On Coorbit Fréchet Spaces

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Abstract

This paper is concerned with a new approach to coorbit space theory. Usually, coorbit spaces are defined by collecting all distributions for which the voice transform associated with a square-integrable group representation possesses a certain decay, usually measured in a Banach space norm such as weighted L_p-norms. Unfortunately, in cases where the representation does not satisfy certain integrability conditions, one is faced with a bottleneck, namely that the discretization of the coorbit spaces is surprisingly diffficult. It turns out that in these cases the construction of coorbit spaces as Fréchet spaces is much more convenient since then atomic decompositions can be established in a very natural way.

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1 Introduction

One of the most important tasks in applied mathematics is the analysis of signals. These signals might be given explicitly, e.g., in image analysis, or implicitly, as solutions of operator equations, say. The first step is always to apply a suitable transformation. By now, an impressive amount of different transformations exists, such as the Fourier transform, the Gabor transform, the wavelet transform, or the shearlet transform, just to name a few. Which one to choose clearly depends on the information one wants to extract from the signal An important observation is that many of these transforms stem from a square-integrable group representation, e.g., the wavelet transform is associated with the affine group or $ax + b$-group. At this point, the very important coorbit theory comes into play which has been developed by Feichtinger and Gröchenig in a series of papers [5, 7, 8, 10]. We also refer to [2], Chapter 3.2 and [1] for an overview. This theory allows for a unified treatment of the different transformations. Moreover, it provides the construction of natural smoothness spaces, the coorbit spaces, where smoothness is measured by the decay of the voice transform associated with the group representation. In addition, by discretizing the representation, atomic

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decompositions and Banach frames for the coorbit spaces can be obtained. To apply this theory, besides the square integrability, a second very important condition has to be satisfied, namely, the reproducing kernel associated with the representation has to be contained in a weighted L_1-space. Unfortunately, in some natural cases, this assumption is not satisfied, see, e.g., Section 5. Nevertheless, in [4], it has been shown that the coorbit theory can also be generalized to this case. Then, the space of distributions from which the coorbit spaces are constructed, has to be modified. Instead of using the dual of a Banach space, it is more suitable to choose the dual space of a Frechet space. By proceeding this way, coorbit spaces can again be constucted. So far, so good. But this is only half the truth. When it comes to practical applications, clearly only discrete data can be handled, and therefore a suitable discretization, e.g., an atomic decomposition and/or a frame is needed. For the Frechet setting, this turns out to be surprisingly difficult. This problem has been intensively studied in [3], but the results look in a certain sense ugly and suboptimal. Then the question arises what might be the reason for this. One conjecture could be: the test and the distribution spaces are Frechet spaces, but the coorbit spaces are Banach spaces. Maybe it is more natural to define the coorbits also as Frechet spaces? In this paper, we follow this line of research. And indeed, it turns out that under some reasonable conditions the existence of an atomic decomposition can be established in quite a natural way. This supports our feeling that we follow a feasible path.

Outline The paper is structured in the following way. We start by recalling basic facts from coorbit theory on Fréchet spaces in Section 2. Then, in Section 3, we introduce our new concept of coorbit spaces as Frechet spaces. Then, in Section 4, we state and prove our main result which provides atomic decompositions in Frechet coorbit spaces. In Section 5, we provide and analyze two examples of coorbit Fréchet spaces, namely Shannon wavelet spaces and generalized modulation spaces. Some useful facts are proved in the appendix.

2 An Overview

In this section we introduce the notation and the basics of coorbit theory on Fréchet spaces, as introduced in [4] and then summarized in [3]. For details the reader is referred to the aforementioned literature.

Throughout this paper G denotes a fixed locally compact second countable group with a left Haar measure and with modular function Δ. We denote by $\int_G f(x)\,\mathrm{d}x$ the integrals with respect to the Haar measure and by $L_0(G)$ the space of Borel-measurable functions. Given $f \in L_0(G)$ the functions $\check{f}$ and $\overline{f}$ are

$$\check{f}(x) = f(x^{-1}), \qquad \overline{f}(x) = \overline{f(x)},$$

and for all $x \in G$ the left and right regular representations λ and ρ act on f as

$$\begin{aligned} \lambda(x)f(y) &= f(x^{-1}y) && \text{a.e } y \in G, \\ \rho(x)f(y) &= f(yx) && \text{a.e } y \in G. \end{aligned}$$

The convolution $f * g$ between $f, g \in L_0(G)$ is the function

$$f * g(x) = \int_G f(y)g(y^{-1}x)\ dy = \int_G f(y) \cdot (\lambda(x)\check{g})(y)\ dy \qquad \text{a.e. } x \in G,$$

provided that the function $y \mapsto f(y) \cdot (\lambda(x)\check{g})(y)$ is integrable for almost all $x \in G$.

We fix a continuous weight $w : G \to (0, \infty)$ satisfying

$$w(xy) \le w(x)w(y), \tag{1a}$$

$$w(x) = w(x^{-1}) \tag{1b}$$

for all $x, y \in G$. It is worthwhile observing that the above properties imply that

$$\inf_{x \in G} w(x) \geq 1.$$

For all $p \in [1, \infty)$ the corresponding weighted Lebesgue space is the separable Banach space

$$L_{p,w}(G) = \left\{ f \in L_0(G) \;\middle|\; \int_G |w(x) f(x)|^p \, dx < \infty \right\}$$

with norm

$$\|f\|_{L_{p,w}}^p = \int_G |w(x) f(x)|^p \, dx,$$

and the obvious modifications for $L_\infty(G)$, which however is not separable. When $w \equiv 1$, that is, in the unweighted case, we simply write $L_p(G)$.

With terminology as in [4] we choose, as a *target space* for the coorbit theory, the space

$$\mathcal{T}_w = \bigcap_{1<p<\infty} L_{p,w}(G).$$

We recall some basic properties of $\mathcal{T}_w$ (see Theorem 4.3 of [4], which is based on results in [6]). We endow $\mathcal{T}_w$ with the (unique) topology such that a sequence $(f_n)_{\in \mathbb{N}}$ in $\mathcal{T}_w$ converges to 0 if and only if $\lim_{n\to+\infty} \|f_n\|_{L_{p,w}} = 0$ for all $1 < p < \infty$. With this topology, $\mathcal{T}_w$ becomes a reflexive Fréchet space. The (an ti)-linear dual space of $\mathcal{T}_w$ can be identified with

$$\mathcal{U}_w = \operatorname{span} \bigcup_{1<q<\infty} L_{q,w^{-1}}(G)$$

under the pairing

$$\int_G \Phi(x) \overline{f(x)} \, dx = \langle \Phi, f \rangle_w, \qquad \Phi \in \mathcal{U}_w, \ f \in \mathcal{T}_w. \tag{2}$$

Take now a (strongly continuous) unitary representation π of G acting on a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ linear in the first entry. We assume that π is reproducing, namely that there exists a vector $u \in \mathcal{H}$ such that the corresponding voice transform

$$Vv(x) = \langle v, \pi(x) u \rangle_{\mathcal{H}}, \qquad v \in \mathcal{H}, \ x \in G,$$

is an isometry from $\mathcal{H}$ into $L_2(G)$. In this case, u is referred to as an *admissible vector*. We observe that this implies that V is injective, when $\operatorname{span} \{\pi(x) u\}_{x \in G}$ is dense in $\mathcal{H}$. We denote by K the reproducing kernel associated to u, namely

$$K(x) = Vu(x) = \langle u, \pi(x) u \rangle_{\mathcal{H}}, \qquad x \in G, v \in \mathcal{H}.$$

It is a bounded continuous function and enjoys the fundamental properties:

$$\begin{aligned}
&\overline{K} = \check{K}, \\
&\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i^{-1} x_j) \geq 0, \qquad c_1, \ldots, c_n \in \mathbb{C}, \ x_1, \ldots, x_n \in G, \\
&K * K = K \in L_2(G).
\end{aligned}$$

For the remainder of the paper, we will work under the following basic hypothesis.

Assumption 1. *We assume* $K \in \mathcal{T}_w$, i.e*,*

$$K \in L_{p,w}(G) \text{ for all } 1 < p < \infty. \tag{4}$$

Observe that if w^{-1} belongs to $L_q(G)$ for some $1 < q < \infty$, then Hölder's inequality shows $K \in L_1(G)$, but in general $K \notin L_{1,w}(G)$. Indeed, in many interesting examples w is independent of one or more variables, so that $w^{-1} \notin L_q(G)$ for all $1 < q < \infty$. Thus, in these circumstances there is no guarantee that $K \in L_1(G)$ and in fact in some known instances (see below) this does not happen.

Another key ingredient of the coorbit theory is given by the *test space* $\mathcal{S}_w$, defined as

$$\mathcal{S}_w = \left\{ v \in \mathcal{H} \;\middle|\; Vv \in L_{p,w}(G) \text{ for all } 1 < p < \infty \right\},$$

which becomes a locally convex topological vector space with the family of semi-norms

$$\|v\|_{p,\mathcal{S}_w} = \|Vv\|_{L_{p,w}}.$$

We recall the main properties of $\mathcal{S}_w$.

Theorem 1 (Theorem 4.4 of [4]). *Under the assumption* (4)*, the following properties hold true.*

1. *The space $\mathcal{S}_w$ is a reflexive Fréchet space, continuously and densely embedded in $\mathcal{H}$.*
2. *The representation π leaves $\mathcal{S}_w$ invariant and its restriction to $\mathcal{S}_w$ is a continuous representation.*
3. *The space $\mathcal{H}$ is continuously and densely embedded into the (anti)-linear dual $\mathcal{S}'_w$, where both spaces are endowed with the weak topology.*
4. *The restriction of the voice transform $V : \mathcal{S}_w \to \mathcal{T}_w$ is a topological isomorphism from $\mathcal{S}_w$ onto the closed subspace $\mathcal{M}^{\mathcal{T}_w}$ of $\mathcal{T}_w$, given by*

   $$\mathcal{M}^{\mathcal{T}_w} = \{ f \in \mathcal{T}_w \mid f * K = f \},$$

   and it intertwines π and λ.
5. *For every $f \in \mathcal{T}_w$, there exists a unique element $\pi(f)u \in \mathcal{S}_w$ such that*

   $$\langle \pi(f)u, v\rangle_{\mathcal{H}} = \int_G f(x)\langle \pi(x)u, v\rangle_{\mathcal{H}}\, dx = \int_G f(x)\overline{Vv(x)}\, dx, \qquad v \in \mathcal{H}.$$

   Furthermore, for every $f \in \mathcal{T}_w$

   $$V\pi(f)u = f * K,$$

   the map

   $$\mathcal{T}_w \ni f \mapsto \pi(f)u \in \mathcal{S}_w$$

   is continuous and its restriction to $\mathcal{M}^{\mathcal{T}_w}$ is the inverse of V.

Here and in what follows the notation $\pi(f)u$ is motivated by the fact that any function $f \in L_1(G)$ defines a bounded operator $\pi(f)$ on $\mathcal{H}$, which is weakly given by

$$\langle \pi(f)v, v'\rangle_{\mathcal{H}} = \int_G f(x)\langle \pi(x)v, v'\rangle_{\mathcal{H}}\, dx, \qquad v, v' \in \mathcal{H},$$

see for example Sect. 3.2 of [9]. However, if $f \notin L_1(G)$, then in general $\pi(f)v$ is well defined only if $v = u$, where u is an admissible vector for the representation π.

Recalling that the (anti-)dual of $\mathcal{T}_w$ is $\mathcal{U}_w$ under the pairing (2), we denote by ${}^tV$ the contragradient map ${}^tV : \mathcal{U}_w \to \mathcal{S}'_w$ given by

$$\langle {}^tV\Phi, v\rangle_{\mathcal{S}_w} = \langle \Phi, Vv\rangle_w, \qquad \Phi \in \mathcal{U}_w,\ v \in \mathcal{S}_w.$$

As usual, we extend the voice transform from $\mathcal{H}$ to the (anti-)dual $\mathcal{S}'_w$ of $\mathcal{S}_w$, where $\mathcal{S}'_w$ plays the role of the space of distributions. For all $T \in \mathcal{S}'_w$ we define the *extended voice transform* of T by

$$V_eT(x) = \langle T, \pi(x)u\rangle_{\mathcal{S}_w}, \qquad x \in G,$$

which is a continuous function on G by item 2) of the previous theorem and $\langle\cdot,\cdot\rangle_{\mathcal{S}_w}$ denotes the pairing between $\mathcal{S}_w$. Here $\mathcal{S}'_w$, whereas $\langle\cdot,\cdot\rangle_w$ is the pairing between $\mathcal{T}_w$ and $\mathcal{U}_w$.

We summarize the main properties of the extended voice transform in the next theorem.

Theorem 2 (Theorem 4.4 of [4]). *Under assumption* (4)*, the following facts hold true.*

1. *For every* $\Phi \in \mathcal{U}_w$ *there exists a unique element* $\pi(\Phi)u \in \mathcal{S}'_w$ *such that*

$$\langle \pi(\Phi)u, v\rangle_{\mathcal{S}_w} = \int_G \Phi(x)\langle \pi(x)u, v\rangle_{\mathcal{H}}\, dx = \int_G \Phi(x)\overline{Vv(x)}\, dx, \qquad v \in \mathcal{S}_w.$$

Furthermore, it holds that

$$V_e\pi(\Phi)u = \Phi * K.$$

2. *For all* $T \in \mathcal{S}'_w$ *the extended voice transform* V_eT *is in* $\mathcal{U}_w$ *and satisfies*

$$V_eT = V_eT * K, \tag{5}$$

$$\langle T, v\rangle_{\mathcal{S}_w} = \langle V_eT, Vv\rangle_w, \qquad v \in \mathcal{S}_w. \tag{6}$$

3. *The extended voice transform* V_e *is injective, it is continuous from* $\mathcal{S}'_w$ *into* $\mathcal{U}_w$ *(when both spaces are endowed with the strong topology), its range is the closed subspace*

$$\mathcal{M}^{\mathcal{U}_w} = \{\Phi \in \mathcal{U}_w \mid \Phi * K = \Phi\} = \operatorname{span} \bigcup_{p\in(1,\infty)} \mathcal{M}^{L_{p,w}(G)} \subset L_{\infty,w^{-1}}(G)$$

and it intertwines the contragradient representation of $\pi_{|\mathcal{S}_w}$ *and* $\lambda_{|\mathcal{U}_w}$.

4. *The map*

$$\mathcal{M}^{\mathcal{U}_w} \ni \Phi \mapsto \pi(\Phi)u \in \mathcal{S}'_w$$

is the left inverse of V_e *and coincides with the restriction of the map* ${}^tV$ *to* $\mathcal{M}^{\mathcal{U}_w}$*, namely*

$$V_e({}^tV\Phi) = V_e\pi(\Phi)u = \Phi, \qquad \Phi \in \mathcal{M}^{\mathcal{U}_w}.$$

5. *Concerning the double inclusion* $\mathcal{S}_w \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}'_w$*, we have*

$$\mathcal{S}_w = \left\{T \in \mathcal{S}'_w \mid V_eT \in \mathcal{T}_w\right\} = \left\{\pi(f)u \,\middle|\, f \in \mathcal{M}^{\mathcal{T}_w}\right\}.$$

Item 2) of the previous theorem states that the voice transform of any distribution $T \in \mathcal{S}'_w$ satisfies the reproducing formula (5) and uniquely defines the distribution T by means of the reconstruction formula (6), i.e.

$$T = \int_G \langle T, \pi(x)u\rangle_{\mathcal{S}_w} \pi(x)u\; dx,$$

where the integral is a Dunford-Pettis integral with respect to the duality between $\mathcal{S}_w$ and $\mathcal{S}'_w$, see, for example, Appendix 3 of [9].

3 Fréchet Coorbit Spaces

In this section, we present our new construction of coorbit spaces as Frechet spaces. For simplicity, we will restrict ourselves to the case where the coorbit spaces are related with intersections of weighted L_p-spaces. In particular, we will show that the most important result in the realm of coorbit space theory, the Correspondence Principle, also holds in our case, see Theorem 4 below.

We fix a w-moderate weight m, i.e. a continuous function $m : G \to (0,\infty)$ such that

$$m(xy) \le w(x)\cdot m(y) \tag{7}$$

$$m(xy) \le m(x)\cdot w(y).$$

The above conditions are equivalent to

$$m(xyz) \leq w(x) \cdot m(y) \cdot w(z) \quad \text{for all } x, y, z \in G$$

up to the constant $w(e)$. It is worth observing that if m is a w-moderate weight on G, then so is m^{-1}, see Lemma 4.1 in [3].

Appealing again to the terminology used in [4], we choose as a *model space* Y for the coorbit space theory the Fréchet space (see Theorem 3 below)

$$Y = \bigcap_{1<p<\infty} L_{p,m}(G).$$

instead of the single Banach space $L_{r,m}(G)$ as it is done in [3]. The space Y is endowed with the initial topology that makes all the inclusions $\iota_p \colon Y \hookrightarrow L_{p,m}(G)$ continuous. It is worth pointing out that the notation $\mathcal{T}_m$ to denote Y would be improper because m is neither necessarily submultiplicative nor symmetric. In particular, the general properties enjoyed by $\mathcal{T}_w$ are not automatically satisfied by Y, but hold in the sense clarified in the next result.

Theorem 3. *1. The space Y is a Fréchet space continuously embedded into $L_0(G)$ via the natural inclusion $j \colon Y \hookrightarrow L_0(G)$. Furthermore*

$$Y' = Y^\sharp = \bigcup_{1<p<\infty} L_{p,m^{-1}}(G),$$

where $Y^\sharp = \left\{g \in L_0(G) \,\middle|\, g j(f) \in L^1(G) \text{ for all } f \in Y\right\}$ is the Köthe dual of Y.

2. The left regular representation leaves Y invariant and its restriction to Y is a continuous representation.

3. For all $f \in \mathcal{T}_w$ and $F \in Y$, it holds that $fF \in L_1(G)$.

Proof. 1. This is a consequence of Theorem 4.3 in [4] with w replaced by m because the proof does not depend on the properties (1a) and (1b) of w.

2. This is a consequence of Theorem 4.3 together with Lemma 4.2 in [4]. The proof of the latter is based on the sub-multiplicative property (1a) of w, which may be replaced by (7), namely

$$\int_G |m(y) f(x^{-1}y)|^p \,\mathrm{d}y = \int_G |m(xy) f(y)|^p \,\mathrm{d}y \leq w(x)^p \int_G |m(y) f(y)|^p \,\mathrm{d}y.$$

3. Theorem 4.3 in [4] shows that $Y \subseteq \mathcal{U}_w = \mathcal{T}_w^\sharp$. Further, the symmetry property (1b) gives

$$m(e) = m(xx^{-1}) \leq m(x) w(x^{-1}) = m(x) w(x)$$

and hence

$$w^{-1}(x) \leq \frac{m(x)}{m(e)}.$$

The definition of Köthe dual then entails that $fF \in L_1(G)$ for every $f \in \mathcal{T}_w$ and every $F \in Y$. □

Following the usual steps of the theory, the coorbit space corresponding to Y is defined by

$$\mathrm{Co}(Y) = \left\{T \in \mathcal{S}'_w \,\middle|\, V_e T \in Y\right\},$$

i.e., $\mathrm{Co}(Y)$ is the vector space

$$\mathrm{Co}(Y) = \left\{T \in \mathcal{S}'_w \,\middle|\, V_e T \in \bigcap_{1<p<\infty} L_{p,m}(G)\right\} = \bigcap_{1<p<\infty} \mathrm{Co}(L_{p,m}(G)).$$

We summarize the main properties of $\mathrm{Co}(Y)$ in the following result, which is an adaptation of Proposition 4.1 of [3] to the present setup.

Theorem 4. *1. The image under the extended voice transform of the coorbit space is*

$$V_e(\mathrm{Co}(Y)) = \{F \in Y \mid F * K = K\},$$

a closed subspace of Y *denoted* $\mathcal{M}^Y$.

2. *For all* $f \in \mathcal{M}^Y$*, the Fourier transform* $\pi(f)u$ *given as in Theorem* (2) *by*

$$\langle \pi(f)u, v\rangle_{\mathcal{S}_w} = \int_G f(x)\langle \pi(x)(u), v\rangle_{\mathcal{H}}\,\mathrm{d}x, \qquad v \in \mathcal{S}_w,$$

belongs to the coorbit space $\mathrm{Co}(Y)$.

3. *The extended voice transform on* $\mathrm{Co}(Y)$ *and the Fourier transform on* $\mathcal{M}^Y$ *at* u *satisfy*

$$V_e\,(\pi(f)u) = f, \qquad f \in \mathcal{M}^Y$$
$$\pi(V_eT)u = T, \qquad T \in \mathrm{Co}(Y).$$

4. V_e *is a bijection from* $\mathrm{Co}(Y)$ *onto* $\mathcal{M}^Y$ *whose inverse is the map* $f \mapsto \pi(f)u$.

Proof. Proposition 2.2 in [4] ensures that for every $f \in Y$ the convolution $f * K$ is well defined. Indeed, the asumption that $K \in Y^\sharp$ used in the proof is equivalent to assuming that $fK \in L_1(G)$ for every $f \in Y$, and this is satisfied by the first item in Theorem 3 because $K \in \mathcal{T}_w$. Further, Proposition 2.4 in [4], whereby one has to take $E = \mathcal{S}_w$, ensures that $\pi(f)u$ is a well defined element of $\mathcal{S}_w'$ for every $f \in Y$. Indeed, the requirement (13) of that proposition is satisfied because it consists in the fact that $fVv \in L_1(G)$ for all $f \in Y$ and every $v \in \mathcal{S}_w$, which is implied by the last item in Theorem 3 because $Vv \in \mathcal{T}_w$. The content of the present theorem is then the same as that of Proposition 2.6 in [4], where again one has to put $E = \mathcal{S}_w$ and also $F = Y$. Indeed, as already observed, condition (13) in [4] is satisfied. Hence (15) in [4] holds true for every $f \in Y$, namely $V_e\,(\pi(f)u) = f * K$, but since $\mathcal{M}^Y \subseteq Y$ we have

$$V_e\,(\pi(f)u) = f * K = f$$

for every $f \in \mathcal{M}^Y$. Therefore $V_e\,(\pi(f)u) \in Y$ and hence $\pi(f)u \in \mathrm{Co}(Y)$ for every $f \in \mathcal{M}^Y$. This proves 2 and the first equality in 3. In order to conclude it is enough to show that $V_eT * K = V_eT$ for every $T \in \mathrm{Co}(Y)$. If so, then Lemma 2.5 in [4] tells us that this is equivalent to the second equality in 3. The fact that $V_eT * K = V_eT$ for every $T \in \mathrm{Co}(Y)$ is exactly (21a) in Proposition 2.6 in [4]. □

By means of the above theorem, which will be referred to as the Correspondence Principle, it is possible to endow $\mathrm{Co}(Y)$ with a natural Fréchet topology, simply by transfering via (the inverse of) the extended voice transform V_e the Fréchet topology that $\mathcal{M}^Y$ inherits as closed subspace of Y. The idea of transfering the topology using the extended voice transform mimics the strategy that is taken in the Banach space setup where the coorbit norm is defined using the Correspondence Principle.

4 Atomic Decomposition

After the topology and the properties of the generalized coorbit spaces $\mathrm{Co}(Y)$ have been established, the next task clearly is the construction of suitable discretizations of these spaces, i.e., to provide some kind of an atomic decomposition. The first question is what an atomic decomposition of a Fréchet space should be. In the classical coorbit setting, this means that there exists a countable family of functions in the coorbit space, the atoms, such that every element can be written as a linear combination of these atoms, and that norm equivalences of the coorbit norms and weighted sequence norms of the coefficients hold. Now, these norm equivalences can be interpreted as the fact that linear mappings from the coorbit spaces to the weighted sequence spaces are continuous, and vice versa. On a Fréchet space, one does not have a norm, but nevertheless a topology so that continuity of mappings is well-defined. Therefore, the following definition arises naturally.

Definition 1. *Let $\mathcal{F}$ be a Fréchet space, and let $b(\Lambda)$ be a sequence space. A set $\{a_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{F}$ gives rise to an* atomic decomposition *if every $f \in \mathcal{F}$ has an expansion*

$$f = \sum_{\lambda\in\Lambda} c_\lambda(f) a_\lambda$$

and the mappings

(i) Analysis:

$$A \ : \ \mathcal{F} \longrightarrow b(\Lambda), \qquad A(f) = \{c_\lambda(f)\}_{\lambda\in\Lambda}$$

(ii) Synthesis:

$$S \ : \ b(\Lambda) \longrightarrow \mathcal{F}, \qquad S(\{d_\lambda\}_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} d_\lambda a_\lambda$$

are well-defined and continuous.

To establish such an atomic decomposition for $\mathrm{Co}(Y)$, observe that by Theorem 4 , and thus by the topology defined on $\mathrm{Co}(Y)$, it is enough to prove an atomic decomposition for $\mathcal{M}^Y$. As in the classical setting, we are looking at atoms of the form $L_{g_i}K$ where $\{g_i\}_{i\in I}$ is a suitable family of elements in G and the corresponding coefficients are given by

$$a_i(f) = \langle f, \phi_i\rangle,$$

where $\{\phi_i\}_{i\in}$ is a suitable partition of unity of G. To this end, we start with a Q-dense set in the group G, whereby Q is a compact set with nonvoid interior and $e \in Q$. Then, a countable family $X = (g_i)_{i\in\mathcal{I}}$ is said to be Q*-dense* if $\cup_{i\in\mathcal{I}} g_i Q = G$. Moreover, let $\Phi = (\phi_i, X, Q)$ denote a partition of unity subordinate to the Q-dense set X, i.e.,

$$\mathrm{supp}\phi_i \subset g_i Q, \quad 0 \le \phi_i \le 1 \text{ for all } i \in \mathcal{I}, \quad \sum_{i\in\mathcal{I}} \phi_i = 1.$$

Based on such a partition of unity, our discretization operator is defined as follows:

$$J_\phi(F) := \sum_{i\in\mathcal{I}} \langle F, \phi_i\rangle L_{g_i} K$$

The first step is to show the continuity of this operator. We need the so-called Q*-oscillation*, which is defined by:

$$\mathrm{osc}_Q(F) := \sup_{q\in Q} |F(qg) - F(g)|.$$

Lemma 5. *Let $\|\mathrm{osc}_Q(K)\|_{L_{q,w}} < \infty$ for all $1 < q < \infty$. Then the map $J_\phi \ : \ \mathcal{M}^Y \to \mathcal{M}^Y$ defined by*

$$J_\phi(F) := \sum_{i\in\mathcal{I}} \langle F, \phi_i\rangle L_{g_i} K$$

is continuous.

Proof. Let $F \in \mathcal{M}^Y$. Then $F \in L_{p,m}$ for all $p > 1$. If we show that $Id - J_\phi$ is continuous, then clearly also J_ϕ is continuous. We start with a pointwise estimate. The reproducing kernel property and the fact that Φ

is a partition of unity imply

$$\begin{aligned}
|(Id - J_\phi)(F)(g)| &= |\int_G F(h)K(h^{-1}g)dh - \sum_{i\in\mathcal{I}} \int_G F(h)\phi_i(h)K(g_i^{-1}g)dh| \\
&= |\int_G F(h)(\sum_{i\in\mathcal{I}} \phi_i(h)(K(h^{-1}g) - K(g_i^{-1}g)dh)| \\
&\le \int_G (\sum_{i\in\mathcal{I}} |F(h)|\phi_i(h) \sup_{q\in Q} |(K(h^{-1}g) - K(qh^{-1}g)|dh) \\
&\le \int_G |F(h)| \sum_{i\in\mathcal{I}} \phi_i(h)\mathrm{osc}_Q(K)(h^{-1}g)dh \\
&\lesssim (|F| * \mathrm{osc}_Q(K))(g).
\end{aligned}$$

Then by the generalized weighted Young inequality (49) we obtain, for $1/p + 1/q = 1 + 1/r$,

$$\|F - J_\phi(F)\|_{L_{r,m}} \lesssim \||F| * \mathrm{osc}_Q(K)\|_{L_{r,m}} \lesssim \|F\|_{L_{p,m}} \| \mathrm{osc}_Q(K)\|_{L_{q,w}}.$$

By our assumptions on $\mathrm{osc}_Q(K)$ and the definition of Fréchet topology the result follows. □

Now, we are in position to state and to prove the main result of this paper.

Theorem 6. *Assume that J_Φ is injective with continuous left inverse J_Φ^{-1}. Then*

(i) Every $T \in \mathrm{Co}(Y)$ can be represented as

$$T = \sum_{i\in\mathcal{I}} \underbrace{\langle V_\psi(T), \phi_i\rangle}_{coefficients} \underbrace{V_\psi^{-1} J_\Phi^{-1} L_{g_i} K}_{atoms}. \tag{13}$$

(ii) Set

$$b(\Lambda) = \bigcap_{1<p<\infty} \ell_{p,m}.$$

If in addition $\|\mathrm{osc}_Q(J_\Phi^{-1}K)\|_{L_{p,w}} < \infty$ for all $1 < p < \infty$, then

$$\{V_\psi^{-1} J_\Phi^{-1} L_{g_i} K\}_{i\in\mathcal{I}}$$

gives rise to an atomic decomposition of $\mathrm{Co}(Y)$.

Proof. We start by showing (i).

$$\begin{aligned}
F &= J_\phi^{-1} J_\phi F \\
&= J_\phi^{-1} \left(\sum_{i\in\mathcal{I}} \langle F, \phi_i\rangle L_{g_i} K \right) \\
&= \sum_{i\in\mathcal{I}} \langle F, \phi_i\rangle J_\phi^{-1} L_{g_i} K \ ,
\end{aligned}$$

where $F \in \mathcal{M}_m$, the reproducing kernel space. Then for $T \in \mathrm{Co}(Y)$ we get by the Correspondence Principle i .e., by applying Theorem 4,

$$\begin{aligned}
V_\psi(T) &= J_\phi^{-1} J_\phi V_\psi(T) \\
&= J_\phi^{-1} \left(\sum_{i\in\mathcal{I}} \langle V_\psi(T), \phi_i\rangle L_{g_i} K \right) \\
&= \sum_{i\in\mathcal{I}} \langle V_\psi(T), \phi_i\rangle J_\phi^{-1} L_{g_i} K
\end{aligned}$$

and therefore,

$$\begin{aligned} T &= V_\psi^{-1} V_\psi(T) \\ &= \sum_{i\in\mathcal{I}} \underbrace{\langle V_\psi(T), \phi_i\rangle}_{\text{coefficients}} \underbrace{V_\psi^{-1} J_\phi^{-1} L_{g_i} K}_{\text{atoms}} \end{aligned}$$

and (i) is shown.

The next step is to prove (ii). We have to show that:

$$\begin{aligned} S: \ b(\Lambda) &\longrightarrow \mathcal{M}^Y \\ \{c_i\}_{i\in\mathcal{I}} &\mapsto \sum_{i\in\mathcal{I}} c_i J_\phi^{-1} L_{g_i} K \\ &\quad \text{is well-defined and continuous} \\ A: \ \mathrm{Co}(Y) &\longrightarrow b(\Lambda) \\ T &\mapsto \{\langle V_\psi T, \phi_i\rangle\}_{i\in\mathcal{I}} \\ &\quad \text{is continuous.} \end{aligned}$$

If this is established, then the result follows by the Correspondence Principle. We start with the operator A. By [2] p. 98, eq. (3.27), we observe that

$$\|\{\langle F, \phi_i\rangle\}_{i\in\mathcal{I}}\|_{\ell_{p,m}} \lesssim \|F\|_{L_{p,m}}.$$

Then,

$$\|\{\langle V_\psi(T), \phi_i\rangle\}_{i\in\mathcal{I}}\|_{\ell_{p,m}} \lesssim \|V_\psi(T)\|_{L_{p,m}} \lesssim \|T\|_{\mathcal{H}_{p,m}} \ .$$

Now we consider the operator S. At first, we observe

$$|\sum_{i\in\mathcal{I}} c_i (J_\phi^{-1}K)(g_i^{-1}\circ l)| \lesssim \left(\sum_{i\in\mathcal{I}} |c_i| \chi_{g_i Q}\right) * \left(\mathrm{osc}_Q(J_\phi^{-1}K) + |J_\phi^{-1}K|\right)(l)$$

see [2], p. 100, Lemma 3.18 for details. Then,

$$\|\sum_{i\in\mathcal{I}} c_i (J_\phi^{-1}K)(g_i^{-1}\circ \cdot)\|_{L_{p,m}} \lesssim \left\|\left(\sum_{i\in\mathcal{I}} |c_i| \chi_{g_i Q}\right) * \left(\mathrm{osc}_Q(J_\phi^{-1}K) + |J_\phi^{-1}K|\right)(\cdot)\right\|_{L_{p,m}} ,$$

and applying the generalized weighted Young inequality with $1/p + 1 = 1/q + 1/q'$ yields

$$\begin{aligned} &\left\|\left(\sum_{i\in\mathcal{I}} |c_i| \chi_{g_i Q}\right) * \left(\mathrm{osc}_Q(J_\phi^{-1}K) + |J_\phi^{-1}K|\right)(\cdot)\right\|_{L_{p,m}} \\ &\qquad \lesssim \left\|\sum_{i\in\mathcal{I}} |c_i| \chi_{g_i Q}(\cdot)\right\|_{L_{q,m}} \cdot \left\|\mathrm{osc}_Q(J_\phi^{-1}K) + |J_\phi^{-1}K|(\cdot)\right\|_{L_{q',w}} . \end{aligned}$$

By our assumptions, $\|\mathrm{osc}_Q(J_\phi^{-1}K)\|_{L_{q',w}}$ and $\|(J_\phi^{-1}K)(\cdot)\|_{L_{q',w}}$ are finite. Therefore, by using Lemma 3.18 in [2], we obtain

$$\|\sum_{i\in\mathcal{I}} c_i (J_\phi^{-1}K)(g_i^{-1}\circ \cdot)\|_{L_{p,m}} \lesssim \|(c_i)_{i\in I}\|_{\ell_{q,m}}$$

and (ii) is shown. □

Remark 7. *(i) It is one of the advantages of our approach that the norms of the Q-oscillations corresponding to K and $J_\phi^{-1}K$, respectively, do not have to be uniformly bounded, they just have to be finite.*

(ii) In many cases, an explicit formula for J_ϕ^{-1} will not be available, so that, at first glance, it seems to be diifficult to estimate the norms of the Q-oscillation of $J_\phi^{-1}K$. However, although the coorbit spaces might consist of ugly distributions, the reproducing kernel spaces are usually spaces of nice, smooth functions, so that in many cases the norms of the Q-oscillations turn out to be finite for all functions in the reproducing kernel space which is clearly sufficient. We also observe this fact in the examples discussed in Section 5.

(iiii) It is clearly a drawback that currently no general conditions that guarantee the injectivity of J_ϕ are available. Fortunately, usually this property can be checked directly, see Section 5. In the classical setting ot coorbit Banach spaces, injectivity and even surjectivity is proved by a Neumann series argument. Of course, the Neumann series setting can be generalized to Fréchet spaces, but in our case this series would not converge for this would require a uniform bound of the Q-oscillations.

(iv) From the examples studied in the following section, we observe that, to show the injectivity of J_ϕ, it is not absolutely necessary to choose a very small neighbourhood Q. Nevertheless, in practical applications, a small set Q (which implies a denser set $X = (g_i)_{i\in I}$) might be advantageous since the constants depend, among other things, on $\|\mathrm{osc}_Q(J_\phi^{-1}K)\|_{L_{q',w}}$ which most likely will grow as Q gets larger. Moreover, as we will see in Section 5 , to show the continuity of the left inverse, sometimes a denser sampling is helpful.

5 Examples of Coorbit Fréchet Spaces

5.1 The Shannon Case

As a first example, we study spaces of band-limited functions. Let G denote the additive group $\mathbb{R}$ with Lebesgue measure. Then G acts on the Paley–Wiener space

$$\mathcal{H} = B^2_\Omega = \{f \in L_2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq \Omega\}, \ \Omega \text{ compact interval}$$

by translations, $\pi(b)v(x) = v(x - b)$ The following facts are well-known:

Theorem 8. *It holds*

i) $\psi \in B^2_\Omega$ is admissible if and only if $|\widehat{\psi}| = 1$ almost everywhere on Ω. Then, the reproducing kernel is given by

$$K = \langle \psi, \pi(\cdot)\psi\rangle_{\mathcal{H}} = \mathcal{F}^{-1}\chi_\Omega.$$

ii) If $\Omega = [-\omega, \omega]$

$$K(b) = 2\omega\,\mathrm{sinc}(2\omega b),$$

if we use the standard notation $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

Consequently, the underlying representation is square-integrable, but of course it is not integrable. Therefore, it fits into our setting.

5.1.1 Fréchet coorbit spaces

The Fréchet coorbit spaces are given by choosing $Y = \bigcap_{p>1} L_{p,m}(G)$, namely:

$$\mathrm{Co}(Y) = \{f \in \mathcal{S}' \mid V_\psi f \in \bigcap_{p>1} L_{p,m}(G)\}$$

and the resulting reproducing kernel spaces are

$$\mathcal{M}^Y := \{F \in \bigcap_{p>1} L_{p,m}(G) \mid F * K = F\}.$$

Remark 9. *Since the voice transform $V : \mathcal{H} \to L_2(\mathbb{R})$ is the canonical inclusion, π is the restriction of L to $\mathcal{H}$ and $\mathrm{Co}(Y) = \mathcal{M}^Y$, compare with Proposition 4.8 of [4]. For simplicity, for any $\phi \in L_2(\mathbb{R})$ and any $\ell \in \mathbb{Z}$ we write $\pi(\ell)\phi = L_\ell \phi = \phi_\ell$.*

As a consequence of the results of the previous section we have the following theorem, whose proof is the content of the next sections.

Theorem 10. *Fix the weights $m = w = 1$. If $\psi \in L_2(\mathbb{R})$ is such that $|\widehat{\psi(x)}| = 1$ almost everywhere on Ω, then $\{\pi_\ell \psi\}_{\ell \in \mathbb{Z}}$ is an atomic decomposition of $\mathcal{H}_m$ and the coefficients are given by*

$$a_\ell(f) = \langle V_e f, \phi_\ell \rangle \qquad \ell \in \mathbb{Z}$$

where $\phi = (\chi_{[0,1)} * \chi_{[0,1)})(\cdot + 1)$ is the centralized cardinal B-spline N_2 and, for any $\ell \in \mathbb{Z}$, $\phi_\ell = L_\ell \phi$.

Remark 11. *(i) The case of weighted spaces can also be treated, at least for the case $w = m$. (Although this might be of limited use since, due to the bad decay properties of the Shannon kernel, only weights of logartihmic type can be handeld.) Only the estimation of the norms of the weighted Q-oscillation requires some care and is achieved using the Sobolev embedding theorem in the way that is explained in the next section. Hence the appropriate inequalities are detatiled below.*

(ii) In [4], it has been shown that in our setting the L_p-Banach coorbit spaces coincide with the Paley-Wiener spaces,

$$\mathrm{Co}(L_p(\mathbb{R})) = B^p_\Omega.$$

Therefore, we observe the interesting fact that

$$\mathrm{Co}(Y) = \bigcap_{p>1} B^p_\Omega.$$

Consequently, although the discretization of $\mathrm{Co}(L_p(\mathbb{R})) = B^p_\Omega$ turned out to be rather complicated, see again [3] for details, an atomic decomposition for their intersection can be constructed in quite a natural way.

From now on $w = m = 1$. We choose $Q = -[1, 1]$, so that the Q-dense set is simply given by the integers $\mathbb{Z}$. The associated partition of unity can be constructed by the integer translates of the centralized cardinal B-spline $N_2 := (\chi_{[0,1)} * \chi_{[0,1)})(\cdot + 1)$, i.e, $\phi_k = N_2(\cdot - k), k \in \mathbb{Z}$. We observe *en passant* that of course $\widehat{\phi}(\xi) = \mathrm{sinc}^2(\pi\xi)$.

5.1.2 Injectivity

The first step is to show that

$$J_\phi = \sum_{i \in \mathcal{I}} \langle F, \phi_i \rangle L_{g_i} K$$

is injective. Suppose that

$$0 = J_\phi(F) = \sum_{k \in \mathbb{Z}} \langle F, \phi_k \rangle \frac{\sin(2\pi(\cdot - k))}{2\pi(\cdot - k)} \quad \text{for some} \quad F \in \mathcal{M}^Y.$$

Applying the Fourier transform yields

$$0 = \sum_{k \in \mathbb{Z}} \langle F, \phi_k \rangle \mathrm{e}^{-2\pi i k \xi} \chi_{[-1,1)}(\xi) \frac{1}{2}, \quad \text{for almost all} \quad \xi \in \mathbb{R},$$

and therefore

$$\langle F, \phi_k \rangle = 0 \quad \text{for all} \quad k \in \mathbb{Z}.$$

Then, by using the reproducing kernel property and Plancherel's theorem we get

$$\begin{aligned} 0 &= \langle F, \phi_k \rangle = \langle F * K, \phi_k \rangle = \langle \widehat{F * K}, \widehat{\phi_k} \rangle \\ &= \langle \hat{F}\hat{K}, \hat{\phi} \mathrm{e}^{-2\pi i k (\cdot)} \rangle = \int_{-1}^{1} \hat{F}(\xi)(\operatorname{sinc} \xi)^2 \mathrm{e}^{2\pi i k \xi} \mathrm{d}\xi. \end{aligned}$$

This means that all Fourier coefficients of $\hat{F}(\xi)(\operatorname{sinc}\xi)^2|_{[-1,1]}$ vanish, so that $\hat{F}(\xi)(\operatorname{sinc}\xi)^2|_{[-1,1]} = 0$. But since $(\operatorname{sinc}\xi)^2$ is positive and F is band-limited, $\hat{F}$ is zero, so that finally $F = 0$. Therefore, J_ϕ is injective.

Remark 12. *From the above calculations, it it clear that also the case of a finer lattice, say $h\mathbb{Z}$, can be handled in a similar fashion. Moreover, other values of ω are no problem.*

5.1.3 Continuity of the Left Inverse

In order to apply Theorem 6, we have also to verify that the left inverse J_Φ^{-1} is continuous. To this end, we consider a specific partition of unity Φ and therewith a concrete, sufficiently dense sampling setting. For this specific case we will be able to provide an explicit description of J_Φ enabling us to derive a bound for J_Φ^{-1}.

The partition of unity is given as above but for more flexibility we introduce a variable interval width τ, and write

$$\varphi(x) = \chi_\tau * \chi_\tau(x),$$

where

$$\chi_\tau(x) = \begin{cases} 1 & \text{if } x \in [-\tau/2, \tau/2] \\ 0 & \text{elsewhere,} \end{cases}$$

so that clearly $\operatorname{supp}(\varphi) = [-\tau, \tau]$. We define the partition of unity by normalizing φ, namely

$$\phi_\tau(x) := \frac{1}{\tau}\varphi(x) = \frac{1}{\tau}\chi_\tau * \chi_\tau(x).$$

Let now

$$\phi_k(x) := \phi_\tau(x - g_k).$$

Then, upon setting $g_k := \tau k$, we have

$$\sum_{k \in \mathbb{Z}} \phi_k(x) \equiv 1$$

and thus, the family $\{\phi_k\}_{k \in \mathcal{I}}$ forms a partition of unity. It follows that

$$\mathcal{F}(\phi_k)(\xi) = \mathcal{F}(\phi_\tau)(\xi)\mathrm{e}^{-2\pi i g_k \xi} = \tau \operatorname{sinc}^2(\tau\xi)\mathrm{e}^{-2\pi i g_k \xi}.$$

We with help of Theorem 8 we have

$$J_\phi F(b) = \sum_{k \in \mathcal{I}} \langle F, \phi_k \rangle L_{g_k} K(b) = \sum_{k \in \mathcal{I}} \langle F, \phi_k \rangle K(b - g_k) = \sum_{k \in \mathcal{I}} \langle F, \phi_k \rangle \, 2\omega \operatorname{sinc}(2\omega\,(b - g_k)) \,.$$

The choice of the partition of unity (grid density) has a significant impact on the boundedness of J_Φ^{-1}. We pick $\tau = \frac{1}{2\omega}$, which implies $g_k = \frac{k}{2\omega}$. Notice that a grid with e.g. $\tau = 1/\omega$ would not yield the desired result, as a consequence of the Poisson summation formula. Our specific choice leads to

$$\widehat{\phi_k}(\xi) = \frac{1}{2\omega}\operatorname{sinc}^2\left(\frac{1}{2\omega}\xi\right)\mathrm{e}^{-2\pi i \frac{k}{2\omega}\xi}$$

and therefore, as $e_k(\xi) := \frac{1}{\sqrt{4\omega}} \mathrm{e}^{-2\pi i \frac{k}{2\omega}\xi}$ forms an orthogonal basis for $L_2(\Omega)$, which is actually not normalized because $\|e_k\|^2_{L_2(\Omega)} = 1/2$, we obtain

$$2\omega\langle F, \phi_k\rangle = \int_{-\omega}^{\omega} \underbrace{\hat{F}(\xi)\mathrm{sinc}^2\left(\frac{1}{2\omega}\xi\right)}_{=:\hat{\tilde{F}}(\xi)} \mathrm{e}^{2\pi i \frac{k}{2\omega}\xi} \mathrm{d}\xi = \tilde{F}\left(\frac{k}{2\omega}\right).$$

It follows, similarly to the Shannon-Nyquist sampling theorem, that

$$J_\phi F(b) = \sum_{k\in\mathbb{Z}} \tilde{F}\left(\frac{k}{2\omega}\right) \mathrm{sinc}\left(2\omega\left(b - \frac{k}{2\omega}\right)\right) = 2\tilde{F}(b) = 4\omega\,(F * \phi_{\frac{1}{2\omega}})(b)$$

or, equivalently,

$$\widehat{J_\Phi F} = 4\omega\,\hat{F}\,\widehat{\phi_{\frac{1}{2\omega}}}\,.$$

As $F \in B^2_\Omega$ we have that $\tilde{F} \in B^2_\Omega$ and hence $J_\Phi F \in B^2_\Omega$, and therefore,

$$\hat{F} = \frac{1}{4\omega}\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2} \overline{\widehat{\phi_{\frac{1}{2\omega}}}}\ \widehat{J_\Phi F}\ , \tag{34}$$

which exists since $|\widehat{\phi_{\frac{1}{2\omega}}}|^2$ has its zeros outside $[-\omega,\omega]$. (Observe that for this argument the denser sampling is necessary). Due to (34) the (left/right) inverse J_Φ^{-1} of some $G \in B^2_\Omega$ can be expressed as a convolution of the form

$$\begin{aligned} J_\Phi^{-1}G &= \mathcal{F}^{-1}\left(\frac{1}{4\omega}\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\overline{\widehat{\phi_{\frac{1}{2\omega}}}}\right) * G \\ &= \frac{1}{4\omega}\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right) * \overline{\phi_{\frac{1}{2\omega}}(-\cdot)} * G\ . \end{aligned}$$

Applying the generalized Young inequality (49) yields for $1 + 1/r = 1/p + 1/q$ and $1 + 1/p = 1/t + 1/s$

$$\|J_\Phi^{-1}G\|_{L_r} \leq \frac{1}{4\omega}\left\|\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right)\right\|_{L_t}\left\|\phi_{\frac{1}{2\omega}}\right\|_{L_s}\|G\|_{L_q}\ .$$

We observe by the support properties of $\phi_{\frac{1}{2\omega}}$ that

$$\left\|\phi_{\frac{1}{2\omega}}\right\|_{L_s} \leq (4\omega)^{1/s}$$

and if $t > 1$ that (see Appendix 6.3)

$$\left\|\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right)\right\|_{L_t} \leq C < \infty\ . \tag{38}$$

Therefore, we finally obtain

$$\|J_\Phi^{-1}G\|_{L_r} \leq \frac{(4\omega)^{1/s}}{4\omega}\,C\,\|G\|_{L_q}$$

for which we have the relation $2 + 1/r = 1/t + 1/s + 1/q$. As $\phi_{\frac{1}{2\omega}}$ belongs also to L_1 it can be simplyfied to $1 + 1/r = 1/t + 1/q$ or equivalenty $r = tq/(t + q - tq)$. For each $r > 1$ we find some $q > 1$ while fulfilling $t > 1$. Setting $t = q = 1 + \varepsilon$, we have $r = (1+\varepsilon)/(1-\varepsilon) > 1$ while ensuring $0 < \varepsilon < 1$ and we obtain $\varepsilon = (r-1)/(r+1)$ and hence $q = 2r/(r+1) > 1$.

Now, we obtain a first intermediate result: the existence and continuity of the left inverse imply by Theorem 6 the existence of an expansion of the form (13) .

5.1.4 Atomic Decomposition

The next step is to establish the atomic decomposition property. Due to Theorem 6 , we have to show that the L_p-norms of the Q-oscillation of $J_\phi^{-1}K$ are finite. To this end, we prove that this property holds for *all* elements in the reproducing kernel space. We start with some useful observations.

i) Let $F \in \mathcal{M}^Y$, then $F \in C^\infty$, since

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} F = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(F * K) = F * \frac{\mathrm{d}^n}{\mathrm{d}x^n} K \tag{39}$$

ii) In Appendix 6.2 , we show that the Shannon kernel satisfies

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} K \in L_p \ , \qquad \text{for all} \quad p > 1 \ . \tag{40}$$

iii) Then $\frac{\mathrm{d}^n}{\mathrm{d}x^n} F \in \mathcal{M}^Y$, since

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} F = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(F * K) = \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} F\right) * K,$$

and, , by the Young inequality,

$$\left\|\frac{\mathrm{d}^n}{\mathrm{d}x^n} F\right\|_{L_r} = \left\|F * \frac{\mathrm{d}^n}{\mathrm{d}x^n} K\right\|_{L_r} \lesssim \|F\|_{L_p} \left\|\frac{\mathrm{d}^n}{\mathrm{d}x^n} K\right\|_{L_q} , \quad 1/p + 1/q = 1 + 1/r, \tag{41}$$

and for any $r > 1$, we find p, q such that $1/p + 1/q = 1 + 1/r$.

Therefore, by combining (39), (40) and (41), we obtain

$$\mathcal{M}^Y \subset \bigcap_{p>1} \bigcap_{k \geq 0} W_p^k$$

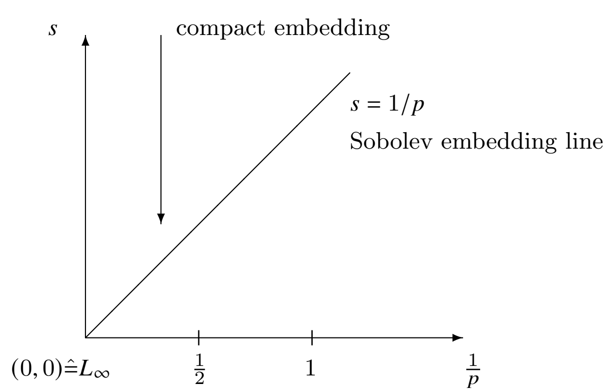
where W_p^k denotes the L_p-Sobolev space of smoothness k. We know that

$$\|\mathrm{osc}_Q(F)\|_{L_p} < \infty$$

if

$$F \in \mathcal{M}_Q^\rho(L_{p,w}) = \left\{F \in L_0(G) \mid \ \|F\|_{L_\infty(Q\cdot)} \in L_{p,w}\right\} \ ,$$

see [3] p.86, Lemma 4.3. (fortunately, it is not needed that $\|\mathrm{osc}_Q(F)\|_{L_p}$ is small, it only has to be finite). We need to show $\|F\|_{L_\infty(Q\cdot)} \in L_p$. To this end, we use the Sobolev embedding theorem as illustrated, in the DeVore-Triebel diagram:



Here, the point $(1/p,s)$ corresponds to the Besov space $B^s_{p,p}$, where $B^s_{p,p}=W^s_p$ for $s\notin\mathbb{N}$. We see that for all $p>1$ the spaces W^2_p embed into L_∞. Therefore we get for any function F in the reproducing kernel space

$$\begin{aligned}
\int \left(\|F\|_{L_\infty(Q\cdot x)}\right)^p \mathrm{d}x &\lesssim \int \left(\|F\|_{W^2_p(Q\cdot x)}\right)^p \mathrm{d}x \\
&\lesssim \int \sum_{k=0}^{2} \left\|\frac{\mathrm{d}^k}{\mathrm{d}x^k}F\right\|^p_{L_p(Q\cdot x)} \mathrm{d}x \\
&= \sum_{k=0}^{2} \int \int_{Q\cdot x} \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k}F\right|^p \mathrm{d}u\mathrm{d}x \\
&= \sum_{k=0}^{2} \int \int_{Q} \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k}F(u+x)\right|^p \mathrm{d}u\mathrm{d}x \\
&= \sum_{k=0}^{2} \int_Q \int_{\mathbb{R}} \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k}F(u+x)\right|^p \mathrm{d}x\mathrm{d}u \\
&= \int_Q \|F\|^p_{W^2_p}\mathrm{d}u = \|F\|^p_{W^2_p}\int_Q \mathrm{d}u < \infty\ .
\end{aligned}$$

This is true for all $F\in\mathcal{M}^Y$. But $J_\phi^{-1}K$ is contained in $\mathcal{M}^Y$ and therefore,

$$\|\mathrm{osc}_Q(J_\phi^{-1}K)\|_{L_{q'}} < \infty,$$

and we are done.

Remark 13. *As mentioned earlier, we are now in a position to show how to handle the case $w=m$. We know that*

$$\left\|\mathrm{osc}_Q(F)\right\|_{L_{p,w}} < \infty$$

if $F\in\mathcal{M}^\rho_Q(L_{p,w})$. Using as before the Sobolev embedding theorem, we get

$$\begin{aligned}
\left\|\|F\|_{L_\infty(Q\cdot)}\right\|_{L_{p,w}} &= \int_{\mathbb{R}} \|F\|^p_{L_\infty(Q\cdot x)} w(x)^p \mathrm{d}x \\
&\lesssim \sum_{n=0}^{2} \int_{\mathbb{R}}\int_Q \left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}F(u+x)\right|^p \mathrm{d}u\, w(x)^p\mathrm{d}x \\
&\lesssim \sum_{n=0}^{2} \int_Q\int_{\mathbb{R}} \left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}F(u+x)\right|^p w(x)^p\mathrm{d}x\,\mathrm{d}u \\
&= \sum_{n=0}^{2} \int_Q\int_{\mathbb{R}} \left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}F(x')\right|^p w(x'-u)^p\mathrm{d}x'\,\mathrm{d}u \\
&\lesssim \int_Q w(-u)^p\mathrm{d}u \sum_{n=0}^{2} \int_{\mathbb{R}} \left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}F(x')\right|^p w(x')^p\mathrm{d}x' \\
&\leq \int_Q w(-u)^p\mathrm{d}u\ \|F\|^p_{W^2_{p,w}}\ .
\end{aligned}$$

The weighted Young inequality, applied to the case $m = w$, implies that

$$\|F\|^p_{W^2_{p,w}} < \infty \quad \forall F \in \mathcal{M}^Y \ .$$

Since $J_\phi^{-1} K \in \mathcal{M}^Y$, the result follows.

5.2 Modulation Spaces

Modulation spaces are based on the translation and modulation operators on functions, namely

$$T_a f(t) = f(t-a), \quad M_b f(t) = \mathrm{e}^{2\pi \mathrm{i} bt} f(t)$$

with corresponding Fourier transforms

$$\widehat{T_a f} = M_{-a}\hat{f}, \quad \widehat{M_b f} = \hat{f}(T_b \cdot).$$

The *reduced Heisenberg group* $\mathbb{H}_r$ is the locally compact group $\mathbb{H}_r = \mathbb{R}^{2d} \times \mathbb{T}$ with multiplication and inversion

$$(x,\omega,z)(x',\omega',z') = (x+x', \omega+\omega', zz'\mathrm{e}^{i\pi(x'\omega - x\omega')}), \quad (x,\omega,z)^{-1} = (-x,-\omega,\bar{z}).$$

The (group) convolution of $f, g \in L_1(\mathbb{H}_r)$ is given by

$$\begin{aligned}(f*g)(x,\omega,z) &= \int_{\mathbb{H}_r} f(x',\omega',z') g\left((x',\omega',z')^{-1}(x,\omega,z)\right) \mathrm{d}x'\mathrm{d}\omega'\mathrm{d}z' \\ &= \int_{\mathbb{H}_r} f(x',\omega',z') g\left(x-x', \omega-\omega', z\overline{z'}\mathrm{e}^{\pi \mathrm{i}(x'\omega - \omega' x)}\right) \mathrm{d}x'\mathrm{d}\omega'\mathrm{d}z'.\end{aligned}$$

Introducing the mapping $j\colon \mathbb{C}^{\mathbb{R}^2} \to \mathbb{C}^{\mathbb{H}_r}$ defined on $F\colon \mathbb{R}^2 \to \mathbb{C}$ by

$$(jF)(x,\omega,z) = \bar{z}\mathrm{e}^{\pi \mathrm{i} x\omega} F(x,\omega),$$

which induces an isometry from $L_2(\mathbb{R}^2)$ into $L_2(\mathbb{H}_r)$ the convolution can be rewritten as

$$\begin{aligned}(jF * jG)(x,\omega,z) &= \int_{\mathbb{H}_r} \overline{z'}\mathrm{e}^{\pi \mathrm{i} x'\omega'} F(x',\omega') G(x-x',\omega-\omega')\bar{z}z'\mathrm{e}^{-\pi \mathrm{i}(x'\omega - \omega' x)}\mathrm{e}^{\pi \mathrm{i}(x-x')(\omega-\omega')}\mathrm{d}x'\mathrm{d}\omega'\mathrm{d}z' \\ &= \mathrm{e}^{\pi \mathrm{i} x\omega}\bar{z}\int_{\mathbb{R}^2} F(x',\omega')G(x-x',\omega-\omega')\mathrm{e}^{2\pi \mathrm{i} x'(\omega'-\omega)}\mathrm{d}x'\mathrm{d}\omega' \\ &= j(F \odot G)(x,\omega,z)\end{aligned}$$

where $F \odot G$ is given, for $F, G \in L_1(\mathbb{R}^2)$, by:

$$(F \odot G)(x,\omega) = \int_{\mathbb{R}^2} F(x',\omega')G(x-x',\omega-\omega')\mathrm{e}^{2\pi \mathrm{i} x'(\omega'-\omega)}\mathrm{d}x'\mathrm{d}\omega'.$$

Therefore the mapping j intertwines the group convolution on $\mathbb{H}_r$ with $\odot$, that is

$$jF * jG = j(F \odot G)$$

for, say, $F, G \in L_1(\mathbb{R}^2)$. The Schrödinger representation $\pi : \mathbb{H}_r \to \mathcal{U}(L_2(\mathbb{R}))$ is the unitrary representation given by

$$(\pi_{(x,\omega,z)} f)(t) = z\mathrm{e}^{-\pi \mathrm{i} x\omega}\mathrm{e}^{2\pi \mathrm{i} t\omega} f(t-x).$$

The corresponding voice transform is

$$
\begin{aligned}
V_g(f)(x,\omega,z) = \langle f, \pi_{(x,\omega,z)} g\rangle &= \int_{\mathbb{R}} f(t)\overline{z\mathrm{e}^{-\pi\mathrm{i}x\omega}\mathrm{e}^{2\pi\mathrm{i}\omega t} g(t-x)}\,\mathrm{d}t \\
&= \bar{z}\mathrm{e}^{\pi\mathrm{i}x\omega}\int_{\mathbb{R}} f(t)\overline{\mathrm{e}^{2\pi\mathrm{i}\omega t} g(t-x)}\,\mathrm{d}t \\
&= j(U_g f)(x,\omega,z),
\end{aligned}
$$

where

$$
U_g(f)(x,\omega) = \int_{\mathbb{R}} f(t)\overline{\mathrm{e}^{2\pi\mathrm{i}\omega t} g(t-x)}\,\mathrm{d}t = \langle f, M_\omega T_x g\rangle.
$$

For the function $g = \chi_{[-\frac{1}{2},\frac{1}{2}]} = \check{g}$, we consider the reproducing kernel

$$
K(x,\omega) = U_g g(x,\omega) = \int_{\mathbb{R}} \underbrace{g(t)\overline{g(t-x)}}_{h_x(t)}\,\mathrm{e}^{-2\pi\mathrm{i}t\omega}\,\mathrm{d}t = \widehat{h_x}(\omega).
$$

Its Fourier transform is given by

$$
\begin{aligned}
\hat{K}(\xi,\eta) &= \int_{\mathbb{R}^2} \hat{h}_x(\omega)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{e}^{-2\pi\mathrm{i}\eta\omega}\,\mathrm{d}x\mathrm{d}\omega = \int_{\mathbb{R}} h_x(-\eta)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} g(-\eta)\overline{g(-\eta-x)}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = g(-\eta)\int_{\mathbb{R}} \overline{g(x-\eta)}\mathrm{e}^{2\pi\mathrm{i}x\xi}\,\mathrm{d}x \\
&= g(-\eta)\,\widehat{\overline{g}}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}\eta\xi} \\
&= \chi_{[-\frac{1}{2},\frac{1}{2}]}(\eta)\,\mathrm{sinc}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}\eta\xi}.
\end{aligned} \tag{48}
$$

We have that $K \in L_p(\mathbb{R}^2)$ for all $p > 1$, but $K \notin L_1(\mathbb{R}^2)$.

Remark 14. *From a practical point of view, our choice of the window function g might not be optimal. It possesses a very good time localization, but the frequency localization is bad. Very often, one strives fo a compromize and uses, e.g., the Gaussian window. Nevertheless, as a test example for our theory, our choice is fine.*

Let

$$
\mathcal{M} = \{F \in L_2(\mathbb{R}^2) : F \odot K = F\}.
$$

We will need the Fourier transform of functions in $\mathcal{M}$.

Lemma 15. *For every $F \in \mathcal{M}$ it holds that*

$$
\widehat{F \odot K}(\xi,\eta) = \mathrm{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}} \hat{F}(\xi',\eta)\,\mathrm{sinc}(\xi')\mathrm{e}^{-2\pi\mathrm{i}\xi'\eta}\,\mathrm{d}\xi'.
$$

Proof. Since $F \in \mathcal{M}$, we have

$$
\begin{aligned}
\hat{F}(\xi,\eta) &= \widehat{F \odot K}(\xi,\eta) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F(x',\omega')K(x-x',\omega-\omega')\mathrm{e}^{2\pi\mathrm{i}x'(\omega'-\omega)}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{e}^{-2\pi\mathrm{i}\omega\eta}\,\mathrm{d}x\mathrm{d}\omega\mathrm{d}x'\,\mathrm{d}\omega' \\
&= \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2} K(x-x',\omega-\omega')\mathrm{e}^{-2\pi\mathrm{i}x'\omega}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{e}^{-2\pi\mathrm{i}\omega\eta}\,\mathrm{d}x\mathrm{d}\omega\Big)\mathrm{e}^{2\pi\mathrm{i}x'\omega'}F(x',\omega')\,\mathrm{d}x'\mathrm{d}\omega' \\
&= \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2} K(x,\omega)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{e}^{-2\pi\mathrm{i}\omega(x'+\eta)}\,\mathrm{d}x\mathrm{d}\omega\Big)F(x',\omega')\mathrm{e}^{-2\pi\mathrm{i}x'\xi}\mathrm{e}^{-2\pi\mathrm{i}\omega'\eta}\,\mathrm{d}x'\mathrm{d}\omega' \\
&= \int_{\mathbb{R}^2} \hat{K}(\xi,x'+\eta)F(x',\omega')\mathrm{e}^{-2\pi\mathrm{i}x'\xi}\mathrm{e}^{-2\pi\mathrm{i}\omega'\eta}\,\mathrm{d}x'\mathrm{d}\omega'.
\end{aligned}
$$

Using now (48), we obtain

$$\begin{aligned}\hat{F}(\xi,\eta) &= \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}^2} F(x',\omega')\chi_{[-\frac{1}{2},\frac{1}{2}]}(x'+\eta)\mathrm{e}^{-2\pi\mathrm{i}\eta\omega'}\,\mathrm{d}x'\mathrm{d}\omega' \\ &= \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}^2} F(x'-\eta,\omega')\chi_{[-\frac{1}{2},\frac{1}{2}]}(x')\mathrm{e}^{-2\pi\mathrm{i}\eta\omega'}\,\mathrm{d}x'\mathrm{d}\omega' \\ &= \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}} \mathcal{F}_\omega F(x'-\eta,\eta)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x')\mathrm{d}x' \\ &= \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}^2} \chi_{[-\frac{1}{2},\frac{1}{2}]}(x')\,\mathrm{e}^{2\pi\mathrm{i}\xi'(x'-\eta)}\hat{F}(\xi',\eta)\,\mathrm{d}x'd\xi' \\ &= \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\int_{\mathbb{R}} \hat{F}(\xi',\eta)\operatorname{sinc}(\xi')\mathrm{e}^{-2\pi\mathrm{i}\xi'\eta}\,\mathrm{d}\xi',\end{aligned}$$

as desired. □

Remark 16. *There is a remarkable difference from the results for the Shannon case. There, the resulting coorbit spaces according to Theorem 4 could be identified as intersections of well-known spaces, namely the Paley-Wiener spaces. Here, the coorbit spaces are* not *intersections of classical modulation spaces, simply because for the non-integrable case these spaces do not exist. We end up with a really new class of spaces.*

5.2.1 Atomic decomposition

We are finally in a position to prove the injectivity of J_ϕ, as well as the L_p–boundedness of the Q-oscillation for $p > 1$. The continuity of the left inverse will be the topic of future research.

We choose $Q = (-\frac{1}{2},\frac{1}{2})^2$ and $\mathbb{Z}^2$ as Q-dense set and write

$$\phi(x,\omega) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)\chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega), \quad \phi_{k,l}(x,\omega) = \phi(x-k,\omega-l).$$

Proposition 17. *The operator $J_\phi\colon \mathcal{M}\to\mathcal{M}$ defined by*

$$J_\phi(F) = \sum_{k,l\in\mathbb{Z}} \langle F,\phi_{k,l}\rangle K(x+k,\omega+l)$$

is injective.

Proof. We show that $J_\phi(F) = 0$ if and only if $\langle F,\phi_{k,l}\rangle = 0$ for all $k,l\in\mathbb{Z}$. Using Plancherel's theorem, we obtain

$$\begin{aligned}0 &= \langle F,\phi_{k,l}\rangle = \langle \widehat{F\odot K},\hat{\phi}_{k,l}\rangle \\ &= \int_{\mathbb{R}^3} \operatorname{sinc}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi\eta}\hat{F}(\xi',\eta)\operatorname{sinc}(\xi')\mathrm{e}^{-2\pi\mathrm{i}\xi'\eta}\operatorname{sinc}(\xi)\mathrm{e}^{-2\pi\mathrm{i}k\xi}\operatorname{sinc}(\eta)\mathrm{e}^{-2\pi\mathrm{i}l\eta}\,\mathrm{d}\xi\mathrm{d}\eta\,\mathrm{d}\xi'.\end{aligned}$$

Observe that

$$\int_{\mathbb{R}} \operatorname{sinc}^2(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi(\eta-k)}\,\mathrm{d}\xi = M_2(\eta-k),$$

where M_2 denotes the centered cardinal B-spline of order 2. Thus, we get

$$0 = \int_{\mathbb{R}} M_2(\eta-k)\mathrm{e}^{-2\pi\mathrm{i}l\eta}\operatorname{sinc}(\eta)\Big(\int_{\mathbb{R}}\hat{F}(\xi',\eta)\operatorname{sinc}(\xi')\mathrm{e}^{-2\pi\mathrm{i}\xi'\eta}\,\mathrm{d}\xi'\Big)\,\mathrm{d}\eta.$$

Now $\{M_2(\eta-k)\mathrm{e}^{-2\pi\mathrm{i}l\eta} : k,l\in\mathbb{Z}\}$ is a Gabor frame in $L_2(\mathbb{R})$ so that the above implies

$$0 = \operatorname{sinc}(\eta)\int_{\mathbb{R}}\hat{F}(\xi',\eta)\operatorname{sinc}(\xi')\mathrm{e}^{-2\pi\mathrm{i}\xi'\eta}\,\mathrm{d}\xi'$$

and then

$$0 = \int_{\mathbb{R}} \hat{F}(\xi', \eta)\operatorname{sinc}(\xi')e^{-2\pi i \xi' \eta} d\xi' \quad \eta - a.e.$$

Consequently, by Lemma 15, we have $\hat{F}(\xi, \eta) = 0$ and then $F(x, \omega) = 0$ a.e.. □

The next step is to show the L_p-boundednes of the Q-oscillation. We will follow closely the argument in Section.5.1.4, i.e., we show the boundedness for all functions in the reproducing kernel space by means of Sobolev embeddings. We first observe

$$\frac{\partial K}{\partial \omega}, \frac{\partial K}{\partial x} \in L_p(\mathbb{R}^2) \quad \text{for all } p > 1.$$

This becomes clear when writing a more explicit expression for K. We have

$$h_x(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\chi_{[x-\frac{1}{2},x+\frac{1}{2}]}(t) = \begin{cases} 0, & |x| > 1\,, \\ \chi_{[-\frac{1}{2},x+\frac{1}{2}]}(t), & -1 \le x < 0, \\ \chi_{[x-\frac{1}{2},\frac{1}{2}]}(t), & 0 \le x \le 1. \end{cases}$$

From

$$\widehat{\chi}_{[a,b]}(\omega) = (b-a)e^{2\pi i \frac{b+a}{2}\omega}\operatorname{sinc}\big((b-a)\omega\big)$$

this results in the explicit expression

$$\begin{aligned} K(x,\omega) = \widehat{h_x}(\omega) &= \begin{cases} 0, & |x| > 1 \\ e^{2\pi i \frac{x}{2}\omega}\operatorname{sinc}\big((1+x)\omega\big), & -1 \le x < 0 \\ e^{2\pi i \frac{x}{2}\omega}\operatorname{sinc}\big((1-x)\omega\big), & 0 \le x \le 1 \end{cases} \\ &= e^{\pi i x \omega}\operatorname{sinc}\big((1-|x|)\omega\big)\chi_{[-1,1]}(x). \end{aligned}$$

Thus as a function of ω the kernel K is a smooth function, whereas w.r.t. x it is only Lipschitz. However, since K and its weak derivatives are compactly supported w.r.t. x, this suffices to belong to L_p.

Lemma 18. *Young's inequality transfers to the $\odot$-product, i.e. we have*

$$\|F \odot G\|_{L_r} \lesssim \|F\|_{L_p}\|G\|_{L_q}$$

for all $F \in L_p(\mathbb{R}^2)$ and $G \in L_q(\mathbb{R}^2)$, whenever $1/p + 1/q = 1 + 1/r$.

Proof. This follows immediately from the intertwining relation

$$j(F \odot G) = jF * jG$$

for the isometry $j : L_p(\mathbb{R}^2) \to L_p(\mathbb{H}_r)$ and Young's inequality for the group $\mathbb{H}_r$. More precisely

$$\begin{aligned} \|F \odot G\|_{L_r(\mathbb{R}^2)} &= \|j^{-1}(jF * jG)\|_{L_r(\mathbb{R}^2)} = \|jF * jG\|_{L_r(\mathbb{H}_r)} \\ &\lesssim \|jF\|_{L_p(\mathbb{H}_r)}\|jG\|_{L_q(\mathbb{H}_r)} = \|F\|_{L_p(\mathbb{R}^2)}\|G\|_{L_q(\mathbb{R}^2)}\,. \end{aligned}$$

□

Lemma 19. *For $F \in \mathcal{M}$ we have*

$$\frac{\partial F}{\partial x} = F \odot \frac{\partial K}{\partial x} \in L_p(\mathbb{R}^2)$$

for all $p > 1$ as well as

$$\frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \omega} \odot K\,,$$

and consequently $\frac{\partial F}{\partial \omega} \in \mathcal{M}$.

Proof. The first identity follows immediately from the reproducing property of K

$$\frac{\partial}{\partial x}(F \odot G) = F \odot \frac{\partial G}{\partial x}$$

together with Young's inequality. The second identity follows from observing

$$F \odot K(x,\omega) = \int F(x-x',\omega-\omega')K(x',\omega')e^{2\pi i(x'-x)\omega'}\mathrm{d}x'\mathrm{d}\omega',$$

and once more Young's inequality. □

Corollary 20. *For $F \in \mathcal{M}$ we also have $\frac{\partial^2 F}{\partial x \partial \omega} \in L_p(\mathbb{R}^2)$ for all $p > 1$.*

In short: While we no longer have the reproducing property for the x-derivatives, we nevertheless retain the needed integrability properties. Those results can be combined into

Corollary 21. *The space $\mathcal{M}$ embeds into $S_p^1 W(\mathbb{R}^2)$ for all $p > 1$, the first order Sobolev space of dominating mixed smoothness.*

For spaces of dominating mixed smoothness, the counterpart of the classical Sobolev embedding can be formulated as

$$S_p^s W(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \quad \text{provided} \quad s > 1/p\,,$$

see, e.g., [11], Chapter 2.4 for details. Note that the condition is the same as for the univariate embedding. Ultimately, we now conclude

$$\begin{aligned}
\int \left(\|F\|_{L_\infty(Q\cdot(x,\omega))}\right)^p \mathrm{d}x\mathrm{d}\omega &\lesssim \int \left(\|F\|_{S_p^1 W(Q\cdot(x,\omega))}\right)^p \mathrm{d}x\mathrm{d}\omega \\
&\lesssim \int \left(\left\|\frac{\partial F}{\partial x}\right\|^p_{L_p(Q\cdot(x,\omega))} + \left\|\frac{\partial F}{\partial \omega}\right\|^p_{L_p(Q\cdot(x,\omega))} + \left\|\frac{\partial^2 F}{\partial x \partial \omega}\right\|^p_{L_p(Q\cdot(x,\omega))}\right) \mathrm{d}x\mathrm{d}\omega \\
&= \int \int_{Q\cdot(x,\omega)} \left|\frac{\partial F}{\partial x}(u,\eta)\right|^p \mathrm{d}u\mathrm{d}\eta\mathrm{d}x\mathrm{d}\omega + \cdots \\
&= \int \int_Q \left|\frac{\partial F}{\partial x}(u+x,\eta+\omega)\right|^p \mathrm{d}u\mathrm{d}\eta\mathrm{d}x\mathrm{d}\omega + \cdots \\
&= \int_Q \int_{\mathbb{R}^2} \left|\frac{\partial F}{\partial x}F(u+x,\eta+\omega)\right|^p \mathrm{d}x\mathrm{d}\omega\mathrm{d}u\mathrm{d}\eta \\
&= \int_Q \|F\|^p_{S_p^1 W(\mathbb{R}^2)} \mathrm{d}u\mathrm{d}\eta = \|F\|^p_{S_p^1 W(\mathbb{R}^2)} \mathrm{d}u\mathrm{d}\eta \int_Q \mathrm{d}u\mathrm{d}\eta < \infty\,.
\end{aligned}$$

6 Appendix

6.1 Generalized Weighted Young Inequality

Lemma 22. *Let w be the control weight and suppose that m is w-moderate. Then*

$$\|H * F\|_{L_{r,m}} \lesssim \|H\|_{L_{p,m}} \|F\|_{L_{q,w}}\,, \quad \textit{where} \qquad 1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}\,. \tag{49}$$

Proof. The w-moderateness of m implies

$$m(h) = m(gg^{-1}h) \lesssim m(g)w(g^{-1}h) = m(g)w(h^{-1}g)\,.$$

Therefore, it follows

$$\begin{aligned}
\|H * F\|_{L_{r,m}} &= \left(\int_G |(H*F)(h)|^r m(h)^r \mathrm{d}h\right)^{1/r} \\
&= \left(\int_G \left|\int_G H(g)F(h^{-1}g)\mathrm{d}g\right|^r m(h)^r \mathrm{d}h\right)^{1/r} \\
&= \left(\int_G \left|\int_G H(g)F(h^{-1}g)m(h)\mathrm{d}g\right|^r \mathrm{d}h\right)^{1/r} \\
&\lesssim \left(\int_G \left|\int_G w(h^{-1}g)F(h^{-1}g)H(g)m(g)\mathrm{d}g\right|^r \mathrm{d}h\right)^{1/r} \\
&= \left(\int_G |\left((m\cdot H)*(w\cdot F)\right)(h)|^r \mathrm{d}h\right)^{1/r} \\
&\lesssim \|w\cdot F\|_{L_q} \|m\cdot H\|_{L_p} = \|H\|_{L_{p,m}} \|F\|_{L_{q,w}} .
\end{aligned}$$

□

6.2 L_p Estimates of the Shannon Kernel and its Derivatives

In this section we prove that the all the derivatives of the Shannon kernel $x \mapsto 2\omega\mathrm{sinc}(2\omega x)$ are in all L_p-spaces, for $p > 1$. For simplicity, we write

$$K(x) = \frac{\sin x}{x}$$

and prove the statement for K, which is obviously equivalent. We show by induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} K(x) = x^{-2^n} \sum_{k=0}^{2^n-1} x^k p_{k,n}(x) ,$$

where the $p_{k,n}$ are trigonometric polynomials. The case $n = 1$ is true because

$$\frac{\mathrm{d}}{\mathrm{d}x} K = \frac{x\cos x - \sin x}{x^2}.$$

Suppose that (6.2) holds for some n. Then

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} K &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2^n} \sum_{k=0}^{2^n-1} x^k p_{k,n}(x)\right) \\
&= x^{-2^n}\left(\sum_{k=0}^{2^n-1} x^k p'_{k,n}(x) + \sum_{k=1}^{2^n-1} k x^{k-1} p_{k,n}(x)\right) - 2^n x^{-2^n-1} \sum_{k=0}^{2^n-1} x^k p_{k,n}(x) \\
&= \frac{x^{2^n}\left(\sum_{k=0}^{2^n-1} x^k p'_{k,n}(x) + \sum_{k=1}^{2^n-1} k x^{k-1} p_{k,n}(x) - 2^n \sum_{k=0}^{2^n-1} x^{k-1} p_{k,n}(x)\right)}{\left(x^{2^n}\right)^2} \\
&= \frac{\sum_{k=0}^{2^n-1} x^{2^n+k} p'_{k,n}(x) + \sum_{k=1}^{2^n-1} k x^{2^n+k-1} p_{k,n}(x) - 2^n \sum_{k=0}^{2^n-1} x^{2^n+k-1} p_{k,n}(x)}{\left(x^{2^n}\right)^2} \\
&= \frac{\sum_{k=0}^{2^{n+1}-1} x^k p_{k,n+1}(x)}{x^{2^{n+1}}} .
\end{aligned}$$

Hence, (6.2) is proved. Since the powers of x that appear in the just derived expressions of $K^{(n)}(x)$ are $x^{-\ell}$ with $\ell \geq 1$, it follows for $|x| \geq C$ that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}x^n} K\right| = \left|\frac{1}{x} \sum_{k=0}^{2^n-1} x^{k-(2^n-1)} p_{k,n}(x)\right| \leq \left|\frac{1}{x}\right| \underbrace{\left|\sum_{k=0}^{2^n-1} x^{k-(2^n-1)} p_{k,n}(x)\right|}_{\leq C'}$$

and therefore

$$\left\|\frac{\mathrm{d}^n}{\mathrm{d}x^n} K\right\|_{L_p} \leq C_p \quad , \qquad p > 1.$$

A yet more explicit formula holds. Indeed, from the product rule

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)},$$

choosing $u(x) = \sin(x)$ and $v(x) = x^{-1}$ it follows from $(v(x))^{(n)} = (-1)^n n! x^{-(n+1)}$ that

$$\left(\frac{\sin x}{x}\right)^{(n)} = x^{-n} \sum_{k=0}^{n} \frac{n!}{k!} (-1)^{n-k} (\sin(x))^{(k)} x^{k-1}.$$

6.3 A Building Block for the Continuity of J_ϕ^{-1}

We have to bound (38). To this end, we split the L_t-norm as follows

$$\begin{aligned} J \quad = \quad & \left\|\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right)\right\|_{L_t}^t = \underbrace{\int_{-\varepsilon}^{\varepsilon} |\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}(x)\right)|^t \mathrm{d}x}_{=:J_1} \\ & + \underbrace{\int_{-\infty}^{-\varepsilon} |\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}(x)\right)|^t \mathrm{d}x}_{=:J_2} + \underbrace{\int_{\varepsilon}^{\infty} |\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}(x)\right)|^t \mathrm{d}x}_{=:J_3} \; . \end{aligned}$$

First, we consider

$$\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right)(x) = \int_{-\omega}^{\omega} |\widehat{\phi_{\frac{1}{2\omega}}}(\xi)|^{-2} \, \mathrm{e}^{2\pi i x \xi} \mathrm{d}\xi$$

and with

$$|\widehat{\phi_{\frac{1}{2\omega}}}(\xi)|^{-2}\big|_{[-\omega,\omega]} = |1/(2\omega)\,\mathrm{sinc}^2\,(\xi/(2\omega))\,|^{-2}\big|_{[-\omega,\omega]} \leq (2\omega)^2 \mathrm{sinc}^{-4}(1/2) = \frac{\omega^2\pi^4}{4}$$

we obtain

$$J_1 \leq \int_{-\varepsilon}^{\varepsilon} \left|\int_{-\omega}^{\omega} \frac{\omega^2\pi^4}{4} \mathrm{d}\xi\right|^t \mathrm{d}x = 2\varepsilon \left(\frac{\omega^3\pi^4}{2}\right)^t \quad .$$

To estimate J_2 we apply integration by parts and obtain

$$\begin{aligned}
\mathcal{F}^{-1}\left(\left(\chi_{[-\omega,\omega]}|\widehat{\phi_{\frac{1}{2\omega}}}|\right)^{-2}\right)(x) &= \int_{-\omega}^{\omega} \underbrace{|\widehat{\phi_{\frac{1}{2\omega}}}(\xi)|^{-2}}_{u(\xi)} \underbrace{\mathrm{e}^{2\pi i x\xi}}_{v'(\xi)} \mathrm{d}\xi \\
&= \frac{1}{2\pi i x}\left\{ |\widehat{\phi_{\frac{1}{2\omega}}}(\xi)|^{-2}\,\mathrm{e}^{2\pi i x\xi}\Big|_{-\omega}^{\omega} + 2\int_{-\omega}^{\omega} \frac{\Re\mathrm{e}(\widehat{\phi_{\frac{1}{2\omega}}}'\,\overline{\widehat{\phi_{\frac{1}{2\omega}}}})}{|\widehat{\phi_{\frac{1}{2\omega}}}|^4}\,\mathrm{e}^{2\pi i x\xi}\,\mathrm{d}\xi \right\} \\
&= \frac{1}{2\pi i x}\left\{ \frac{\omega^2\pi^4}{2} 2i\sin(2\pi\omega x) + 2\int_{-\omega}^{\omega} \underbrace{\frac{\Re\mathrm{e}(\widehat{\phi_{\frac{1}{2\omega}}}'\,\overline{\widehat{\phi_{\frac{1}{2\omega}}}})}{|\widehat{\phi_{\frac{1}{2\omega}}}|^4}}_{\leq C<\infty}\,\mathrm{e}^{2\pi i x\xi}\,\mathrm{d}\xi \right\} \\
&\leq \frac{1}{\pi|x|}\left\{\frac{\omega^2\pi^4}{2} + 2\omega C\right\}
\end{aligned}$$

resulting in

$$J_2 \quad \leq \quad \left(\frac{\omega^2\pi^2}{2} + \frac{2\omega C}{\pi}\right)^t \int_{-\infty}^{-\varepsilon} \frac{1}{|x|^t}\mathrm{d}x = \left(\frac{\omega^2\pi^2}{2} + \frac{2\omega C}{\pi}\right)^t \frac{1}{t-1}\frac{1}{\varepsilon^{t-1}} \; .$$

As J_3 is treated analougsly, we finally obtain

$$J \leq 2\varepsilon\left(\frac{\omega^3\pi^4}{2}\right)^t + 2\left(\frac{\omega^2\pi^2}{2} + \frac{2\omega C}{\pi}\right)^t \frac{1}{t-1}\frac{1}{\varepsilon^{t-1}} < \infty$$

and we are done.

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