where $\delta > 8$, $\gamma \geq 4$, $h \in L^1(\mathbb{R})$, and C a constant, and suppose that $\tilde{\psi}$ and $\breve{\psi}$ satisfy analogous conditions with the obvious change of coordinates. Further, suppose that the shearlet system $SH(\phi, \psi, \tilde{\psi}, \breve{\psi}; c, \alpha)$ forms a frame for $L^2(\mathbb{R}^3)$.

Then, for any $\beta \in (1, 2]$ and $\nu > 0$, the frame $SH(\phi, \psi, \tilde{\psi}, \breve{\psi}; c, \alpha)$ provides almost optimally sparse approximations of functions $f \in \mathcal{E}_3^{\alpha,\beta}(\nu)$ in the sense that:

$$\|f - f_n\|_2^2 = O(n^{-\min\{\alpha/2-\epsilon, 2\beta/3\}}) \qquad \text{as } n \to \infty,$$

where $\epsilon = \epsilon(\alpha)$ satisfies $\epsilon < 0.04$ and f_n is the nonlinear n-term approximation obtained by choosing the n largest shearlet coefficients of f.

For $\alpha = 2$, we even have $\|f - f_n\|_2^2 = O(n^{-\min\{\alpha/2, 2\beta/3\}} (\log n)^2)$ in Theorem 1 which is optimal (up to a log-factor). We remark that a large class of generators $\psi, \tilde{\psi}$, and $\breve{\psi}$ satisfy the conditions (i) and (ii) in Theorem 1. Theorem 1 is a three-dimensional version of a result from [2]. However, as opposed to the two-dimensional setting, anisotropic structures in three-dimensional data comprise of *two* morphologically different types of structure, namely surfaces *and* curves. It would therefore be desirable to allow our 3D image class to also contain cartoon-like images with *curve* singularities. To achieve this we allow our discontinuity surface ∂B to be a closed, continuous, *piecewise* C^α smooth surface. We denote this function class $\mathcal{E}_3^{\alpha,\beta}(\nu, L)$, where $L \in \mathbb{N}$ is the maximal number of C^α pieces. Surprisingly, the pyramid-adapted shearlet systems still deliver the same almost optimal rate for this extended image class $\mathcal{E}_3^{\alpha,\beta}(\nu, L)$. We refer to [3] for the precise statement of the result.

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An introduction to mocklets

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(joint work with Enesto De Vito)

The *mocklets* are the admissible vectors for a class of representations of suitable semi-direct products which generalize the metaplectic representation of the symplectic group as restricted to its (standard) parabolic subgroups. The setup is the following:

i) the Hilbert space of signals is $L^2(\mathbb{R}^d)$, regarded in the frequencies domain;

ii) the parameter space G is the semi-direct product $G = \mathbb{R}^n \rtimes H$, where H is a locally compact second countable group with an n-dimensional representation $h \mapsto M_h$ (hence M_h is an $n \times n$ matrix);

iii) the group H acts also on $\mathbb{R}^d$ by a C^∞ action $(h,x) \mapsto h.x$ in such a way that there exists a positive character β of H for which

$$\int_{\mathbb{R}^d} \varphi(h^{-1}.x) = \beta(h) \int_{\mathbb{R}^d} \varphi(x)dx \qquad \varphi \in C_c(X).$$

Thus, for $h \in H$ the Jacobian of the map $x \mapsto h.x$ does not depend on x;

iv) a C^∞ map $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ such that, for all $x \in X$ and $h \in H$

$$\Phi(h.x) = h[\Phi(x)] = {}^t M_h^{-1} \Phi(x).$$

The *mock-metaplectic* representation is the representation that acts on $L^2(\mathbb{R}^d)$ as

$$(U_{(a,h)} f)(x) = \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(x), a\rangle} f(h^{-1}.x).$$

Our results in [1] are based on the following two assumptions:

H1 the H-orbits in the dual group $\mathbb{R}^n$ are locally closed;

H2 for almost all $y \in \Phi(\mathbb{R}^d)$ the stability subgroup of y is compact.

The first result gives a necessary condition to have a reproducing formula: the "translations" group $\mathbb{R}^n$ needs to be smaller than the space where the signals are defined.

Theorem 1. *If U is a reproducing representation, then the set of critical points of Φ has zero Lebesgue measure, hence $n \leq d$.*

From now on we assume the existence of an open H-invariant subset $X \subset \mathbb{R}^d$ on which the Jacobian of Φ is strictly positive and whose complement is negligible. As a consequence, $Y = \Phi(X)$ is an H-invariant open set of $\mathbb{R}^n$ and each level set $\Phi^{-1}(y)$ is a Riemannian submanifold (with Riemannian measure dv_y). The coarea formula gives the following disintegration formula for the d-dimensional Lebesgue measure dx: there exists a family $\{\nu_y\}_{y\in Y}$ of Radon measures on X such that

a) for all $y \in Y$ the measure ν_y is concentrated on $\Phi^{-1}(y)$;

b) for all $\varphi \in C_c(X)$ $\displaystyle\int_X \varphi(x) d\nu_y(x) = \int_{\Phi^{-1}(y)} \varphi(x) \frac{dv_y(x)}{(J\Phi)(x)}$;

c) for all $\varphi \in C_c(X)$ $\displaystyle\int_X \varphi(x)dx = \int_Y \Big(\int_X \varphi(x) d\nu_y(x) \Big) dy.$

The next step is to label the H-orbits of Y. The natural choice of the quotient space Y/H with the quotient topology can give rise to pathological spaces. However, H1 implies that there exist a locally compact second countable space Z with a Radon measure dz, a Borel map $\pi : Y \to Z$ and a family of Radon measures $\{\tau_z\}_{z\in Z}$ on Y with the following properties:

a) $\pi(y) = \pi(y')$ if and only if y and y' belongs to the same orbit and there is a Borel map $z \mapsto o(z)$ from $\Phi(Z)$ to Y such that $\pi(o(z)) = z$, so that $\pi^{-1}(z) = H[o(z)]$;

b) for all $z \in Z$, τ_z is concentrated on $\pi^{-1}(z)$ and is a relatively invariant measure with character $|\det(M_h)|^{-1}$;

c) for all $\varphi \in C_c(Y)$ $\displaystyle\int_Y \varphi(x)dy = \int_Z \Big(\int_Y \varphi(x) d\tau_z(y) \Big) dz.$

For all $z \in \pi(Y)$, we think of $o(z)$ as the origin of the orbit $\pi^{-1}(z) = H[o(z)]$ and we denote by $H_z = H_{o(z)}$ the stability subgroup of $o(z)$ and by $\nu_z = \nu_{o(z)}$ the H_z-invariant measure on the level set $\Phi^{-1}(o(z))$, which is H_z-invariant. Furthermore we fix the Haar measure ds on H_z is such a way that Weil's formula holds true

$$\int_H \varphi(h)|\det(M_h)|^{-1}\, dh = \int_Y \Big(\int_{H_z} \varphi(h_y\, s) ds\Big) d\tau_z(u) \qquad \varphi \in C_c(H),$$

where $h_y \in H$ is such that $h_y[o(z)] = y$ for all $y \in H[o(z)]$. Define π^z be the quasi-regular representation of H_z acting on $L^2(X, \nu_{o(z)})$ by

$$(\pi^z_s f_{o(z)})(x) = f_{o(z)}(s^{-1}.x) \qquad \nu_y\text{-a.e. } x \in \Phi^{-1}(o(z)).$$

Assumption H2 ensures that, up the a negligible set, H_z is compact, so that π^z is completely reducible

$$\pi^z \simeq \bigoplus_{i\in I} m_i\, \pi^{z,i} \qquad L^2(X, \nu_{o(z)}) \simeq \bigoplus_{i\in I} m_i\, \mathbb{C}^{d^z_i}$$

where each $\pi^{z,i}$ is an irreducible representation of H_z acting on some $\mathbb{C}^{d^z_i}$ and the cardinal $m_i \in \mathbb{N} \cup \{\aleph_0\}$ is the multiplicity of $\pi^{z,i}$ into π^z. It is possible to choose the index set I in such a way that m_i is independent of z (it can happen that $d^z_i = 0$).

We are now ready to characterize the existence of admissible vectors for the representation U, namely for the existence of mocklets.

Theorem 2. *Suppose that the set of critical points of Φ is negligible and Assumptions $H1$ and $H2$ hold true. If G is non-unimodular, then U is reproducing whereas if G is unimodular, U is reproducing if and only if*

$$\int_Z \frac{\operatorname{card}\Phi^{-1}(o(z))}{\operatorname{vol}(H_z)}\, dz < +\infty \qquad \text{where } \operatorname{vol} H_z = \int_{H_z} ds$$
$$m_i \le \dim(L^2(Y, \tau_z, \mathbb{C}^{d_{i,z}})) \qquad \forall i \in I\ ,\ \text{almost every } z \in Z.$$

In [1] an explicit characterization of the admissible vectors is given.

An example. One of the main motivations for our construction is the connection with the continuous shearlet transform that we now illustrate. The full shearlet group $\mathbb{R}^2 \rtimes (\mathbb{R} \rtimes \mathbb{R}_+)$, with scaling γ, can be realized as a subgroup of the symplectic group $Sp(2, \mathbb{R})$ as follows. Fix $\gamma \in \mathbb{R}$ (in the usual shearlet literature we have $\gamma = 1/2$) and define

$$\sigma_t = \begin{bmatrix} t_1 & t_2 \\ t_2 & 0 \end{bmatrix}, \qquad M_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}, \qquad M_a = \begin{bmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2-\gamma} \end{bmatrix},$$

where $t = (t_1, t_2) \in \mathbb{R}^2$, $s \in \mathbb{R}$ and $a \in \mathbb{R}_+$. The matrices

$$g(a, s, t) = \begin{bmatrix} M_s M_a & 0 \\ \sigma_t M_s M_a & {}^t(M_s M_a)^{-1} \end{bmatrix}$$

form a subgroup G of $Sp(2, \mathbb{R})$ which is isomorphic to the shearlet group. It falls in the setup described above as follows. First of all, $H = \{M_s M_a\}$ and $n = d = 2$.

On the one hand $\mathbb{R}^2 = \mathbb{R}^d$ must be interpreted as frequency space (and its points are denoted by ω), and on the other hand $\mathbb{R}^2 = \mathbb{R}^n$ is the dual of the representation space of H (and its points are denoted by y). The H action on the frequency space $\mathbb{R}^2$ is $h.\omega = M_s M_a \omega$, whereas the contragredient representation is

$$h[y] = \begin{bmatrix} a^{-1} & 0 \\ -sa^{-1} & a^{-\gamma} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The intertwining map is $\Phi(\omega_1, \omega_2) = -\frac{1}{2}(\omega_1^2, \omega_1\omega_2)$. It maps both half planes $X_L = \{\omega \in \mathbb{R}^2 : \omega_1 < 0\}$ and $X_R = \{\omega \in \mathbb{R}^2 : \omega_1 > 0\}$ onto X_L. By means of the restriction $\Phi_L = \Phi|_{X_L}$ we define the map $\Psi : L^2(X_L) \to L^2(X_L)$ by

$$\Psi f(y) = |J_{\Phi_L^{-1}}(y)|^{1/2} f(\Phi_L^{-1}(y)).$$

The inverse Fourier transform $\mathcal{F}^{-1}$ maps $L^2(X_L)$ onto a closed proper subspace $\mathcal{S} \subset L^2(\mathbb{R}^2)$ and it is easy to see that for $F \in \mathcal{S}$ we have

$$S_{a,s,t}\, F(\omega) = \mathcal{F}^{-1}\left(\Psi U_{g(a,s,t)} \Psi^{-1} \mathcal{F} F\right)(\omega),$$

where

$$S_{a,s,t}F(\omega) = a^{-3/4} F\left(\begin{bmatrix} a^{-1} & -sa^{-1} \\ 0 & a^{-\gamma} \end{bmatrix} \begin{bmatrix} \omega_1 - t_1 \\ \omega_2 - t_2 \end{bmatrix}\right)$$

is the continuous shearlet representation (see e.g. [2]) and where

$$U_{g(a,s,t)} f(\omega) = a^{\gamma/2} e^{i\pi\langle\sigma_t\omega,\omega\rangle} f\Big(\begin{bmatrix} a^{1/2} & 0 \\ sa^{\gamma-1/2} & a^{\gamma-1/2} \end{bmatrix}\omega\Big).$$

is the restriction of the metaplectic representation to G. All the hypotheses that we have introduced are easily satisfied. By applying our results one obviously finds the well-known conditions on admissible vectors, that is, on shearlets.

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Shearlet Multiresolution and Adaptive Directional Multiresolution

Tomas Sauer

(joint work with Gitta Kutyniok, Angelika Kurtz)

Like with the wavelet transform there are several ways and conceptional concepts to develop numerical implementations of the shearlet transform. The first, maybe more straightforward concept, is to sample the continuous transformation at a finite set of parameters chosen such that the resulting discrete transformation (hopefully) captures all information of the underlying transformed functions. In both cases, the numerical computation can be accelerated by using a Fast Fourier Transform to evaluate the underlying convolutions, cf[1]. The second approach, on the other hand, is entirely discrete and relies on filter banks and the concept