \in J} w_i |(\phi_i, T\phi)|^2 \\ &\leq \sum_{i \in I} w_i |(\phi_i, T\phi)|^2 \\ &= (T\phi, WT\phi) \\ &= (\phi, P\phi). \end{aligned} \tag{2}$$

Since $\operatorname{Dom} T$ is dense in $\mathcal{K}$ and $\psi_i \in \mathcal{K}$, relation (2) shows that $\left(\sum_{i \in J} (\psi_i, \cdot)\psi_i\right)$ is a monotone increasing net of continuous operators bounded by P, so that it converges weakly to a bounded operator and, using (1) and the first two lines of (2), its limit is P, that is

$$\mathrm{w} - \sum_{i \in I} (\psi_i, \cdot)\psi_i = P.$$

From this relation it follows that the map j from $\mathcal{K}$ to $\ell^2(I)$

$$j(\phi) = ((\psi_i, \phi))_{i \in I} \quad \phi \in \mathcal{K}$$

is a well defined isometry. Moreover the adjoint of j is the map π from $\ell^2(I)$ onto K explicitly given by

$$\pi((x_i)_{i \in I}) = \sum_{i \in I} x_i \psi_i \quad (x_i)_{i \in I} \in \ell^2(I),$$

where the sum is in the weak topology of $\mathcal{H}$. We observe that $j \circ \pi$ is the projector onto $j(\mathcal{K})$.

If $(f_i)_{i \in I}$ is the canonical basis of $\ell^2(I)$, then $\pi f_i = \psi_i$ for all $i \in I$.

Since $\ell^2(I) = j(\mathcal{K}) \oplus j(\mathcal{K})^\perp$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{N}$ and $\dim \mathcal{N} = \infty$, there exists an (in general not unique) isometry U from $\ell^2(I)$ to $\mathcal{H}$ such that $PU = \pi$. For all $i \in I$ let $e_i = Uf_i$. Since $\mathcal{K} \subset U\ell^2(I)$, then $\{e_i\}_{i \in I}$ is nondegenerate with respect to P. Moreover, for all $i \in I$,

$$W^{\frac{1}{2}} e_i = W^{\frac{1}{2}} P e_i = W^{\frac{1}{2}} PU f_i = W^{\frac{1}{2}} \pi f_i = W^{\frac{1}{2}} \psi_i = w_i^{\frac{1}{2}} \phi_i,$$

so that $\{e_i\}_{i \in I}$ gives the decomposition $\{(P[\phi_i], w_i)\}_{i \in I}$, as claimed. □

Remark 1. Given two different sets nondegenerate with respect to P, $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$, let $\{(P_i, w_i)\}_{i \in I}$ and $\{(Q_i, v_i)\}_{i \in I}$ be the corresponding decompositions of W.

Hence $P_i = Q_i$ for all $i \in I$ if and only if $Pe_i = \alpha_i P f_i$ for some $\alpha_i \in \mathbb{C}$. In this case the two decompositions are equal if and only if $|\alpha_i| = 1$ for all $i \in I$.

The "if" parts of both claims are trivial computations. Conversely, if $P_i = Q_i$ for all $i \in I$, then there exist a family of nonzero numbers $(\alpha_i)_{i\in I}$ such that $W^{\frac{1}{2}} f_i = \alpha_i W^{\frac{1}{2}} e_i$, for all $i \in I$. The claim follows observing that $TW^{\frac{1}{2}} = P$. Moreover, if $w_i = v_i$ for all $i \in I$, then

$$w_i = v_i = \left\| W^{\frac{1}{2}} f_i \right\|^2 = |\alpha_i|^2 \left\| W^{\frac{1}{2}} e_i \right\|^2 = |\alpha_i|^2 w_i,$$

hence $|\alpha_i| = 1$.

Remark 2. Obviously a nondegenerate set can be completed to a Hilbert basis of $\mathcal{H}$. Conversely, from any Hilbert basis we can *extract* a nondegenerate set.

In fact, let $\{f_k\}_{k\geq 1}$ be a basis of $\mathcal{H}$ and $J = \{k \geq 1 \,:\, Pe_k \neq 0\}$. Define for all $k \in J$

$$\begin{aligned} v_k &= (f_k, W f_k) \\ \phi_k &= w_k^{-\frac{1}{2}} W^{\frac{1}{2}} f_k. \end{aligned}$$

Mimicking the first part of the proof of theorem 1, we have that

$$W = \sum_{k\in J} v_k P[\phi_k],$$

so that the *set* $\mathcal{P} = \{P[\phi_k] \,:\, k \in J\}$ decomposes W, but it could happen that $P[\phi_i] = P[\phi_j]$ for some $i \neq j \in J$.

A nondegenerate set $\{e_i\}_{i\in I}$ giving rise to the previous decomposition set $\mathcal{P}$ can be constructed with the following procedure. Let $\{J_i\}_{i\in I}$ be the partition of J such that

1. if $k, k' \in J_i$ then $Pe_k = \alpha P e_{k'}$ for some $\alpha \in \mathbb{C}$;
2. if $k \in J_i$, $k' \in J_{i'}$, $i \neq i'$, then Pe_k is not collinear with $Pe_{k'}$.

For all $i \in I$ define $V_i = \overline{\mathrm{span}}\{f_k \,:\, k \in J_i\}$ and Q_i the orthogonal projector onto V_i. For all $i \in I$ choose an index $k_i \in J_i$, since $Pf_{k_i} \neq 0$ we have that $Q_i P f_{k_i} \neq 0$. Define

$$e_i = \frac{Q_i P f_{k_i}}{\|Q_i P f_{k_i}\|},$$

then $\{e_i\}_{i\in I}$ is obviously non degenerate with respect to P and, by construction, generates the decomposition $\{(P[\phi_{k_i}], w_i)\}_{i\in I}$ where $w_i = \sum_{k\in J_i} v_k$. Moreover $\{P_i\}_{i\in I} = \mathcal{P}$ as claimed.

Remark 3. Theorem 1 and the previous remark give a classification of the decompositions of W by means of the Hilbert bases of $\mathcal{H}$. The same results could be obtained by replacing $\mathcal{H}$ with any other complex, separable, infinite dimensional Hilbert space containing $\mathcal{K}$ as a closed subspace and having

$$\operatorname{codim} \mathcal{K} = \infty. \tag{3}$$

This would only introduce notational complications.

Moreover the first statement of theorem 1 holds without the assumption (3) on the codimension of $\mathcal{K}$. Hence, a decomposition of W corresponds to *any* basis of *any* complex separable Hilbert space containing $\mathcal{K}$ as a closed subspace. The condition (3) assures that the Hilbert space is big enough to give *all* decompositions of W.

Remark 4. It follows from the previous theorem that one pure state $P[\phi]$ is an element of some decomposition of W if and only if $\phi \in \operatorname{Ran} W^{\frac{1}{2}}$. This partial result has been found by Hadjisavvas in [3].

Remark 5. Let A be a simple selfadjoint operator on $\mathcal{H}$ with a pure point spectrum and $\{e_i\}_{i\geq 1}$ a basis of eigenvectors of A ($Ae_i = \lambda_i e_i$). Then, due to remark 2, $\{e_i\}_{i\geq 1}$ gives rise to a decomposition (possibly with some repeated pure state) of the state W whose weights are just the probabilities of obtaining the value λ_i while measuring the physical quantity A when the system is prepared in the state W. This result holds without assumptions on the codimension of $\mathcal{K}$ (see remark 3).

3. Irreducible decompositions

In this section we describe a particular class of decompositions, already studied in [3] and called *irreducible*. For finite decompositions, they are exactly the ones with the same number of elements of the spectral one.

We begin with the notion of irreducible family of vectors. A family $(\phi_i)_{i\in I}$ in $\mathcal{H}$ is *irreducible* if

$$\phi_i \notin \overline{\operatorname{span}}\{\phi_j \,:\, j \in I, j \neq i\} \qquad \forall i \in I;$$

in the mathematical literature an irreducible family is often called *topologically free*. The irreducible families are easily characterised by the following condition:

Lemma 1. *Let $(\phi_i)_{i\in I}$ be a family in $\mathcal{H}$. The following facts are equivalent:*

1. *$(\phi_i)_{i\in I}$ is irreducible;*
2. *there is a family $(\theta_i)_{i\in I}$ in $\mathcal{H}$ such that*
$$(\phi_i, \theta_j) = \delta_{ij} \qquad \forall i, j \in I$$
(we call $(\theta_i)_{i\in I}$ a dual family of $(\phi_i)_{i\in I}$).

Moreover we can always choose the vectors θ_i so that

$$\theta_i \in \overline{\mathit{span}}\{\phi_k\} \qquad \forall i \in I \tag{4}$$

and, in this case, the family $(\theta_i)_{i\in I}$ is uniquely determined by $(\phi_i)_{i\in I}$.

Proof. For all $i \in I$ let

$$V_i = \overline{\operatorname{span}}\{\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots\}.$$

Suppose that $\phi_i \notin V_i$ for all $i \in I$. This implies that $V_i^\perp \neq \{0\}$. Define P_i as the projector onto $V_i^\perp$ and $\psi_i = P_i\phi_i$. By assumption, the vector ψ_i is nonzero. For all $i \in I$, let $\theta_i = \frac{\psi_i}{\|\psi_i\|^2}$, then we have that

$$(\theta_i, \phi_j) = \delta_{ij} \quad \forall i \in I,$$

which proves the existence. Moreover, by construction, the vectors θ_i satisfy the condition (4). Conversely, suppose that $(\phi_i)_{i\in I}$ has a dual family $(\theta_i)_{i\in I}$, then $\theta_i \perp V_i$, since $(\theta_i, \phi_j) = 0$ if $j \neq i$. Moreover $(\theta_i, \phi_i) = 1$, so that $\phi_i \notin V_i$. Now, let $(\theta'_i)_{i\in I}$ be an other dual family of $(\phi_i)_{i\in I}$ satisfying (4), then for all $i \in I$,

$$(\theta'_i - \theta_i, \phi_j) = 0 \quad j \in I,$$

so that $\theta'_i - \theta_i \in \overline{\text{span}}\{\phi_i\}^\perp$. Due to (4), it follows that $\theta'_i = \theta_i$. □

Remark 6. Let $(\phi_i)_{i\in I}$ be an irreducible family and $\mathcal{K} = \overline{\text{span}}\{\phi_i\}$. Due to lemma 1, there exists a unique dual family $(\theta_i)_{i\in I}$ of $(\phi_i)_{i\in I}$ in $\mathcal{K}$. Obviously $(\phi_i)_{i\in I}$ is a dual family of $(\theta_i)_{i\in I}$; nevertheless it can happen that $\overline{\text{span}}\{\theta_i\}$ is a proper subspace of $\mathcal{K}$, so that $(\phi_i)_{i\in I}$ is not the dual family of $(\theta_i)_{i\in I}$ with the property $\phi_i \in \overline{\text{span}}\{\theta_k\}$ (see counterexample in remark 9).

A decomposition $\{(P_i, w_i)\}_{i\in I}$ of a state W is irreducible if there is an irreducible family $(\phi_i)_{i\in I}$ of unit vectors in $\mathcal{H}$ such that $P_i = P[\phi_i]$ for all $i \in I$. In this case

$$W\theta_i = w_i\phi_i \quad \forall i \in I.$$

This proves that an irreducible set $\{P_i\}_{i\in I}$ decomposes W with a unique family of weights.

The following proposition characterises the irreducible decompositions of a state in terms of a property of the nondegenerate sets that generate them *via* theorem 1.

Let W be a state, $\mathcal{K}$ the closure of the range of W and P the projection onto $\mathcal{K}$. As in section 2, we assume that codim $\mathcal{K} = \infty$. Let $\{e_i\}_{i\in I}$ be nondegenerate with respect to P and $\{(P[\phi_i], w_i)\}_{i\in I}$ the corresponding decomposition of W given by theorem 1.

Proposition 1. *The following statements are equivalent:*

1. $\{(P[\phi_i], w_i)\}_{i\in I}$ *is an irreducible decomposition of* W*;*
2. *for all* $i \in I$, $e_i \in \mathrm{Ran}\, W^{\frac{1}{2}}$.

Moreover, in this case, $\{e_i\}_{i\in I}$ *is a Hilbert basis of* $\mathcal{K}$ *uniquely defined by the family* $(\phi_i)_{i\in I}$.

Proof. If $\{(P[\phi_i], w_i)\}_{i\in I}$ is an irreducible decomposition of W, then there is a dual family $(\theta_i)_{i\in I}$ of $(\phi_i)_{i\in I}$ and $W\theta_i = w_i\phi_i$; hence, for all $i \in I$, $\phi_i \in \mathrm{Ran}\, W = \mathrm{Dom}\, T^2$, where T is defined in section 2. By construction

$\phi_i = w_i^{-\frac{1}{2}} W^{\frac{1}{2}} e_i$ so that $Pe_i = w_i^{\frac{1}{2}} T\phi_i \in \operatorname{Ran} W^{\frac{1}{2}}$. Moreover

$$\begin{aligned} \|Pe_i\|^2 &= w_i(T\phi_i, T\phi_i) \\ &= w_i(\phi_i, T^2\phi_i) \\ &= (\phi_i, \theta_i) \\ &= 1, \end{aligned}$$

so that $Pe_i = e_i$ for all $i \in I$ and this proves the first implication. Conversely, if $e_i \in \operatorname{Ran} W^{\frac{1}{2}}$ for all $i \in I$, then $\left(w_i^{\frac{1}{2}} Te_i\right)_{i\in I}$ is a dual family of $\left(w_i^{-\frac{1}{2}} W^{\frac{1}{2}} e_i\right)_{i\in I} = (\phi_i)_{i\in I}$, proving that $(\phi_i)_{i\in I}$ is irreducible.

Finally, since $\operatorname{Ran} W^{\frac{1}{2}} \subset \mathcal{K}$, $e_i \in \mathcal{K}$ for all $i \in I$. By definition of nondegenerate set, $\overline{\operatorname{span}}\{e_i\} \supset \mathcal{K}$ so that $\{e_i\}_{i\in I}$ is a basis of $\mathcal{K}$ and it is clearly uniquely defined by the family $(\phi_i)_{i\in I}$. □

Remark 7. Using this result we have a one to one correspondence (up to phase factors) between the irreducible decomposition of W and the Hilbert bases of $\mathcal{K}$ contained in $\operatorname{Ran} W^{\frac{1}{2}}$. Due to this property, when dealing with irreducible decompositions we can drop the assumption on the codimension of $\mathcal{K}$.

Remark 8. As a particular case of the previous proposition we obtain the following result of Hadjisavvas, [3]: one pure state $P[\phi]$ is an element of some irreducible decomposition of W if and only if $\phi \in \operatorname{Ran} W$.

We now turn to the problem (b) posed in the introduction: given a state W and an irreducible family $(\phi_i)_{i\in I}$ of unit vectors in $\mathcal{H}$, determine whether W can be decomposed in terms of the set of pure states $(P[\phi_i])_{i\in I}$. If this is the case then, as one readily verifies,

$$\overline{\operatorname{span}}\{\phi_i\} = \overline{\operatorname{Ran}}\, W$$

and the corresponding family of weights $(w_i)_{i\in I}$ is unique. Due to remark 7 we can assume without loss of generality that the state is injective.

Let W be an injective state, $T = W^{-\frac{1}{2}}$, $(\phi_i)_{i\in I}$ be an irreducible family of unit vectors of $\mathcal{H}$ such that $\overline{\operatorname{span}}\{\phi_i\} = \mathcal{H}$ and $(\theta_i)_{i\in I}$ be its uniquely defined dual family in $\mathcal{H}$. The following theorem solves the problem of the decomposability of W on the set $\{P[\phi_i]\}_{i\in I}$ under the very weak assumption

$$\overline{\operatorname{span}}\{\theta_i\} = \mathcal{H}. \tag{5}$$

Theorem 2. *With the previous assumption* (5)*, the following facts are equivalent:*

1) *the set* $\{P[\phi_i]\}_{i\in I}$ *decomposes* W*;*
2) $W\theta_i = \alpha_i\phi_i$ *for all* $i \in I$*, with* $\alpha_i \in \mathbb{C}$*;*
3) $\phi_i \in \operatorname{Ran} W$ *and* $(T\phi_i, T\phi_j) = 0$ *if* $i \neq j$*.*

In this case the family of weights of the decomposition is exactly $(\alpha_i)_{i\in I}$*.*

Proof. 1) $\Rightarrow$ 3)**:** Since $W = \sum w_i P[\phi_i]$ for some family of weights $(w_i)_{i\in I}$, we have that $W\theta_i = w_i\phi_i$, which in turn implies $W^{\frac{1}{2}}\theta_i = w_i T\phi_i$. The claim follows from these relations.

3) $\Rightarrow$ 2)**:** Since $\phi_i \in \operatorname{Ran} W = \operatorname{Dom} T^2 \subset \operatorname{Dom} T$ and T is injective, then $T\phi_i \neq 0$ and

$$\frac{1}{\|T\phi_i\|^2}(T^2\phi_i, \phi_j) = \delta_{ij}.$$

As the dual family $(\theta_i)_{i\in I}$ is unique, we have that $\theta_i = \dfrac{T^2\phi_i}{\|T\phi_i\|^2}$. The thesis follows with $\alpha_i = \dfrac{1}{\|T\phi_i\|^2}$.

2) $\Rightarrow$ 1)**:** Since W is positive and injective, then $\alpha_i = (\theta_i, W\theta_i) > 0$. By hypothesis $\phi_i \in \operatorname{Dom} T^2$, hence $e_i = \alpha_i^{\frac{1}{2}} T\phi_i$ is well defined and in $\operatorname{Dom} T$. It is easy to prove that $(e_i)_{i\in I}$ is an orthonormal family, so that, for all finite subset J of I,

$$\begin{aligned}\sum_{i\in J} \alpha_i P[\phi_i] &= \sum_{i\in J} (W^{\frac{1}{2}} e_i, \cdot) W^{\frac{1}{2}} e_i \\ &= W^{\frac{1}{2}} \left(\sum_{i\in J} (e_i, \cdot) e_i \right) W^{\frac{1}{2}}.\end{aligned}$$

This shows that the net $\left(\sum_{i\in J} \alpha_i P[\phi_i]\right)$ converges weakly to a bounded operator A. By construction A equals W on the algebraic span of the vectors θ_i, hence also on its closure that, by assumption, is $\mathcal{H}$. To show that $\{(P[\phi_i], \alpha_i)\}_{i\in I}$ is a decomposition of W we observe that, for all finite subset J of I, $\sum_{i\in J} \alpha_i P_i \leq W$. Taking the trace of both sides one has $\sum_{i\in J} \alpha_i \leq 1$ which shows that $(\alpha_i)_{i\in I}$ is summable, hence $(\alpha_i P_i)_{i\in I}$ is summable in trace norm and its sum is necessarily W. From the continuity of the trace it follows that $(\alpha_i)_{i\in I}$ is a family of weights.

□

We observe that, without the hypothesis (5), the relations among the three statements of theorem 2 are

$$1) \Rightarrow 2) \Leftrightarrow 3).$$

In particular, the following counterexample shows that 2) $\not\Rightarrow$ 1).

Remark 9. Let W be an injective state on an infinite dimensional separable Hilbert space $\mathcal{H}$. Let $\{f_i\}_{i\geq 1}$ be a basis of eigenvectors of W and $(\lambda_i)_{i\geq 1}$ be the corresponding family of eigenvalues. Since W is a positive, trace class operator with trace one, the vector

$$e_1 = \sum_{i\geq 1} \lambda_i^{\frac{1}{2}} f_i$$

is a well defined unit vector in $\mathcal{H}$. A trivial computation shows that $e_1 \notin \operatorname{Ran} W^{\frac{1}{2}}$ (here we use the fact that $\dim \mathcal{H} = \infty$). For all $i \geq 2$, let

$$v_i = \lambda_i^{\frac{1}{2}} f_{i-1} - \lambda_{i-1}^{\frac{1}{2}} f_i,$$

and V be the algebraic span of the vectors v_i. Due to the fact that W is injective, the eigenvalues λ_i are nonzero for all $i \geq 1$; from this it follows that $V \subset \operatorname{Ran} W^{\frac{1}{2}}$ and the set of vectors $\{v_i \ : \ i \geq 2\}$ is linearly independent. Let $\{e_i\}_{i\geq 2}$ be the orthonormal set in V obtained using the Gram-Schmidt procedure on the set $\{v_i\}_{i\geq 2}$. Then $V = \operatorname{span}\{e_i, \ i \geq 2\}$ and, for all $i \geq 2$, $e_i \in \operatorname{Ran} W^{\frac{1}{2}}$. Moreover, by an easy calculation, e_1 is orthogonal to v_i for all $i \geq 2$, so that $\{e_i\}_{i\geq 1}$ is an orthonormal set. We prove that $\{e_i\}_{i\geq 1}$ is a basis of $\mathcal{H}$. In fact, let $x \in \mathcal{H}$ be such that x is orthogonal to e_i for all $i \geq 1$, then x is orthogonal to v_i for all $i \geq 2$. Using this condition it follows that, if $a_i = (f_i, x)$,

$$a_i = \frac{\lambda_i^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}} a_1 \qquad \forall i \geq 2.$$

Moreover, since $(e_1, x) = 0$, we have that $a_1 = 0$, so that $x = 0$.

Now, $\{e_i\}_{i\geq 1}$ is a basis of $\mathcal{H}$, hence, due to theorem 1 and remark 3, it defines a decomposition $\{(P[\phi_i], w_i)\}_{i\geq 1}$ of W where

$$\phi_i = w_i^{-\frac{1}{2}} W^{\frac{1}{2}} e_i.$$

Since $e_1 \notin \operatorname{Ran} W^{\frac{1}{2}}$, the family $(\phi_i)_{i\geq 1}$ is not irreducible, due to proposition 1.

For all $i \geq 2$, $e_i \in \operatorname{Ran} W^{\frac{1}{2}}$, so that the vectors

$$\theta_i = w_i^{\frac{1}{2}} T e_i$$

are well defined and the following relations hold

$$\begin{aligned} (\phi_i, \theta_j) &= \delta_{ij} \quad i \geq 1, j \geq 2 \\ W\theta_i &= w_i \phi_i \quad i \geq 2. \end{aligned} \tag{6}$$

Since W is injective, $\overline{\operatorname{span}}\{\phi_i, \ i \geq 1\} = \mathcal{H}$. Due to (6), $(\phi_i)_{i\geq 2}$ is irreducible, whereas $(\phi_i)_{i\geq 1}$ is not irreducible, so that $\phi_1 \in \overline{\operatorname{span}}\{\phi_i, \ i \geq 2\}$, hence

$$\overline{\operatorname{span}}\{\phi_i, \ i \geq 2\} = \mathcal{H}.$$

Now, we have an irreducible family $(\phi_i)_{i\geq 2}$ such that

$$\begin{aligned} \overline{\operatorname{span}}\{\phi_i, \ i \geq 2\} &= \mathcal{H} \\ W\theta_i &= w_i \phi_i \quad \forall i \geq 2, \end{aligned}$$

nevertheless $W \neq \sum_{i\geq 2} w_i P[\phi_i]$.

Moreover, due to (6), $(\theta_i, \phi_1) = 0$ for all $i \geq 2$, so that

$$\overline{\operatorname{span}}\{\theta_i, \ i \geq 2\} \neq \mathcal{H}.$$

Remark 10. It is worth noting that the statement 1) of theorem 2 does not imply the assumption (5). In fact, let $(\phi_i)_{i\geq 2}$ be the irreducible family of the previous remark and $(\nu_i)_{i\geq 2}$ be a family of weights, then, obviously, $\{(P[\phi_i], \nu_i)_{i\geq 2}\}$ is an irreducible decomposition of the state $\sum_{i\geq 2} \nu_i P[\phi_i]$, but the condition (5) does not hold because

$$\overline{\text{span}}\{\theta_i,\ i \geq 2\} \quad \neq \quad \mathcal{H}.$$

As a particular case, we observe that if the family $(\phi)_{i\in I}$ is a Schauder basis of $\mathcal{H}$, then both $(\phi)_{i\in I}$ is irreducible and the condition (5) holds.

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References

[1] P. Bush, P.J. Lahti, P. Mittelstaedt, “The Quantum Theory of Measurement”, Springer-Verlag, Berlin, 1991.

[2] B.C. van Frassen, “Quantum Mechanics: an Empiricist View”, Oxford University Press, Oxford, 1991.

[3] N. Hadjisavvas, Properties of mixtures of non-orthogonal states, *Lett. Math. Phys.*, **5** (1981), 327-332.

[4] L.P. Hughston, R. Jozsa, W.K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Phys. Lett.*, **183A** (1993), 14-18.

[5] E.T. Jaynes, Information Theory and Statistical Mechanics. II, *Phys. Rev.*, **108** (1957), 171-190.

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